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SEVERAL NEW PRACTICAL CRITERIA FOR NONSINGULAR H-MATRICES

HONGMIN MO

ABSTRACT. H-matrix is a special class of matrices with wide applications in engineering and scientific computation,how to judge if a given matrix is an H-matrix is very important,especially for large scale matrices.In this paper,we obtain several new practical criteria for judging nonsingular H-matrices by using the partitioning technique and Schur complement of matrices.Their effectiveness is illustrated by numerical examples.

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1. Introduction

H-matrix is a special class of matrices with wide applications in numerical algebra,cybernetics theory,electrical system theory and so on,it has close relations with diagonally dominant matrices,M-matrices,positive definite matrices,positive stable matrices.At present,many scholars have obtained numerous criteria for judging H-matrices by applying some techniques in matrix theory and inequalities,and iterative algorithm.For details of such criteria,the reader can see [1]-[5].But for large scale matrices,these criteria mentioned above may have some difficulties resulting from matrix calculation.In this paper,we use the partitioning technique and Schur complement of matrices to reduce the order of matrices,and get several new practical criteria for judging H-matrices.As a special case,we also get two necessary and sufficient conditions for an Z-matrix being an H-matrix.In the last of this paper,we present two numerical examples to illustrate the effectiveness of our results.

Let $M_n(C)(M_n(R))$ denote the set of all $n \times n$ complex(real) matrices, $N = \{1, 2, \ldots, n\}$. For $A \in M_n(C)$,nonempty index sets $\alpha, \beta \subseteq N$,we denote by $A(\alpha, \beta)$ that submatrix of A lying in the rows indicated by α and the columns

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indicated by β,especially the submatrix $A(\alpha,\alpha)$ is abbreviated $A(\alpha)$, $|\alpha|$ equals the cardinality of α.

$A=(a_{ij})\in M_n(C),\mu(A)=(m_{ij})\in M_n(R)$ is called the comparison matrix of A,where $m_{ii}=|a_{ii}|, m_{ij}=-|a_{ij}|, i\neq j, i,j\in N$.

Definition 1[1]. $Z_n=\{A=(a_{ij})\in M_n(R): a_{ij}\leq 0, i\neq j, i,j\in N\}$.If $A\in Z_n$,then A is called an Z-matrix.

Definition 2[1]. $M_n=\{A\in M_n(R): A\in Z_n, A^{-1}\geq 0\}$.If $A\in M_n$,then A is called an nonsingular M-matrix.

Definition 3[6]. Let $A\in M_n(C),\alpha\subset N,\alpha'=N-\alpha, A(\alpha)$ be nonsingular,then

$$A/\alpha \triangleq A/A(\alpha)=A(\alpha')-A(\alpha',\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha')$$

is called the Schur complement with respect to $A(\alpha)$,in addition,we appoint that $A/\emptyset=A$.

Definition 4[2]. Let $A=(a_{ij})\in M_n(C)$,if $|a_{ii}|>\sum\limits_{j\in N-\{i\}}|a_{ij}|,\forall i\in N$,then A is called a strictly diagonally dominant matrix;if there exists a positive diagonal matrix D such that AD is a strictly diagonally dominant matrix,then A is called a generalized strictly diagonally dominant matrix.

It is well known that nonsingular H-matrices are equal to generalized strictly diagonally dominant matrices.

2. Main results

At first,we introduce the following lemmas.

Lemma 1[1]. *Let $A\in M_n(C)$,then A is an nonsingular H-matrix if and only if $\mu(A)$ is an nonsingular M-matrix.*

Lemma 2[1]. *If $A\in M_n, B\in Z_n$ and $B\geq A$,then $B\in M_n$.*

Lemma 3[2]. *Let $A\in M_n(C)$ be an nonsingular H-matrix,then strictly diagonal dominance holds for at least one index $i\in N$.*

Theorem 1. *Let $A\in M_n(C)$ be partitioned as*

$$A=\begin{pmatrix} A(\alpha) & A(\alpha,\alpha') \\ A(\alpha',\alpha) & A(\alpha') \end{pmatrix} \tag{1}$$

Then the necessary condition for A being an nonsingular H-matrix is $A(\alpha), A/\alpha$ are nonsingular H-matrices. Where $\emptyset\neq\alpha\subset N,\alpha'=N-\alpha$,and $A(\alpha)$ is nonsingular.

Proof. By Lemma 1, A is an nonsingular H-matrix,then $\mu(A)$ is an nonsingular M-matrix,therefore $\mu(A)(\alpha)$,$\mu(A)/\alpha$ are nonsingular M-matrices(see[1]).

Obviously $\mu(A)(\alpha) = \mu(A(\alpha))$,noting that $\mu(A)(\alpha',\alpha) \leq 0, \mu(A)(\alpha,\alpha') \leq 0, [\mu(A)(\alpha)]^{-1} \geq 0$,then $\mu(A)(\alpha',\alpha)[\mu(A)(\alpha)]^{-1}\mu(A)(\alpha,\alpha') \geq 0$,thus

$$\begin{array}{rcl} \mu(A)/\alpha & = & \mu(A)(\alpha') - \mu(A)(\alpha',\alpha)[\mu(A)(\alpha)]^{-1}\mu(A)(\alpha,\alpha') \\ & = & \mu[A(\alpha')] - \mu(A)(\alpha',\alpha)\{\mu[A(\alpha)]\}^{-1}\mu(A)(\alpha,\alpha') \\ & \leq & \mu\{A(\alpha') - A(\alpha',\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha')\} \\ & = & \mu(A/\alpha) \end{array}$$

By Lemma 2,we get that $\mu(A(\alpha)), \mu(A/\alpha)$ are nonsingular M-matrices.Further by Lemma 1,we obtain that $A(\alpha), A/\alpha$ are nonsingular H-matrices. □

Remark 1. The sufficiency of Theorem 1 does not hold,i.e.,$A(\alpha), A/\alpha$ are nonsingular H-matrices,A unlikely is an nonsingular H-matrix.For example,let

$$A = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ -1 & 1 & 3 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

Taking $\alpha = \{1,2\}, \alpha' = \{3,4\}$,it is easy to know that $A(\alpha) = \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$, $A/\alpha = \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$ are nonsingular H-matrices,but by Lemma 3,A is not an nonsingular H-matrix.

Corollary 1. *Let $A \in Z_n$ be partitioned as (1),then A is an nonsingular H-matrix if and only if $A(\alpha), A/\alpha$ are nonsingular H-matrices.*

Proof. Necessity: From Theorem 1, it is obviously hold.
Sufficiency: Since $A \in Z_n$, then $A(\alpha) \in Z_{|\alpha|}$.So if $A(\alpha)$ is an nonsingular H-matrix,then $A(\alpha)$ is an nonsingular M-matrix,and $[A(\alpha)]^{-1} \geq 0$. At the same time noting that $A(\alpha') \in Z_{n-|\alpha|}, A(\alpha',\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha') \geq 0$, thus $A/\alpha = A(\alpha') - A(\alpha',\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha') \in Z_{n-|\alpha|}$.Therefore A/α is an nonsingular H-matrix,A/α is also an nonsingular M-matrix,and $(A/\alpha)^{-1} \geq 0$.

Taking

$$U_1 = \begin{pmatrix} I_{|\alpha|} & 0 \\ -A(\alpha',\alpha)[A(\alpha)]^{-1} & I_{n-|\alpha|} \end{pmatrix}, U_2 = \begin{pmatrix} I_{|\alpha|} & -[A(\alpha)]^{-1}A(\alpha,\alpha') \\ 0 & I_{n-|\alpha|} \end{pmatrix},$$

then

$$U_2^{-1}A^{-1}U_1^{-1} = (U_1AU_2)^{-1} = \begin{pmatrix} A(\alpha) & 0 \\ 0 & A/\alpha \end{pmatrix}^{-1} = \begin{pmatrix} (A(\alpha))^{-1} & 0 \\ 0 & (A/\alpha)^{-1} \end{pmatrix}.$$

Hence

$$A^{-1} = U_2 \begin{pmatrix} (A(\alpha))^{-1} & 0 \\ 0 & (A/\alpha)^{-1} \end{pmatrix} U_1.$$

By the definitions of U_1, U_2 and the previous proof,we easy to know that $A^{-1} \geq 0$.From Definition 2,A is an nonsingular M-matrix,thus A is an nonsingular H-matrix. □

Theorem 2. *Let $A \in M_n(C)$ be partitioned as*

$$A = \begin{pmatrix} A(\alpha) & A(\alpha,\beta) & A(\alpha,\gamma) \\ A(\beta,\alpha) & A(\beta) & A(\beta,\gamma) \\ A(\gamma,\alpha) & A(\gamma,\beta) & A(\gamma) \end{pmatrix} \tag{2}$$

Then the necessary condition for A being an nonsingular H-matrix is that $A(\alpha)$, $H_{11}, H_{22} - H_{21}H_{11}^{-1}H_{12}$ are nonsingular H-matrices, where

$$\begin{aligned} H_{11} &= A(\beta) - A(\beta,\alpha)[A(\alpha)]^{-1}A(\alpha,\beta), \\ H_{22} &= A(\gamma) - A(\gamma,\alpha)[A(\alpha)]^{-1}A(\alpha,\gamma), \\ H_{12} &= A(\beta,\gamma) - A(\beta,\alpha)[A(\alpha)]^{-1}A(\alpha,\gamma), \\ H_{21} &= A(\gamma,\beta) - A(\gamma,\alpha)[A(\alpha)]^{-1}A(\alpha,\beta), \end{aligned}$$

$\emptyset \neq \alpha, \beta, \gamma \subset N, \alpha \cup \beta \cup \gamma = N.$

Proof. If A is partitioned as (2),by the definition of Schur complement,we have

$$\begin{aligned} A/\alpha &= \begin{pmatrix} A(\beta) & A(\beta,\gamma) \\ A(\gamma,\beta) & A(\gamma) \end{pmatrix} - \begin{pmatrix} A(\beta,\alpha) \\ A(\gamma,\alpha) \end{pmatrix} (A(\alpha))^{-1} \begin{pmatrix} A(\alpha,\beta) & A(\alpha,\gamma) \end{pmatrix} \\ &= \begin{pmatrix} A(\beta) - A(\beta,\alpha)[A(\alpha)]^{-1}A(\alpha,\beta) & A(\beta,\gamma) - A(\beta,\alpha)[A(\alpha)]^{-1}A(\alpha,\gamma) \\ A(\gamma,\beta) - A(\gamma,\alpha)[A(\alpha)]^{-1}A(\alpha,\beta) & A(\gamma) - A(\gamma,\alpha)[A(\alpha)]^{-1}A(\alpha,\gamma) \end{pmatrix} \\ &= \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \\ &\triangleq H \end{aligned}$$

By Theorem 1,A is an nonsingular H-matrix,then $A(\alpha), A/\alpha = H$ are nonsingular H-matrices. Further by Theorem 1,H is an nonsingular H-matrix,then $H_{11}, H/H_{11} = H_{22} - H_{21}H_{11}^{-1}H_{12}$ are also nonsingular H-matrices. □

Corollary 2. *Let $A \in Z_n$ be partitioned as (2). Then A is an nonsingular H-matrix if and only if $A(\alpha), H_{11}, H_{22} - H_{21}H_{11}^{-1}H_{12}$ are nonsingular H-matrices, where $H_{11}, H_{12}, H_{21}, H_{22}$ are the same as in Theorem 2.*

Proof. Necessity: From Theorem 2,it is obviously hold.
Sufficiency: Since $A \in Z_n$,then $A(\alpha) \in Z_{|\alpha|}$. So if $A(\alpha)$ is an nonsingular H-matrix, then $A(\alpha)$ is a nonsingular M-matrix,and $[A(\alpha)]^{-1} \geq 0$. Noting that $A(\beta) \in Z_{|\beta|}, A(\beta,\alpha) \leq 0, A(\alpha,\beta) \leq 0$, therefore $H_{11} = A(\beta) - A(\beta,\alpha)[A(\alpha)]^{-1} A(\alpha,\beta) \in Z_{|\beta|}$.

With the same reason, we easy to know that $H_{22} = A(\gamma) - A(\gamma,\alpha)[A(\alpha)]^{-1} A(\alpha,\gamma) \in Z_{|\gamma|}, H_{12} = A(\beta,\gamma) - A(\beta,\alpha)[A(\alpha)]^{-1}A(\alpha,\gamma) \leq 0, H_{21} = A(\gamma,\beta) - A(\gamma,\alpha)[A(\alpha)]^{-1}A(\alpha,\beta) \leq 0$.Consequently, $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \in Z_{n-|\alpha|}$. By Corollary 1,if $H_{11}, H_{22}-H_{21}H_{11}^{-1}H_{12} = H/H_{11}$ are nonsingular H-matrices, then $H = A/\alpha$ is an nonsingular H-matrix. $A(\alpha)$ is an nonsingular H-matrix,further by Corollary 1, we get that A is an nonsingular H-matrix. □

Remark 2. By Corollary 2, if $A \in Z_n$ is partitioned as (2),then to judge if the given matrix A is an H-matrix,just to judge if $A(\alpha), H_{11}, H_{22} - H_{21}H_{11}^{-1}H_{12}$ are H-matrices. In order to escape computing the inverse of matrices, we let $A(\alpha), H_{11}$ be 1×1 matrices,thus only to make some subtraction,multiplication of matrices, we can judge if the matrix A is an H-matrix. In addition,for large scale matrices, noting that to judge if $A(\alpha), H_{11}, H_{22} - H_{21}H_{11}^{-1}H_{12}$ are H-matrices is independent, so we also take account of using parallel algorithm, judging if $A(\alpha), H_{11}, H_{22} - H_{21}H_{11}^{-1}H_{12}$ are H-matrices respectively. In general, if $n = 3k(3k+1 or 3k+2)$, we let $A(\alpha), A(\beta)$ be $k \times k$ matrices,$A(\gamma)$ be $(k+1) \times (k+1)$ or $(k+2) \times (k+2)$ matrix.

3. Numerical examples

In this part,we illustrate the following examples to show the effectiveness of our results.

Example 1.

$$A = \begin{pmatrix} 6 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 \\ -3 & -3 & 6 & -3 \\ -3 & -3 & -3 & 6 \end{pmatrix}$$

Let $A(\alpha), H_{11}$ be 1×1 matrices, by direct calculations with MATLAB 7.1, we have

frequency	$A(\alpha)$	H_{11}	$H_{22} - H_{21}H_{11}^{-1}H_{12}$
first	6.0000	5.8333	$\begin{pmatrix} 4.8000 & -4.2000 \\ -4.2000 & 4.8000 \end{pmatrix}$

By Corollary 2, it is obviously that A is an H-matrix.

Example 2.

$$A = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & 0 & -2 \\ -1 & 2 & 0 & 0 & 0 & 0 & -1 \\ -1 & -2 & 5 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 9 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 6 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & 4 & 0 \\ 0 & -1 & 0 & -2 & 0 & 0 & 10 \end{pmatrix}$$

Firstly,as Example 1,we let $A(\alpha), H_{11}$ be 1×1 matrices,by direct calculations with MATLAB 7.1,we get the following table:

frequency	$A(\alpha)$	H_{11}	$H_{22} - H_{21}H_{11}^{-1}H_{12}$
first	3.0000	1.6667	$\begin{pmatrix} 5.0000 & -2.0000 & -0.8000 & 0 & -3.0000 \\ -1.0000 & 9.0000 & 0 & -1.0000 & 0 \\ -1.0000 & 0 & 6.0000 & -1.0000 & -1.0000 \\ 0 & -1.0000 & -0.4000 & 4.0000 & -1.0000 \\ 0 & -2.0000 & -0.2000 & 0 & 9.0000 \end{pmatrix}$
second	5.0000	8.6000	$\begin{pmatrix} 5.8326 & -1.0465 & -1.6279 \\ -0.4186 & 3.8837 & -1.0698 \\ -0.2372 & -0.2326 & 8.8605 \end{pmatrix}$
third	5.8326	3.8086	8.7086

By the definition of H-matrices and Corollary 2, it is obviously that A is an H-matrix.

Next, we use the parallel algorithm mentioned in Remark 2 to judge if A matrix is an H-matrix.

Let $A(\alpha), A(\beta), A(\gamma)$ be $2 \times 2, 2 \times 2, 3 \times 3$ matrices respectively, by direct calculations with MATLAB 7.1, we have

matrices	$A(\alpha)$	H_{11}	$H_{22} - H_{21}H_{11}^{-1}H_{12}$
original matrices	$\begin{pmatrix} 3.0000 & -1.0000 \\ -1.0000 & 2.0000 \end{pmatrix}$	$\begin{pmatrix} 5.0000 & -2.0000 \\ -1.0000 & 9.0000 \end{pmatrix}$	$\begin{pmatrix} 5.8326 & -1.0465 & -1.6279 \\ -0.4186 & 3.3387 & -1.0698 \\ -0.2372 & -0.2326 & 8.8605 \end{pmatrix}$
inverse matrices	$\begin{pmatrix} 0.4000 & 0.2000 \\ 0.2000 & 0.6000 \end{pmatrix}$	$\begin{pmatrix} 0.2093 & 0.0465 \\ 0.0233 & 0.1163 \end{pmatrix}$	$\begin{pmatrix} 0.1766 & 0.0499 & 0.0385 \\ 0.0205 & 0.2651 & 0.0358 \\ 0.0053 & 0.0083 & 0.1148 \end{pmatrix}$
conclusions	$M - matrix$	$M - matrix$	$M - matrix$

By the Definition 2, Lemma 1 and Corollary 2, we easy to know that A is an H-matrix.

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Hongmin Mo received master's degree from Xiangtan University and Ph.D at Central South University under the direction of Professor Shuhuang Xiang. Since 1993 I have been working at Jishou University, My research interests mainly focus on the matrix theory and its applitions and linear complementarity problems.

College of Mathematics and Computer Science, Jishou University, Jishou, Hunan 416000, China

e-mail: `mohongmin@163.com`