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FIXED POINT THEOREMS IN d-COMPLETE TOPOLOGICAL SPACES

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ABSTRACT. We prove the existence of common fixed points for three self mappings satisfying contractive conditions in d-complete topological spaces. Our results are generalizations of result of Troy L. Hicks and B. E. Roades[Troy L. Hicks and B. E. Roades, Fixed points for pairs of mappings in d-complete topological spaces, Int. J. Math. and Math. Sci., 16(2)(1993), 259-266].

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1. Introduction and preliminaries

Banach fixed point theorem plays an important role in several branches of mathematics. For instance, it has been used to show the existence of solutions of nonlinear equations, nonlinear Volterra integral equations, nonlinear integro-differential equations and sysytems of linear equations and to show the convergence of algorithms in computational mathematics. Beacause of its importance for mathematical theory, Banach fixed point theorem has been extended in many directions[1,2,3,4,7,11,12,13].

The generalizations to a large class of non-metric spaces which include d-complete symetric(semi-metric)spaces and complete quasi-metric spaces are enormouse too. One of these generalizations is the following Theorem 1.1.

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Hicks[8] and Hicks and Rhoades[9,10] gave several fixed point theorems in d-complete topological spaces. In [10], Hicks and Rhoades obtained the following theorem.

Theorem 1.1. *Let (X,t) be a Hausdorff d-complete topological space and $f : X \to X$ be w-continuous functions. Then f has a fixed point in X if and only if there exist a $k \in (0,1)$ and a w-continuous function $g : X \to X$ which commutes with f and satisfies*
(i) $g(X) \subset f(X)$,
(ii) $d(gx, gy) \leq kd(fx, fy)$ for all $x, y \in X$.
Indeed, if (i) and (ii) hold then f and g have a unique common fixed point in X.

In this paper we give generalizations of above theorem.

Let (X,t) be a topological space and $d : X \times X \to [0,\infty)$ ba a function such that $d(x,y) = 0$ if and only if $x = y$. A topological space (X,t) is said to be d-*complete* if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that the sequence is convergent to some $x \in X$ in (X,t). Complete metric spaces and Complete quasi-metric spaces are examples of d-complete topological spaces.

A function $f : (X,t) \to (X,t)$ is w-*continuous* at $x \in X$ if $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} fx_n = fx$.

Denote Λ as the family of all nondecreasing and continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that
(ϕ1) $\phi(0) = 0$ and $0 < \phi(t) < t$ for all $t > 0$,
(ϕ2) $\Sigma_{n=0}^{\infty} \phi^n(t) < \infty$ for all $t > 0$,
where $\phi^n(t)$ is n-th iteration of $\phi(t)$.

Note that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$.

Lemma 1.2. *Let (X,t) be a d-complete topological space and let $\{x_n\}$ be a sequence in X. If, for $\phi \in \Lambda$, $d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n))$ for $n = 1, 2, 3, \cdots$, then $\{x_n\}$ converges to a point x in (X,t).*

Proof. Assume that $d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n))$, for $n = 1, 2, 3, \cdots$. Then we have

$$d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n)) \leq \phi^2(d(x_{n-2}, x_{n-1})) \leq \cdots \leq \phi^n(d(x_0, x_1)).$$

Thus we obtain $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$ by (ϕ2). It follows from the d-completeness of X that there exists an $x \in X$ such that $\lim_{n\to\infty} x_n = x$. □

2. Fixed point theorems

Our first main result is the following.

Theorem 2.1. *Let (X, t) be a Hausdorff d-complete topological space and $f, h : X \to X$ be w-continuous functions. Then f and h have a common fixed point in X if and only if there exist a $\phi \in \Lambda$ and w-continuous function $g : X \to X$ such that*

(i) g commute with f and g commute with h,

(ii) $g(X) \subset f(X) \cap h(X)$,

(iii) $d(gx, gy) \leq \phi\Big(max\{d(fx, hy), d(hx, gx), d(fy, gy)\}\Big)$, for all $x, y \in X$.

Indeed, if (i), (ii) and (iii) hold then f, g and h have a unique common fixed point in X.

Proof. Let $z \in X$ be a common fixed point of f and h. Put $g(x) = z$ for all $x \in X$. Then g is a w-continuous function. Also, $g(X) \subset f(X) \cap h(X)$, and g commute with f and g commute with h.

For $\phi \in \Lambda$, $d(gx, gy) = d(z, z) = 0 \leq \phi\Big(max\{(fx, hy), d(hx, gx), d(fy, gy)\}\Big)$, for all $x, y \in X$.

Suppose that there exist a $\phi \in \Lambda$ and w-continuous function $g : X \to X$ such that (i), (ii) and (iii) are satisfied.

Let $x_0 \in X$ and x_n be such that $gx_n = fx_{n+1} = hx_{n+1}$.
If $d(gx_{n_0-1}, gx_{n_0}) < d(gx_{n_0}, gx_{n_0+1})$ for some n_0, then

$$\begin{aligned}
&d(gx_{n_0}, gx_{n_0+1})\\
&\leq \phi\Big(max\{d(fx_{n_0}, hx_{n_0+1}), d(hx_{n_0}, gx_{n_0}), d(fx_{n_0+1}, gx_{n_0+1}\}\Big)\\
&\leq \phi\Big(max\{d(gx_{n_0-1}, gx_{n_0}), d(gx_{n_0-1}, gx_{n_0}), d(gx_{n_0}, gx_{n_0+1})\}\Big)\\
&\leq \phi\Big(\{d(gx_{n_0}, gx_{n_0+1})\}\Big)\\
&< d\Big(gx_{n_0}, gx_{n_0+1}\Big)
\end{aligned}$$

which is a contradiction. Thus we have $d(gx_n, gx_{n+1}) \leq d(gx_{n-1}, gx_n)$ for all n. Then we have

$$\begin{aligned}
&d(gx_n, gx_{n+1})\\
&\leq \phi\Big(max\{d(fx_n, hx_{n+1}), d(hx_n, gx_n), d(fx_{n+1}, gx_{n+1}\}\Big)\\
&\leq \phi\Big(max\{d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\}\Big)\\
&\leq \phi\Big(max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\}\Big)\\
&\leq \phi\Big(d(gx_{n-1}, gx_n)\}\Big).
\end{aligned}$$

By Lemma 1.2, there exists a $p \in X$ such that $\lim_{n\to\infty} gx_n = p$. Then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = p$.

By w-continuity of f and g, $\lim_{n\to\infty} fgx_n = fp$ and $\lim_{n\to\infty} gfx_n = gp$. From (i) $fp = gp$ since (X,t) is Hausforff.

Also, $\lim_{n\to\infty} ghx_n = gp$ and $\lim_{n\to\infty} hgx_n = hp$ and so $gp = hp$. Thus $z = fp = gp = hp$.

From (i) we have $ggp = fgp = gfp = ghp = hgp$. Then we have

$$\begin{aligned}
&d(z, gz) \\
&= d(gp, ggp) \\
&\leq \phi\Big(max\{d(fp, hgp), d(hp, gp), d(fgp, ggp\}\Big) \\
&\leq \phi\Big(max\{d(gp, ggp), d(gp, gp), d(ggp, ggp\}\Big) \\
&= \phi\Big(d(gp, ggp)\Big) \\
&= \phi(d(z, gz))
\end{aligned}$$

which implies $d(z, gz) = 0$ and so $z = gz$. Thus we obtain $z = gz = fz = hz$. For the uniqueness, let $u = gu = fu = hu$.

Then

$$\begin{aligned}
&d(z, u) \\
&= d(gz, gu) \\
&\leq \phi\Big(max\{d(fz, hu), d(hz, gz), d(fu, gu\}\Big) \\
&\leq \phi\Big(max\{d(z, u), d(z, z), d(u, u)\}\Big) \\
&= \phi(d(z, u)).
\end{aligned}$$

Thus we obtain $z = u$. □

If we have $\phi(t) = kt, k \in (0,1), t \geq 0$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let (X,t) be a Hausdorff d-complete topological space and $f, h : X \to X$ be w-continuous functions. Then f and h have a common fixed point in X if and only if there exist a $k \in (0,1)$ and w-continuous function $g : X \to X$ such that*

(i) g commute with f and g commute with h,

(ii) $g(X) \subset f(X) \cap h(X)$,

(iii) $d(gx, gy) \leq kmax\Big\{(fx, hy), d(hx, gx), d(fy, gy)\Big\}$ for all $x, y \in X$.

Indeed, if (i), (ii) and (iii) hold then f, g and h have a unique fixed point in X.

By Theorem 2.1 and Corollary 2.2, we obtain the following two corollaries.

Corollary 2.3. *Let (X, t) be a Hausdorff d-complete topological space and $f, h : X \to X$ be w-continuous functions. Then f and h have a common fixed point in X if and only if there exist a $\phi \in \Lambda$ and w-continuous function $g : X \to X$ such that*
(i) g commute with f and g commute with h,
(ii) $g(X) \subset f(X) \cap h(X)$,
(iii) $d(gx, gy) \leq \phi((fx, hy))$ for all $x, y \in X$.
Indeed, if (i), (ii) and (iii) hold then f, g and h have a unique fixed point in X.

Corollary 2.4. *Let (X, t) be a Hausdorff d-complete topological space and $f, h : X \to X$ be w-continuous functions. Then f and h have a common fixed point in X if and only if there exist a $k \in (0, 1)$ and w-continuous function $g : X \to X$ such that*
(i) g commute with f and g commute with h,
(ii) $g(X) \subset f(X) \cap h(X)$,
(iii) $d(gx, gy) \leq k(fx, hy)$ for all $x, y \in X$.
Indeed, if (i), (ii) and (iii) hold then f, g and h have a unique fixed point in X.

If we have $f = h$ in Corollary 2.3, then we obtain the following corollary.

Corollary 2.5. *Let (X, t) be a Hausdorff d-complete topological space and $f : X \to X$ be w-continuous function. Then f has a fixed point in X if and only if there exist a $\phi \in \Lambda$ and w-continuous function $g : X \to X$ such that*
(i) g commute with f ,
(ii) $g(X) \subset f(X)$,
(iii) $d(gx, gy) \leq \phi((fx, fy))$ for all $x, y \in X$.
Indeed, if (i), (ii) and (iii) hold then f and g have a unique fixed point in X.

Remark 3.1. The above corollary generalize Theorem 1.1(theorem 2.1[10]).

Theorem 2.6. *Let (X, t) be a Hausdorff d-complete topological space and $f, h : X \to X$ be w-continuous functions. Then f and h have a common fixed point in X if and only if there exist nonnegative constants a_i satisfying $a_1 + a_2 + a_3 < 1$ and w-continuous function $g : X \to X$ such that*
(i) g commute with f and g commute with h,
(ii) $g(X) \subset f(X) \cap h(X)$,
(iii) $d(gx, gy) \leq a_1 d(fx, hy) + a_2 d(hx, gx) + a_3 d(fy, gy)$, for all $x, y \in X$,

Indeed, if $(i), (ii)$ and (iii) hold, then f, g and h have a unique common fixed point in X.

Proof. Let $z = fz = hz$, and let $gx = z$ for all $x \in X$. Then conditions $(i), (ii)$ and (iii) are satisfied.

Assume that there exist nonnegative constants a_i satisfying $a_1 + a_2 + a_3 < 1$ and w-continuous function $g : X \to X$ such that $(i), (ii)$ and (iii) are satisfied. Then as in the proof of Theorem 2.1, we have a sequence $\{x_n\}$ in X such that $gx_n = fx_{n+1} = hx_{n+1}$. Then we have

$$\begin{aligned} &d(gx_n, gx_{n+1}) \\ &\leq a_1 d(fx_n, hx_{n+1}) + a_2 d(hx_n, gx_n) + a_3 d(fx_{n+1}, gx_{n+1}) \\ &= a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_{n-1}, gx_n) + a_3 d(gx_n, gx_{n+1}) \end{aligned}$$

which implies $d(gx_n, gx_{n+1}) \leq kd(gx_{n-1}, gx_n)$, where $k = \dfrac{a_1 + a_2}{1 - a_3}$. By Lemma 1.2, there exists a $p \in X$ such that $\lim_{n\to\infty} gx_n = p$. Then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = p$.

As in the proof of Theorem 2.1, $z = fp = gp = hp$. From (i) we have $ggp = fgp = gfp = ghp = hgp$. Thus we obtain

$$\begin{aligned} &d(z, gz) \\ &= d(gp, ggp) \\ &\leq a_1 d(fp, hgp) + a_2 d(hp, gp) + a_3 d(fgp, ggp) \\ &= a_1 d(gp, ggp) + a_2 d(gp, gp) + a_3 d(ggp, ggp) \\ &= a_1 d(gp, ggp) \\ &= a_1 d(z, gz) \end{aligned}$$

which implies $d(z, gz) = 0$ and so $z = gz$. Thus we obtain $z = gz = fz = hz$.

Assume that $u = gu = fu = hu$. Then we have

$$\begin{aligned} &d(z, u) \\ &= d(gz, gu) \\ &\leq a_1 d(fz, hu) + a_2 d(hz, gz) + a_3 d(fu, gu) \\ &= a_1 d(z, u) + a_2 d(z, z) + a_3 d(u, u) \\ &= a_1 d(z, u). \end{aligned}$$

Thus we obtain $z = u$. □

If we have $f = h$ in Theorem 2.6, then we obtain the following corollary.

Corollary 2.7. *Let (X,t) be a Hausdorff d-complete topological space and $f: X \longrightarrow X$ be w-continuous function. Then f has a fixed point in X if and only if there exist nonnegative constants a_i satisfying $a_1+a_2+a_3<1$ and w-continuous function $g: X \longrightarrow X$ such that*

(i) g commute with f,

(ii) $g(X) \subset f(X)$,

(iii) $d(gx,gy) \leq a_1 d(fx,fy) + a_2 d(fx,gx) + a_3 d(fy,gy)$, for all $x,y \in X$,
Indeed, if (i), (ii) and (iii) hold, then f and g have a unique common fixed point in X.

Remark 3.2. The above corollary generalize Theorem 1.1(theorem 2.1[10]).

References

1. R. P. Agarawl, D. O. O'Regan, N. Shahzad, *Fixed point theorems for generalized contractive maps of Mei-Keeler type*, Math. Nachr. **276** (2004), 3-12.
2. J. P. Aubin, J. Siegel, *Fixed point and stationary points of dissipative multi-valued maps*, Proc. Amer. Math. Soc. **78** (1980), 391-398.
3. A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci. **29** (2002), 531-536.
4. H. Covitz, S. B. Nadler Jr., *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math. **8** (1970), 5-11.
7. Y. Feng, S. Liu, *Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings*, J. Math. Anal. Appl. **317** (2006), 103-112.
8. Troy L. Hicks, *Fixed point theorems for d-complete topological spaces I*, Int. J. Math. and Math. Sci. **15** (1992), 435-440.
9. Troy L. Hicks and B. E. Roades, *Fixed point theorems for d-complete topological spaces II*, Math. Japonica **37** (1992), 847-853.
10. Troy L. Hicks and B. E. Roades, *Fixed points for pairs of mappings in d-complete topological spaces*, Int. J. Math. and Math. Sci. **16(2)** (1993), 259-266.
11. S. B. Nadler Jr., *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475-478.
12. P. Vijayaraju, B. E. Rhoades, R. Mohanraj, *A fixed point theorem for a pair of maps satisfying a general contracive condition of integral type*, Int. J. Math. Math. Sci. **15** (2005), 2359-2364.
13. T. Wang, *Fixed point theorems and fixed point stability for multivalued mappings on metric spaces*, J. Nanjing Univ. Math. Baq. **6** (1989), 16-23.

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