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ON THE PRECISE ASYMPTOTICS IN COMPLETE MOMENT CONVERGENCE OF NA SEQUENCES†

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ABSTRACT. Let $X_1, X_2, \cdots$ be identically distributed negatively associated random variables with $EX_1 = 0$ and $E|X_1|^3 < \infty$. In this paper we prove $\lim_{\epsilon\downarrow 0} \frac{1}{-\log\epsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} ES_n^2 I\{|S_n| \geq \sigma\epsilon n\} = 2$ and $\lim_{\epsilon\downarrow 0} \epsilon^{2-p} \sum_{n=1}^{\infty} \frac{1}{n^p} E|S_n|^p I\{|S_n| \geq \sigma\epsilon n\} = \frac{2}{2-p}$ for $0 < p < 2$, where $S_n = \sum_{i=1}^{n} X_i$ and $0 < \sigma^2 = EX_1^2 + \sum_{i=2}^{\infty} Cov(X_1, X_i) < \infty$. We consider some results of i.i.d. random variables obtained by Liu and Lin(2006) under negative association assumption.

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1. Introduction and main results

Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We start the definition of negatively associated random variables. A finite sequence $\{X_i,\ 1 \leq i \leq n\}$ of random variables is said to be negatively associated(NA) if for every disjoint subsets A and B of $\{1, 2, \cdots, n\}$, we have

$$Cov\Big(f(X_i; i \in A), g(X_j; j \in B)\Big) \leq 0$$

whenever f on R^A and g on R^B are coordinatewise nondecreasing functions and the covariance exists. An infinite sequence of random variables is NA if every finite subsequence is NA. This concept was introduced by Joag-Dev and

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Proschan(1983). They also showed that many well known multivariate distributions possess the NA property. Some examples include: the multinomial distribution, the multivariate hypergeometric distribution, the Dirichlet distribution, the negatively correlated normal distribution, the permutation distribution, the random sampling without replacement, and etc. Because of its wide application in multivariate statistical analysis and system reliability, the notion of NA has received considerable attention recently. We refer to Joag-Dev and Proschan(1983) for fundamental properties, Newman(1984) for the central limit theorem, Shao and Su(1999) for the law of the iterated logarithm, Shao(2000) for moment equalities and the maximal of the partial sum inequalities, Li and Zhang(2004) for complete moment convergence, and Fu and Zhang(2007) for the precise rates of in the law of the logarithm.

Let $X, X_1, X_2, \cdots$ be a sequence of i.i.d. random variables, and $S_n = X_1 + \cdots + X_n$. Hsu and Robbins(1947) and Erdös(1950) showed that for $\epsilon > 0$ $\sum_{n=1}^{\infty} P(|S_n| \geq \epsilon n) < \infty$ if and only if $EX = 0$ and $EX^2 < \infty$. Later, Spitzer(1956) proved that for $\epsilon > 0$ $\sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| \geq \epsilon n) < \infty$ if and only if $EX_1 = 0$ and $E|X_1| < \infty$. Relying heavily on the technique of Erdös(1950) and Spitzer(1956), Katz(1963) generalized Spitzer's result and proved that, for $p < 2$, $r \geq p$ and $\epsilon > 0$, $\sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P(|S_n| \geq \epsilon n^{\frac{1}{p}}) < \infty$ if and only if $E|X|^r < \infty$ and when $r \geq 1$, $EX = 0$.

There have been extensions in various directions of the Hsu-Robbins-Erdös theorem. Heyde(1975) proved that $\lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \epsilon n) = EX^2$ whenever $EX = 0$ and $EX^2 < \infty$.

For analogous results in the more general case, see Gut and Spătaru(2000), Spătaru(1999), etc. Liu and Lin(2006) also introduced a new kind of complete convergence for i.i.d. random variables and derived some results about precise asymptotics for this kind of complete moment convergence as follows.

Theorem A(Liu and Lin(2006)). *Suppose that $\{X_n,\ n \geq 1\}$ is a sequence of i.i.d. random variables with $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$. Assume $EX_1^2 \log^+ |X_1| < \infty$. Then, we have*

$$\lim_{\epsilon \downarrow 0} \frac{1}{-\log \epsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} ES_n^2 I\{|S_n| \geq \sigma \epsilon n\} = 2. \tag{1.1}$$

Theorem B(Liu and Lin(2006)). *Suppose that $\{X_n,\ n \geq 1\}$ is a sequence of i.i.d. random variables with $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$. Then we have, for*

$1 < p < 2$

$$\lim_{\epsilon\downarrow 0}\epsilon^{2-p}\sum_{n=1}^{\infty}\frac{1}{n^p}E|S_n|^pI\{|S_n|\geq \sigma\epsilon n\} = \frac{2}{2-p}. \tag{1.2}$$

In this paper we show that (1.1) and (1.2) hold for negatively associated random variables under appropriate conditions.

Now we state our main results as follows.

Theorem 1.1. *Let $\{X_n,\ n\geq 1\}$ be a negatively associated sequence of identically distributed random variables with $EX_1 = 0$, $E|X_1|^3 < \infty$. Assume that $0 < \sigma^2 = EX_1^2 + 2\sum_{i=2}^{\infty}Cov(X_1, X_i) < \infty$. Then (1.1) holds.*

Theorem 1.2. *Let $\{X_n,\ n\geq 1\}$ be a sequence of identically distributed and negatively associated random variables with $EX_1 = 0$, $E|X_1|^3 < \infty$. Assume that $0 < \sigma^2 = EX_1^2 + 2\sum_{i=2}^{\infty}Cov(X_1, X_i) < \infty$. Then (1.2) holds.*

2. Proof of Theorem 1.1

Set $b(\epsilon) = [\epsilon^{-2}]$. Without loss of generality, we assume that $\sigma^2 = 1$.

Note that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}ES_n^2I\{|S_n|\geq \epsilon n\} = \epsilon^2\sum_{n=1}^{\infty}P\{|S_n|\geq \epsilon n\} + \sum_{n=1}^{\infty}\frac{1}{n^2}\int_{\epsilon n}^{\infty}2xP\{|S_n|\geq x\}dx.$$

Since a sequence $\{X_n,\ n\geq 1\}$ of identically distributed NA random variables with $EX_1 = 0$ and $EX_1^2 < \infty$ satisfies $\sum_{n=1}^{\infty}P\left\{\max_{1\leq k\leq n}|\sum_{i=1}^{k}X_i|\geq \epsilon n\right\} < \infty$ for all $\epsilon > 0$, in order to prove Theorem 1.1, we only need to show that

$$\lim_{\epsilon\downarrow 0}\frac{1}{-\log\epsilon}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{\epsilon n}^{\infty}2xP\{|S_n|\geq x\} = 2. \tag{2.1}$$

Proposition 2.1(Liu and Lin(2006)). *Let N be the standard normal random variable. Then, we have*

$$\lim_{\epsilon\downarrow 0}\frac{1}{-\log\epsilon}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{\epsilon n}^{\infty}2xP\{|N|\geq x/\sqrt{n}\}dx = 2. \tag{2.2}$$

Proof. See the proof Proposition 3.1 in Liu and Lin(2006). □

Lemma 2.2(Newman(1984)). *Let $\{X_n,\ n\geq 1\}$ be a strictly stationary sequence of negatively associated random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. If*

$0 < \sigma^2 = EX_1^2 + 2\sum_{i=2}^{\infty} Cov(X_1, X_i) < \infty$, *then*

$$\frac{S_n}{\sigma\sqrt{n}} \to^{\mathcal{D}} N(0,1) \text{ as } n \to \infty, \tag{2.3}$$

where $\to^{\mathcal{D}}$ *indicates convergence in distribution, N is a standard normal distribution and* $S_n = \sum_{i=1}^{n} X_i$.

Lemma 2.3(Shao, 2000). *Let $\{Y_i,\ 1 \le i \le n\}$ be a sequence of negatively associated random variables with mean zeros and finite variances. Denote* $S_n = \sum_{k=1}^{n} Y_k,\ B_n = \sum_{i=1}^{n} EY_i^2$. *Then, for any $u > 0,\ v > 0$,*

$$P\left\{\max_{1\le k\le n} |S_k| \ge u\right\}$$
$$\le 2P\left\{\max_{1\le k\le n} |Y_k| \ge v\right\} + 4\exp\left\{-\frac{u^2}{8B_n}\right\} + 4\left(\frac{B_n}{4(uv + B_n)}\right)^{\frac{u}{12v}}.$$

Lemma 2.4(Shao and Su(1999)). *Let $q \ge 2$ and let $\{X_n,\ n \ge 1\}$ be a strictly stationary sequence of negatively associated random variables with $EX_i = 0$ and $E|X_i|^q < \infty$. Then, there exists a constant $A_q > 0$ such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^q \le A_q\left\{\left(\sum_{i=1}^{n} EX_i^2\right)^{q/2} + \sum_{i=1}^{n} E|X_i|^q\right\}.$$

Proposition 2.5. *Let $\{X_n; n \ge 1\}$ be a strictly stationary sequence of negatively associated random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. Assume that* $\sigma^2 = EX_1^2 + 2\sum_{i=2}^{\infty} Cov(X_1, X_i) = 1$ *and $E|X_1|^3 < \infty$. Then*

$$\lim_{\epsilon\downarrow 0} \frac{1}{-\log\epsilon} \sum_{n=1}^{b(\epsilon)} \left|\int_{\epsilon n}^{\infty} 2xP\{|S_n| \ge x\}dx - \int_{\epsilon n}^{\infty} 2xP\{|N| \ge x/\sqrt{n}\}dx\right| = 0. \tag{2.4}$$

Proof. Write $\Delta_n = \sup_x \left|P\{|S_n| \ge x\sqrt{n}\} - P\{|N| \ge x\}\right|$. Note that $P\{|N| \ge x\}$ is a continuous function for $x \ge 0$ and this combined with Lemma 2.2 yields $\lim_{n\to\infty} \Delta_n = 0$ for any $x \ge 0$. From the proof of Proposition 3.2 of Liu and

Lin(2006) we have

$$\begin{aligned}
&\sum_{n=1}^{b(\epsilon)} \frac{1}{n^2}\left|\int_{\epsilon n}^{\infty} 2xP\{|S_n|\geq x\}dx - \int_{\epsilon n}^{\infty} 2xP\{|N|\geq x/\sqrt{n}\}dx\right| \\
\leq\ & \sum_{n=1}^{b(\epsilon)} \frac{1}{n}\int_0^{1/\sqrt{n}\Delta_n^{\frac{1}{4}}} 2n(x+\epsilon)P\{|S_n|\geq n(x+\epsilon)\} - P\{|N|\geq \sqrt{n}(x+\epsilon)\}|dx \\
+\ & \sum_{n=1}^{b(\epsilon)} \frac{1}{n}\Big[\int_{1/\sqrt{n}\Delta_n^{\frac{1}{4}}}^{\infty} 2n(x+\epsilon)P\{|S_n|\geq n(x+\epsilon)\}dx \qquad (2.5)\\
+\ & \int_{1/\sqrt{n}\Delta_n^{\frac{1}{4}}}^{\infty} 2n(x+\epsilon)P\{|N|\geq \sqrt{n}(x+\epsilon)\}dx\Big] \\
:=\ & \sum_{n=1}^{b(\epsilon)} \frac{1}{n}(\Delta_{n_1}+\Delta_{n_2}+\Delta_{n_3}).
\end{aligned}$$

Since $n \leq b(\epsilon)$ implies $\epsilon\sqrt{n}\leq 1$, we have

$$\Delta_{n_1} \leq \int_0^{1/\sqrt{n}\Delta_n^{\frac{1}{4}}} 2n(x+\epsilon)\Delta_n dx \leq n\Delta_n\left(\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{4}}}+\epsilon\right)^2 \leq \left(\Delta_n^{\frac{1}{4}}+\Delta_n^{\frac{1}{2}}\right)^2. \quad (2.6)$$

For Δ_{n_2}, by Lemma 2.3, we have

$$\begin{aligned}
\Delta_{n_2} \leq\ & n\int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{4}}}}^{\infty} 4n(x+\epsilon)|P\left\{|X_1| > \frac{1}{24}n(x+\epsilon)\right\}dx \\
& + \int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{4}}}}^{\infty} 8n(x+\epsilon)\exp\left\{-\frac{\theta^2 n(x+\epsilon)^2}{8}\right\}dx \\
& + \int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{4}}}}^{\infty} 8n(x+\epsilon)\left(\frac{6}{\theta^2 n(x+\epsilon)^2}\right)^2 dx \\
:=\ & I_1+I_2+I_3.
\end{aligned}$$

$$I_1 \leq \int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{4}}}}^{\infty} 4n^2(x+\epsilon)\frac{(24)^3}{n^3(x+\epsilon)^3}dx \leq (\frac{c}{n})\int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{4}}}}^{\infty}\frac{1}{(x+1)^2}\}dx \leq C(\Delta_n^{\frac{1}{2}}). \quad (2.7)$$

$$I_2 = \int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{4}}}}^{\infty} 8n(x+\epsilon)\exp\{-\frac{\theta^2 n(x+\epsilon)^2}{8}\}dx \leq (\frac{32}{\theta^2})e^{-\frac{\theta^2}{8\sqrt{\Delta_n}}}. \quad (2.8)$$

$$I_3 \leq C\int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{4}}}}^{\infty} 8n(x+\epsilon)(\frac{6}{n(x+\epsilon)^2\theta^2})^2 dx \leq C\Delta_n^{\frac{1}{2}}. \quad (2.9)$$

Now, we estimate Δ_{n_3}. By Markov's inequality, we have

$$\begin{aligned}\Delta_{n_3} &= \int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{4}}}}^{\infty} 2n(x+\epsilon)P\{|N| \geq \sqrt{n}(x+\epsilon)\}dx \qquad (2.10)\\ &\leq \int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{4}}}}^{\infty} 2n(x+\epsilon)\frac{E|N|^4}{n^2(x+\epsilon)^4}dx \leq C\Delta_n^{\frac{1}{2}}.\end{aligned}$$

From (2.6) to (2.10), we get (2.4). □

Proposition 2.6. *Let $\{X_n; n \geq 1\}$ be a strictly stationary sequence of negatively associated random variables with $EX_1 = 0$ and $E|X_1|^3 < \infty$. Assume that $\sigma^2 = EX_1^2 + 2\sum_{i=2}^{\infty} Cov(X_1, X_i) = 1$. Then we have, for*

$$\lim_{\epsilon \downarrow 0} \frac{1}{-\log \epsilon} \sum_{n=b(\epsilon)+1}^{\infty} \frac{1}{n^2}\left|\int_{\epsilon n}^{\infty} 2xP\{|S_n| \geq x\}dx - \int_{\epsilon n}^{\infty} 2xP\{|N| \geq x/\sqrt{n}\}dx\right| = 0.$$

Proof.

$$\begin{aligned}&\sum_{n=b(\epsilon)+1}^{\infty} \frac{1}{n^2}\left|\int_{\epsilon n}^{\infty} 2xP\{|S_n| \geq x\}dx - \int_{\epsilon n}^{\infty} 2xP\{|N| \geq x/\sqrt{n}\}dx\right|\\ &\leq \sum_{n=b(\epsilon)+1}^{\infty} \int_0^{\infty} 2(x+\epsilon)P\{|S_n| \geq n(x+\epsilon)\}dx \qquad (2.11)\\ &+ \sum_{n=b(\epsilon)+1}^{\infty} \int_0^{\infty} 2(x+\epsilon)P\{|N| \geq \sqrt{n}(x+\epsilon)\}dx := I_4 + I_5.\end{aligned}$$

We first estimate I_5.

$$I_5 \leq \sum_{n=b(\epsilon)+1}^{\infty} \int_0^{\infty} 2(x+\epsilon)\frac{E|N|^4}{n^2(x+\epsilon)^4}dx \leq C\sum_{n=b(\epsilon)+1}^{\infty} \frac{1}{n^2\epsilon^2} < \infty. \qquad (2.12)$$

By Lemma 2.4 we have

$$\begin{aligned}I_4 &\leq \sum_{n=b(\epsilon)+1}^{\infty} \int_0^{\infty} 2(x+\epsilon)P\{|S_n| \geq n(x+\epsilon)\}dx\\ &\leq \sum_{n=b(\epsilon)+1}^{\infty} \int_0^{\infty} 2(x+\epsilon)\frac{E|S_n|^3}{n^3(x+\epsilon)^3}dx \qquad (2.13)\\ &\leq \sum_{n=b(\epsilon)+1}^{\infty} \left[-\frac{n^{\frac{3}{2}}(E|X_1|^2)^{\frac{3}{2}} + nE|X_1|^3}{n^3(x+\epsilon)}\right]_0^{\infty}\end{aligned}$$

$$= \sum_{n=b(\epsilon)+1}^{\infty} \frac{(E|X_1|^2)^{\frac{3}{2}}}{n^{\frac{3}{2}}\epsilon} + \sum_{n=b(\epsilon)+1}^{\infty} \frac{E|X_1|^3}{n^2\epsilon} < \infty.$$

From (2.12) and (2.13) we have $\lim_{\epsilon\downarrow 0} \frac{1}{-\log\epsilon}(I_4 + I_5) = 0$ and complete the proof. □

Proof of Theorem 1.1: The proof of Theorem 1.1 is complete by Propositions 2.1, 2.5 and 2.6. □

3. Proof of Theorem 1.2

Note that, for $0 < p < 2$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} E|S_n|^p I\{|S_n| \geq \epsilon n\} = \epsilon^p \sum_{n=1}^{\infty} P(|S_n| \geq \epsilon n)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n^p} \int_{\epsilon n}^{\infty} px^{p-1} P(|S_n| \geq x)dx.$$

We assume that $\sigma^2 = 1$. As in the proof of Theorem 1.1, we only need to show that

$$\lim_{\epsilon\downarrow 0} \epsilon^{2-p} \sum_{n=1}^{\infty} \frac{1}{n^p} \int_{\epsilon n}^{\infty} px^{p-1} P(|S_n| \geq x)dx = \frac{p}{2-p}. \tag{3.1}$$

Proposition 3.1 (Liu and Lin(2006)). *Let N be the standard normal random variable. Then, we have*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2-p} \sum_{n=1}^{\infty} \frac{1}{n^p} \int_{\epsilon n}^{\infty} px^{p-1} P(|N| \geq x/\sqrt{n})dx = \frac{p}{2-p}.$$

Proposition 3.2. *Let $\{X_n; n \geq 1\}$ be a sequence of identically distributed NA random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. Then*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2-p} \sum_{n=1}^{Mb(\epsilon)} \frac{1}{n^p} \left| \int_{\epsilon n}^{\infty} px^{p-1} P(|S_n| \geq x)dx - \int_{\epsilon n}^{\infty} px^{p-1} P(|N| \geq x/\sqrt{n})dx \right| = 0. \tag{3.2}$$

Proof. It is easy to see that

$$\sum_{n=1}^{Mb(\epsilon)} \frac{1}{n^p} \left| \int_{\epsilon n}^{\infty} px^{p-1} P(|S_n| \geq x)dx - \int_{\epsilon n}^{\infty} px^{p-1} P(|N| \geq x/\sqrt{n})dx \right|$$

$$\leq \sum_{n=1}^{Mb(\epsilon)} \frac{1}{n^{\frac{p}{2}}} n^{\frac{p}{2}} \int_0^{\infty} p(x+\epsilon)^{p-1} \left| P(|S_n| \geq (x+\epsilon)n) - P(|N| \geq \sqrt{n}(x+\epsilon)) \right| dx$$

$$\leq \sum_{n=1}^{Mb(\epsilon)} \frac{1}{n^{\frac{p}{2}}} (\Delta'_{n_1} + \Delta'_{n_2}),$$

where

$$\Delta'_{n_1} = n^{\frac{p}{2}} \int_0^{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{2p}}}} p(x+\epsilon)^{p-1} \left|P(|S_n| \geq (x+\epsilon)n) - P(|N| \geq \sqrt{n}(x+\epsilon))\right| dx,$$

$$\Delta'_{n_2} = n^{\frac{p}{2}} \int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{2p}}}}^{\infty} p(x+\epsilon)^{p-1} \left|P(|S_n| \geq (x+\epsilon)n) - P(|N| \geq \sqrt{n}(x+\epsilon))\right| dx.$$

Write $\Delta_n = \sup_x \left|P\{|S_n| \geq x\sqrt{n}\} - P\{|N| \geq x\}\right|$. Then $\Delta_n \to 0$ *as* $n \to \infty$(See proof of Proposition 2.5).

Since $n \leq Mb(\epsilon)$ implies $\epsilon\sqrt{n} \leq \sqrt{M}$, one can easily obtain that

$$\Delta'_{n_1} \leq \Delta_n n^{\frac{p}{2}} \left(\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{2p}}} + \epsilon\right)^p \leq \left(\Delta_n^{1/2p} + \sqrt{M}\Delta_n^{1/p}\right)^p. \tag{3.3}$$

By Markov's inequality and Lemma 2.4, we have for $p < 2$

$$\Delta'_{n_2} \leq n^{\frac{p}{2}} \left(\int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{2p}}}}^{\infty} \frac{pE|S_n|^2}{(x+\epsilon)^{3-p}n^2} dx + \int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{2p}}}}^{\infty} \frac{pE|N|^2}{(x+\epsilon)^{3-p}n} dx\right) \tag{3.4}$$

$$\leq Cn^{\frac{p}{2}} \left(\int_{\frac{1}{\sqrt{n}\Delta_n^{\frac{1}{2p}}}}^{\infty} \frac{E|X_1|^2 + E|N|^2}{(x+\epsilon)^{3-p}n}\right) dx \leq C\Delta_n^{\frac{1}{p}-\frac{1}{2}}$$

By combining (3.3) and (3.4) we get (3.2). □

Proposition 3.3. *Let* $\{X_n; n \geq 1\}$ *be a sequence of identically distributed NA random variables with* $EX_1 = 0$ *and* $E|X_1|^3 < \infty$. *Then we have, for* $0 < p < 2$

$$\lim_{M\to\infty} \limsup_{\epsilon\downarrow 0} \epsilon^{2-p} \sum_{n=Mb(\epsilon)+1}^{\infty} \frac{1}{n^p} \left|\int_{\epsilon n}^{\infty} px^{p-1}P(|S_n| \geq x)dx - \int_{\epsilon n}^{\infty} px^{p-1}P(|N| \geq x/\sqrt{n})dx\right| = 0.$$

Proof.

$$\sum_{n=Mb(\epsilon)+1}^{\infty} \frac{1}{n^p} \left|\int_{\epsilon n}^{\infty} px^{p-1}P(|S_n| \geq x)dx - \int_{\epsilon n}^{\infty} px^{p-1}P(|N| \geq x/\sqrt{n})dx\right|$$

$$\leq \sum_{n=Mb(\epsilon)+1}^{\infty} \int_0^{\infty} p(x+\epsilon)^{p-1}P(|S_n| \geq (x+\epsilon)n)dx \tag{3.5}$$

$$+ \sum_{n=Mb(\epsilon)+1}^{\infty} \int_0^{\infty} p(x+\epsilon)^{p-1}P(|N| \geq (x+\epsilon)\sqrt{n})dx \leq G_1 + G_2.$$

By Lemma 2.4 we have

$$\begin{aligned} G_1 &\leq C \sum_{n=Mb(\epsilon)+1}^{\infty} \int_0^{\infty} p(x+\epsilon)^{p-1} P(|S_n| \geq (x+\epsilon)n) dx \\ &\leq C \sum_{n=Mb(\epsilon)+1}^{\infty} \int_0^{\infty} \left(\frac{n^{\frac{3}{2}} E|X_1|^2 + nE|X_1|^3}{n^3 (x+\epsilon)^{4-p}} \right) dx \\ &\leq C \sum_{n=Mb(\epsilon)+1}^{\infty} \left(\frac{E|X_1|^2}{n^{3/2}\epsilon^{3-p}} + \frac{E|X_1|^3}{n^2 \epsilon^{3-p}} \right) \qquad (3.6) \\ &\leq C\Big((Mb(\epsilon))^{-\frac{1}{2}} \epsilon^{p-3} E|X_1|^2 + (Mb(\epsilon))^{-1} \epsilon^{p-3} E|X_1|^3\Big) \\ &\leq C\Big(M^{-\frac{1}{2}} \epsilon^{p-2} E|X_1|^2 + M^{-1} \epsilon^{p-1} E|X_1|^3\Big) \end{aligned}$$

and

$$\begin{aligned} G_2 &\leq C \sum_{n=Mb(\epsilon)+1}^{\infty} \int_0^{\infty} p(x+\epsilon)^{p-1} P(|N| \geq (x+\epsilon)\sqrt{n}) dx \\ &\leq C \sum_{n=Mb(\epsilon)+1}^{\infty} \int_0^{\infty} p(x+\epsilon)^{p-1} \frac{E|N|^4 + 1}{(x+\epsilon)^4 n^2} dx \qquad (3.7) \\ &\leq C \sum_{n=Mb(\epsilon)+1}^{\infty} \frac{E|N|^4 + 1}{n^2 \epsilon^{4-p}} \leq CM^{-1} \epsilon^{p-2} (E|N|^4 + 1). \end{aligned}$$

Hence, it follows from (3.6) and (3.7) that $\lim_{M\to\infty} \lim_{\epsilon\downarrow 0} \sup \epsilon^{2-p}(G_1+G_2) = 0$ and the proof completes. □

Proof of Theorem 1.2: The proof is obtained. □

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