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NOETHERIAN RINGS OF KRULL DIMENSION 2*

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ABSTRACT. We prove that a maximal ideal M of $D[x]$ has two generators and is of the form $\langle p, q(x)\rangle$ where p is an irreducible element in a PID D having infinitely many nonassociate irreducible elements and $q(x)$ is an irreducible non-constant polynomial in $D[x]$. Moreover, we find how minimal generators of maximal ideals of a polynomial ring $D[x]$ over a DVR D consist of and how many generators those maximal ideals have.

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1. Introduction

Throughout this paper, we assume that a ring R is a commutative ring with unity 1, $R[x]$ is a polynomial ring over a ring R, and $R[[x]]$ is a power series ring over a ring R (see [1, 3] for more details and their further properties).

In [4], they found the necessary and sufficient condition that every maximal ideal M of a polynomial ring $D[x]$ over a principal ideal domain D (PID for short) has height 2. By Krull's principal ideal theorem in [3], it is well known that if a prime ideal $\wp$ of a Noetherian ring R which is minimal among the prime ideals containing a proper ideal $(x_1, \ldots, x_n)$ in R has height $\leq n$. Hence every maximal ideal of a Noetherian ring of height 2 has at least two minimal generators. However, we don't know when a maximal ideal of height 2 has two generators in general.

In Section 2, we show that a maximal ideal M of $D[x]$ has two generators and is of the form $\langle p, q(x)\rangle$ where p is an irreducible element in a PID D having infinitely many nonassociate irreducible elements and $q(x)$ is an irreducible polynomial in $D[x]$ (see Theorem 2.6).

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In Section 3, we introduce a discrete valuation ring D (DVR for short) and show how minimal generators of maximal ideals of $D[x]$ consist of and how many generators those maximal ideals have (see Theorem 3.10). Furthermore, we give complete descriptions of maximal ideals of a polynomial ring $k[[x]][y]$ over a typical DVR $k[[x]]$ when k is a field (see Corollary 3.11).

2. Maximal ideals of a polynomial ring over a PID

In this section, we shall investigate maximal ideals of a polynomial ring over a PID having infinitely many nonassociate irreducible elements. First of all, we introduce some well-known definitions and preliminary results.

Definition 2.1 ([1]). Let R be a commutative ring with unity 1. Then we denote the collection of all maximal ideals in R by

$$\Omega(R) = \{M \mid M \text{ is a maximal ideal of } R\},$$

and the collection of all prime ideals in R by

$$\mathrm{Spec}(R) = \{\wp \mid \wp \text{ is a prime ideal of } R\}.$$

The *Jacobson radical ideal* $\sqrt{R}$ of a ring R is the intersection of all maximal ideals M in R. That is,

$$\sqrt{R} = \bigcap_{M \in \Omega(R)} M.$$

The *nilradical* $\sqrt{0}$ of R is the intersection of all prime ideals of R. In other words,

$$\sqrt{0} = \bigcap_{\wp \in \mathrm{Spec}(R)} \wp.$$

Definition 2.2 ([1, 3]). Let R be a commutative ring with unity 1. The *height* of a prime ideal $\wp$ is the supremum of the lengths of all the chains

$$\wp_0 \subset \wp_1 \subset \cdots \subset \wp_t = \wp$$

of prime ideals of R that end at $\wp$.

The *Krull dimension* $\dim R$ is the supremum of the lengths of all the chains of primes ideals of $\wp$, or equivalently, the supremum of the heights of all the prime ideals $\wp$ in R.

The following theorem was proved by F. Zanello in [4].

Theorem 2.3. *Let D be a PID. Then the following statements are equivalent.*

(a) *every maximal ideal of $D[x]$ has height* 2.

(b) *D has infinitely many pairwise nonassociate irreducible elements.*

Definition 2.4 ([2]). Let D be a commutative ring with unity 1. D is *Noetherian* if D satisfies the ascending chain condition on ideals, i.e., if for every chain of ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$ of D, there is an integer n such that $I_j = I_n$ for all $j \geq n$.

Let D be a commutative ring with unity 1 and let $D[[x]]$ be the *ring of formal power series* over the ring D. Its elements are called *power series.* The power series in $D[[x]]$ is denoted by the formal sum $\sum_{i=0}^{\infty} a_i x^i$ and the elements a_i are called *coefficients* and a_0 is called the *constant term.*

Remark 2.5 ([2]). Recall that

(a) a commutative ring D with unity 1 is Noetherian if and only if every ideal of D is finitely generated.

(b) If D is a commutative Noetherian ring with unity 1, then both the formal power series ring $D[[x]]$ and the n variable polynomial $D[x_1, \ldots, x_n]$ over the ring D are also Noetherian. Moreover,

$$\dim D[x_1, \ldots, x_n] = \dim D[[x_1, \ldots, x_n]] = \dim D + n.$$

Now we prove the main theorem in this section.

Theorem 2.6. *Let D be a PID having infinitely many nonassociate irreducibles and $D[x]$ be a polynomial ring over D. Then every maximal ideal of $D[x]$ is not principal and of the form $\langle p, q(x)\rangle$ where p is an irreducible of D and $q(x)$ is an irreducible polynomial in $D[x]$ of a positive degree. Conversely, if p is an irreducible of D and $q(x)$ is an irreducible polynomial in $D[x]$ of a positive degree, then every ideal of the form $\langle p, q(x)\rangle$ in $D[x]$ is maximal.*

Proof. Since D is a PID having infinitely many irreducibles, every maximal ideal M of $D[x]$ has height 2 by Theorem 2.3. Hence M cannot be generated by a single element in M.

Since D is a PID (i.e., Noetherian), $D[x]$ is also a Notherian ring by Remark 2.5 (b). By Remark 2.5 (a), every maximal ideal M is generated by a finite number of irreducible polynomials, $p_1(x), \ldots, p_s(x)$ in $D[x]$ with $s \geq 2$, i.e., $M = \langle p_1(x), \ldots, p_s(x)\rangle$. Note that one of the $p_i(x)$'s must have a positive degree.

Let k be a field of quotients of D. Then $k[x]$ is a PID, and M has to be a ring $k[x]$ since $s \geq 2$. In other words, there exist $q_i(x) \in k[x]$ for $i = 1, \ldots, s$ such that

$$p_1(x)q_1(x) + p_2(x)q_2(x) + \cdots + p_s(x)q_s(x) = 1.$$

Moreover, there exist $c_i \in D$ such that $c_i q_i(x) \in D[x]$ for every $i = 1, \ldots, s$. Let $c = c_1 c_2 \cdots c_s$. Then $cq_i(x) \in D[x]$ for every i, and thus

$$p_1(x)(cq_1(x)) + p_2(x)(cq_2(x)) + \cdots + p_s(x)(cq_s(x)) = c \in M.$$

Since c is not a unit and a product of a finite number of irreducibles in D, M contains an irreducible element p in D.

Note that there is a natural isomorphism φ from $D[x]/pD[x]$ to $(D/\langle p\rangle)[x]$ where $\langle p\rangle = \{p\alpha \mid \alpha \in D\}$. In other words, for every $f(x) = \sum a_i x^i \in D[x]$,

$$\varphi\left(f(x) + pD[x]\right) = \sum (a_i + \langle p\rangle)x^i := \bar{f}(x).$$

Since $\langle p\rangle$ is a maximal ideal of D, i.e, $D/\langle p\rangle$ is a field and $\overline{M} = \langle \bar{p}_1(x), \ldots, \bar{p}_s(x)\rangle$ is a maximal ideal of $(D/\langle p\rangle)[x]$, we have $\overline{M} = \langle \bar{p}_1(x), \ldots, \bar{p}_s(x)\rangle = \langle \bar{q}(x)\rangle$ for some irreducible polynomial $q(x) \in M$. Furthermore, since every element $\bar{p}_i(x)$ is a multiple of $\bar{q}(x)$ in $(D/\langle p\rangle)[x]$, we have that

$$\bar{p}_i(x) = \bar{q}(x) \cdot \bar{g}_i(x)$$

for some $g_i(x) \in D[x]$ for every i. In other words, $p_i(x) = q(x)g_i(x) + r_i(x)$ for some $r_i(x) \in pD[x]$ for such i. Hence

$$\begin{aligned} M &= \langle p_1(x), \ldots, p_s(x)\rangle \\ &= \langle q(x) \cdot g_1(x) + r_1(x), \ldots, q(x) \cdot g_s(x) + r_s(x)\rangle \\ &\subseteq \langle p, q(x)\rangle, \end{aligned}$$

and thus $M = \langle p, q(x)\rangle$ since M is a maximal ideal of $D[x]$ and $M \subseteq \langle p, q(x)\rangle \subseteq M$.

Furthermore, if $q(x) = q$ is a constant, then $1 \in \langle p, q\rangle = M$, which is a contradiction. Therefore, $q(x)$ must be an irreducible polynomial in $D[x]$ of a positive degree.

Using the above isomorphism φ, one can see that every ideal of the form $\langle p, q(x)\rangle$ in $D[x]$, where p is an irreducible element in D and $q(x)$ is an irreducible polynomial in $D[x]$ of a positive degree, is maximal, as we desired. □

The following corollary is immediate from Theorem 2.6 since a ring $\mathbb{Z}$ of integers or a ring of Gaussian integers $\mathbb{Z}[i]$ is a PID and has infinitely many primes.

Corollary 2.7. *Let D be either a ring of integers $\mathbb{Z}$ or a ring of Gaussian integers $\mathbb{Z}[i]$. Then every maximal ideal of $D[x]$ is not principal and of the form $\langle p, q(x)\rangle$ where p is an irreducible element in D and $q(x)$ is an irreducible polynomial in $D[x]$ of a positive degree. Conversely, if p is an irreducible element in D and $q(x)$ is an irreducible polynomial in $D[x]$ of a positive degree, then every ideal of the form $\langle p, q(x)\rangle$ in $D[x]$ is maximal.*

3. Maximal ideals of a polynomial ring $D[x]$ over a DVR D

In this section, we give some examples of integral domains of Krull dimension 2, which don't satisfy the condition of Theorem 2.3. In other words, we shall find maximal ideals of $D[x]$ having height 1 or 2 when D is a discrete valuation ring (see Definition 3.1). Moreover, we will give full descriptions of such maximal ideals.

Definition 3.1 ([1]). A local domain D with a unique maximal ideal M is said to be a *discrete valuation ring* (DVR for short) if M is principal.

The following remark is well known, so we recall them without proof here (see [1, 2, 3]).

Remark 3.2. Let D be a commutative ring with unity 1 and $f(x) = \sum_{i=0}^{\infty} a_i x^i \in D[[x]]$.

(a) $f(x)$ is a unit in $D[[x]]$ if and only if a_0 is a unit in D.

(b) If a_0 is irreducible in D, then $f(x)$ is irreducible in $D[[x]]$.
(c) If $f(x)$ is nilpotent in $D[[x]]$, then a_i is nilpotent in D for all $i \geq 0$.
(d) $f(x) \in \sqrt{D[[x]]}$ if and only if $a_0 \in \sqrt{D}$.
(e) The contraction of a maximal ideal M of $D[[x]]$ is a maximal ideal of D. In other words, if $M \in \Omega(D[[x]])$, then $M \bigcap D \in \Omega(D)$. Furthermore, $M = \langle M \bigcap D, x \rangle$.
(f) Every prime ideal of D is the contraction of a prime ideal of $D[[x]]$. In other words,

$$\operatorname{Spec}(D) = \{\wp \cap D \mid \wp \in D[[x]]\}.$$

Remark 3.3. Let k be a field. Then

(a) $k[[x]]$ is a DVR, and so PID whose only ideals are $\{0\}$, $k[[x]]$, and $\langle x^k \rangle$ for some $k \in \mathbb{Z}^+$.
(b) The principal ideal $\langle x \rangle$ is the unique maximal ideal of $k[[x]]$, that is, there is only one irreducible element x in $k[[x]]$.

Example 3.4. (a) Let M be a maximal ideal of $\mathbb{Z}[[x]]$. Then, by Remark 3.2 (e), $M = \langle M \bigcap \mathbb{Z}, x \rangle$. Moreover, by Remark 3.2 (e) or (f), $M \bigcap \mathbb{Z}$ is also a maximal ideal of $\mathbb{Z}$, that is, $M \bigcap \mathbb{Z} = \langle p \rangle$ for some prime number $p \in \mathbb{Z}$. It follows that $M = \langle p, x \rangle$.
(b) Let k be a field. Note that

$$k[[x]][y] \subsetneq k[y][[x]].$$

For example, if

$$f(x) = 1 + xy + x^2y^2 + \cdots + x^ny^n + \cdots = \sum_{i=0}^{\infty} x^i y^i,$$

then $f(x) \in k[y][[x]]$, but $f(x) \notin k[[x]][y]$.

Question 3.5. (a) Let $k[x]$ be a one variable polynomial ring over a field k. *What are maximal ideals M of $k[x][[y]]$?*
(b) More generally, let D be a PID. *What are maximal ideals of $D[[x]]$?*

By Remark 3.2 (e), we can find an answer to Question 3.5 (see Proposition 3.6), and it gives another example of an integral domain of Krull dimension 2 whose all maximal ideals have height 2 and two minimal generators.

Proposition 3.6. *With notations as in Question 3.5, every maximal ideal M of $D[[x]]$ is of the form $\langle p, x \rangle$ for some irreducible element $p \in D$. In particular, a maximal ideal of $k[x][[y]]$ is of the form $\langle p(x), y \rangle$ for some irreducible polynomial $p(x) \in k[x]$.*

Proof. Since M is a maximal ideal of $D[[x]]$, by Remark 3.2 (e), $M = \langle M \bigcap D, x \rangle$ and $M \bigcap D$ is also a maximal ideal of D. Hence $M \bigcap D = \langle p \rangle$ for some irreducible element $p \in D$, that is, $M = \langle p, x \rangle$, as we claimed.

Furthermore, since $k[x]$ is also a PID, every maximal ideal of $k[x][[y]]$ is of the form $\langle p(x), y\rangle$ for some irreducible polynomial $p(x) \in k[x]$, as we wished. This completes the proof. □

The following Corollary 3.7 is immediate from Proposition 3.6 and Remark 3.2 (e), so we omit the proof here.

Corollary 3.7. *With notations as in Question 3.5, every maximal ideal M of $D[[x_1,\dots,x_n]]$ is of the form $\langle p, x_1, \dots, x_n\rangle$ for some irreducible element $p \in D$. In particular, a maximal ideal of $k[x][[y_1,\dots,y_n]]$ is of the form $\langle p(x), y_1, dots, y_n\rangle$ for some irreducible polynomial $p(x) \in k[x]$.*

The following corollary is also obtained from Theorem 2.6, Proposition 3.6, and Remark 3.2 (e).

Corollary 3.8. *Let $k[x_1, x_2]$ be a two variable polynomial ring over a field k and $R := k[x_1, x_2][[y_1,\dots,y_n]]$ be an n-variable power series ring over a ring $k[x_1,x_2]$. Then every maximal ideal M of R is of the form $\langle p(x_1), q(x_1,x_2), y_1, \dots, y_n\rangle$ where $p(x_1)$ is an irreducible polynomial in $k[x_1]$ and $q(x_1,x_2)$ is an irreducible polynomial in $k[x_1,x_2]$ such that $\bar{q}(x_1,x_2) \in (k[x_1]/\langle p(x_1)\rangle[x_2])$ is also irreducible.*

Now we consider a ring $k[[x]][y]$ over a field k. Note that there are two kinds of maximal ideals in $k[[x]][y]$ of either height 1 or 2 by Theorem 2.3 since $k[[x]]$ has only one irreducible element x. Hence we have a natural question as follows.

Question 3.9. What are maximal ideals in $k[[x]][y]$?

Before we give an answer to Question 3.9, we shall prove slightly more general case here.

Theorem 3.10. *Let D be a DVR with a maximal ideal $\langle p\rangle$ and M be a maximal ideal of $D[x]$. If M has height 2, then M is of the form $\langle p, q(x)\rangle$ for some irreducible non-constant polynomial $q(x) \in D[x]$, and if M has height 1, then M is of the form $\langle q(x)\rangle$ for some irreducible non-constant polynomial $q(x) = a_0 + a_1x + \cdots + a_tx^t \in D[x]$ where a_0 is a unit in D, and $p \mid a_i$ for every $i = 1, 2, \dots, t$.*

Proof. Let $\wp = M \cap D$. Then $\wp$ is either $\langle 0\rangle$ or $\langle p\rangle$ since $\langle 0\rangle$ and $\langle p\rangle$ are only prime ideals in D.

First, assume that $\wp = \langle p\rangle$. Let $\varphi : D[x] \to (D/\langle p\rangle)[x]$ be given by

$$\varphi(a_0 + a_1x + \cdots + a_sx^s) = (a_0 + \langle p\rangle) + (a_1 + \langle p\rangle)x + \cdots + (a_s + \langle p\rangle)x^s$$

for $a_0 + a_1x + \cdots + a_sx^s \in D[x]$. Then φ is a ring homomorphism from $D[x]$ onto $(D/\langle p\rangle)[x]$. Since M is a maximal ideal in $D[x]$, $\varphi(M)$ is also a maximal ideal $(D/\langle p\rangle)[x]$. Hence

$$\varphi(M) = \langle\overline{q(x)}\rangle$$

for some irreducible polynomial $q(x) \in D[x]$ since $D/\langle p \rangle$ is a field. We now show that $M = \langle p, q(x) \rangle$. Sine p and $q(x)$ are in M, it is clear that $\langle x, q(x) \rangle \subseteq M$. Conversely, let $f(x) \in M$. Then

$$\varphi(f(x)) = \overline{f(x)} = \overline{q(x)} \cdot \overline{g(x)}$$

for some $g(x) \in D[x]$. In other words,

$$f(x) = p \cdot h(x) + q(x)g(x)$$

for some $h(x) \in D[x]$, and hence $f(x) \in \langle p, q(x) \rangle$. Thus $M = \langle p, q(x) \rangle$, as we claimed.

Now suppose that $\wp = \langle 0 \rangle$. Using the same idea as in the proof of Theorem 2 in [4], we have that $M = \langle q(x) \rangle$ for some irreducible polynomial

$$q(x) = a_0 + a_1 x + \cdots + a_t x^t \in D[x],$$

where a_0 is a unit in D, $p \mid a_i$ for every $i = 1, 2, \ldots, t$, and M has height 1. This completes the proof. □

The following corollary is immediate from Theorem 3.10 since $k[[x]]$ is a DVR with a unique maximal ideal $\langle x \rangle$ and gives the complete answer to Question 3.9.

Corollary 3.11. *With notation as in Question 3.9, let M be a maximal ideal of $k[[x]][y]$. If M contains x, then M is a maximal ideal of height 2 and of the form $\langle x, p(y) \rangle$ for some irreducible polynomial $p(y) \in k[y]$. If M does not contain x, then M is a principal maximal ideal of height 1and of the form $\langle p(y) \rangle$ for some irreducible polynomial $p(y) = p_0(x) + p_1(x)y + \cdots + p_t(x)y^t \in k[[x]][y]$, $p_i(x) \in k[[x]]$ where $p_0(x) = a$ for some nonzero $a \in k$ and $x \mid p_i(x)$ for $i = 1, 2, \ldots, t$.*

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