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WEAK INEQUALITIES WITH CONTROL FUNCTIONS AND FIXED POINT RESULTS

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ABSTRACT. In recent times control functions have been used in several problems of metric fixed point theory. Also weak inequalities have been considered in a number of works on fixed points in metric spaces. Here we have incorporated a control function in certain weak inequalities. We have established two fixed point theorems for mapping satisfying such inequalities. Our results are supported by examples.

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1. Introduction

The Banach contraction mapping principle is widely recognized as the source of metric fixed point theory. A mapping $T : X \to X$ where (X, d) is a metric space is said to be a contraction mapping if for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y), \qquad where\ 0 < k < 1. \tag{1.1}$$

According to the contraction mapping principle, any mapping T satisfying (1.1) in a complete metric space will have a unique fixed point. This principle has been generalised in different directions by mathematicians over the years. Also in the contemporary research it remains a heavily investigated branch. The works noted in [1], [2], [4], [10], [14] and [16] [19] are some examples from this line of research.

In particular, in [1] Alber and Guerre-Delabriere introduced the concept of weak contraction in Hilbert spaces. Rhoades [18] has shown that the result which Alber et al. had proved in [1] is also valid in complete metric spaces. We state the result of Rhoades in the following:

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Definition 1.1 (weakly contractive mapping). A mapping $T : X \to X$ where (X, d) is a metric space is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \tag{1.2}$$

where $x, y \in X$ and $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

If we takes $\phi(t) = kt$ where $0 < k < 1$ then (1.2) reduces to (1.1).

Theorem 1.1 [18]. *If $T : X \to X$ is a weakly contractive mapping where (X,d) is a complete metric space, then T has a unique fixed point.*

Weak inequalities of the above type have been used to establish fixed point results in a number of subsequent works some of which are noted in [5], [6], [13], [21] and [22].

There is another important generalization of the Banach contraction principle given by Khan et al. where they used a control function (which they called altering distance function).

Definition 1.2 (altering distance function)[15]. A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is monotone increasing and continuous
(ii) $\psi(t) = 0$ if and only if $t = 0$.

The following generalisation of the Banach contraction principle was proved by them.

Theorem 1.2 [15]. *Let (X, d) be a complete metric space, ψ be an altering distance function, and $f : X \to X$ be a self mapping which satisfies the following inequality:*

$$\psi(d(fx, fy)) \leq c\psi(d(x, y)) \tag{1.3}$$

for all $x, y \in X$ and for some $0 < c < 1$, then f has a unique fixed point.

In fact Khan et al. proved a more general theorem [15, theorem 2] of which the above result is a corollary. Altering distance has been used in metric fixed point theory in a number of papers. Some of the works utilizing the concept of altering distance function are noted in [17], [19] and [20]. In [7] 2-variable, in [8] 3-variable and in [3] 4-variable generalizations of altering distance function have been introduced and have been applied to fixed point problems. It has also been extended to the case of multivalued and fuzzy mappings [9]. The concept of altering distance function has also been extended to fixed point problems in Menger spaces ([10], [11]).

The purpose of this paper is to work out fixed point results for mappings in metric spaces by use of weak inequalities and altering distance function. We have two theorems both of which are supported by examples. The difference between the contents of the two theorems is envisaged by the observation that the second theorem does not apply to the example associated with the first theorem.

2. Main result

Theorem 2.1 *Let (X, d) be a complete metric space, and $T : X \to X$ be a self mapping which satisfies the following inequality:*

$$\Psi(d(Tx, Ty)) \leq \Psi(M(x, y)) - \Phi(N(x, y)) \tag{2.1}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), \frac{1}{2}(d(x, Tx) + d(y, Ty)), \frac{1}{2}(d(y, Tx) + d(x, Ty))\}, \tag{2.2}$$

$$N(x, y) = \min\{d(x, y), \frac{1}{2}(d(y, Tx) + d(x, Ty))\}, \tag{2.3}$$

$\Phi : [0, \infty) \to [0, \infty)$ is a lower semi continuous function with $\Phi(t) > 0$ for all $t \in (0, \infty)$ and $\Phi(0) = 0$ and $\Psi : [0, \infty) \to [0, \infty)$ is an altering distance function (definition 1.2). Further let for $x \in X$, there exist $N = N(x)$ such that $d(T^n x, T^{n+2} x) \geq 2d(T^{n+1} x, T^{n+2} x)$ for all $n > N$. Then there is a unique fixed point of T.

Proof. Let $x_0 \in X$, we define a sequence $\{x_n\}$ in X, such that for all $n \geq 0$,

$$x_{n+1} = Tx_n. \tag{2.4}$$

If $x_n = x_{n+1}$, then x_n is a fixed point of T. Hence we assume for all $n \geq 0$,

$$x_n \neq x_{n+1} \tag{2.5}$$

Putting $x = x_n$ and $y = x_{n+1}$ in (2.1), we have

$$\Psi(d(x_{n+1}, x_{n+2})) \leq \Psi(M(x_n, x_{n+1})) - \Phi(N(x_n, x_{n+1})). \tag{2.6}$$

Now

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), \frac{1}{2}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})), \frac{1}{2}(d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2}))\} \tag{2.7}$$

and

$$N(x_n, x_{n+1}) = \min\{d(x_n, x_{n+1}), \tfrac{1}{2}(d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2}))\}, \tag{2.8}$$

If possible, let for some n, $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$. By the triangular inequality

$$0 < d(x_{n+1}, x_{n+2}) - d(x_n, x_{n+1}) \leq d(x_n, x_{n+2})$$

Hence $N(x_n, x_{n+1}) > 0$. Then from (2.6), (2.7) and (2.8) we have by the property of Φ-function

$$\Psi(d(x_{n+1}, x_{n+2})) \leq \Psi(d(x_{n+1}, x_{n+2})) - \Phi(N(x_n, x_{n+1})) \leq \Psi(d(x_{n+1}, x_{n+2})).$$

which is a contradiction. Hence for all $n \geq 0$,

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) \tag{2.9}$$

In view of (2.9) we obtain from (2.7) and (2.8), for all $n \geq 0$,

$$M(x_n, x_{n+1}) = d(x_n, x_{n+1}). \tag{2.10}$$

$$N(x_n, x_{n+1}) = \frac{1}{2}\left(d(x_n, x_{n+2})\right). \tag{2.11}$$

Putting (2.10) and (2.11) in (2.6), we have for all $n \geq 0$,

$$\Psi(d(x_{n+1}, x_{n+2})) \leq \Psi(d(x_n, x_{n+1})) - \Phi(\tfrac{1}{2}(d(x_n, x_{n+2}))). \tag{2.12}$$

Again (2.9) implies that the sequence $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists $r \geq 0$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = r.$$

By the conditions of the theorem this implies that there exists N such that for all $n > N$, $|d(x_n, x_{n+2}) - 2r| \leq |d(x_n, x_{n+1}) - r| + |d(x_{n+1}, x_{n+2}) - r| \to 0$ as $n \to \infty$. Thus $\lim_{n \to \infty} d(x_n, x_{n+2}) = 2r$. Making $n \to \infty$ in (2.12) , by the continuity Ψ-function and the lower semi continuity of Φ -function, we have $\Psi(r) \leq \Psi(r) - \Phi(\frac{1}{2}r)$, which by the property of Ψ-function and Φ-function implies that $r = 0$. Hence we have,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{2.13}$$

And also, $\lim_{n \to \infty} d(x_n, x_{n+2}) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, there exists $\epsilon > 0$ and sequences of natural numbers $\{m(k)]$ and $\{n(k)\}$ such that for every natural number k

$$n(k) > m(k) > k \tag{2.14}$$

and

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon. \tag{2.15}$$

Corresponding to $m(k)$ we can choose $n(k)$ to be the smallest integer such that (2.15) is satisfied, and we have

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{2.16}$$

further (15) implies $d(Tx_{m(k)-1}, Tx_{n(k)-1}) \neq 0$. Hence $x_{m(k)-1} \neq x_{n(k)-1}$. Putting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (2.1), (2.2) and (2.3) we have for all k,

$$\Psi(d(Tx_{m(k)-1}, Tx_{n(k)-1})) = \Psi(d(x_{m(k)}, x_{n(k)}))$$
$$\leq \Psi(M(x_{m(k)-1}, x_{n(k)-1})) - \Phi(N(x_{m(k)-1}, x_{n(k)-1})) \tag{2.17}$$

where

$$M(x_{m(k)-1}, x_{n(k)-1}) = \max\{d(x_{m(k)-1}, x_{n(k)-1}), \frac{1}{2}(d(x_{m(k)-1}, x_{m(k)})$$
$$+d(x_{n(k)-1}, x_{n(k)})), \frac{1}{2}(d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)}))\} \tag{2.18}$$

and

$$N(x_{m(k)-1}, x_{n(k)-1}) = \min\{d(x_{m(k)-1}, x_{n(k)-1}),$$
$$\frac{1}{2}(d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)}))\}. \tag{2.19}$$

Then for every positive integer k we have,

$\varepsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \leq \varepsilon + d(x_{n(k)-1}, x_{n(k)})$. [by (2.16)]

Making $k \to \infty$ in the above inequality we obtain by (2.13),

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.20}$$

Again for all k,

$d(x_{m(k)-1}, x_{n(k)-1}) \leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1})$ and $d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})$ Making $k \to \infty$ and using (2.13) and (2.20) in the above two inequalities we obtain

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon. \tag{2.21}$$

Again for all k,

$$d(x_{m(k)-1}, x_{n(k)}) \leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)})$$

and $d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)})$.
Making $k \to \infty$ and using (2.13) and (2.20) in the above inequalities we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon. \tag{2.22}$$

Also for all k, $d(x_{n(k)-1}, x_{m(k)}) \leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)})$ and

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)})$$

Making $k \to \infty$ in the above inequalities we have using (2.13) and (2.20)

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \epsilon. \tag{2.23}$$

Making $k \to \infty$ in (2.17) and using (2.13), (2.18)- (2.23) we have by continuity of Ψ-function and lower semi continuity of Φ-function, $\Psi(\epsilon) \leq \Psi(\epsilon) - \Phi(\epsilon)$.

Then we have by the virtue of the property of Ψ-function and Φ -function it is a contradiction with $\epsilon > 0$. Hence $\{x_n\}$ is a Cauchy sequence and therefore $\{x_n\}$ is convergent in the complete metric space X.

Let

$$x_n \to z \text{ as } n \to \infty. \tag{2.24}$$

By (2.5), there exists a subsequence $\{y_k\}$ of $\{x_n\}$ such that $z \neq y_k$ for all k. Substituting $x = y_k$ and $y = z$ in (2.1), (2.2) and (2.3) we obtain

$$\Psi(d(y_{k+1}, Tz)) \leq \Psi(M(y_k, z)) - \Phi(N(y_k, z)) \tag{2.25}$$

where

$$M(y_k, z) = \max\{d(y_k, z), \frac{1}{2}(d(y_k, y_{k+1}) + d(z, Tz)), \frac{1}{2}(d(y_k, Tz) + d(z, y_{k+1}))\}, \tag{2.26}$$

$N(y_k, z) = \min\{d(y_k, z), \frac{1}{2}(d(y_k, Tz) + d(z, y_{k+1}))\}$.(2.27) Making $k \to \infty$ in the above inequalities, we obtain $\Psi(d(z, Tz)) \leq \Psi(\frac{1}{2}(d(z, Tz)))$, which implies that $d(z, Tz) = 0$, that is, $z = Tz$. Hence z is a fixed point of T. We next establish that the fixed point is unique. Let z_1 and z_2 be two fixed points of T and $z_1 \neq z_2$, then putting $x = z_1$ and $y = z_2$ in (2.1), (2.2) and (2.3) we obtain

$$\Psi(d(z_1, z_2)) \leq \Psi(d(z_1, z_2)) - \Phi(d(z_1, z_2))$$

which by the virtue of the property of Ψ-function and Φ functions implies $d(z_1, z_2)$ $= 0$. That is $z_1 = z_2$. This completes the proof of the theorem. □

Example 2.2. Let $S = [0,1] \cup \{2,3,4,.....\}$ and

$$d(x,y) = \begin{cases} |x-y|, & \text{if x,y} \in [0,1],\ x \neq y \\ x+y, & \text{if at least one of x or y } \notin [0,1], \text{ and } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then (X,d) is a complete metric space.[4]

Let $T : X \to X$ be defined as

$$Tx = \begin{cases} x - \frac{1}{2}x^2, & \text{if } 0 \leq x \leq 1, \\ x-1, & \text{if } x \in \{2,3,4,...\}. \end{cases}$$

Let $\Psi : [0,\infty) \to [0,\infty)$ be defined as

$$\Psi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ t^2, & \text{if } t > 1. \end{cases}$$

Let $\Phi : [0,\infty) \to [0,\infty)$ be defined as

$$\Phi(t) = \begin{cases} \frac{1}{2}t^2, & \text{if } 0 \leq t \leq 1, \\ \frac{1}{2}, & \text{if } t > 1. \end{cases}$$

Other conditions of theorem 2.1 except the inequality (2.1) are clearly satisfied. Without loss of generality we assume that $x > y$ and discuss the following cases.

Case - I: $x \in [0,1]$,

$$\begin{aligned} \Psi(d(Tx,Ty)) &= (x - \frac{1}{2}x^2) - (\ y - \frac{1}{2}y^2) = (x-y) - \frac{1}{2}(x^2 - y^2) \\ &< (x-y) - \frac{1}{2}(x-y)^2 (\text{Since} x+y > x-y \\ &\leq \Psi(M(x,y)) - \Phi(N(x,y)) \end{aligned}$$

Case -II: $x \in \{3,4,5,.....\}$

Then, $d(Tx,Ty) = \begin{cases} x-1+y-\frac{1}{2}y^2, & \text{if } y = [0,1] \\ x-1+y-1, & \text{if } y = \{2,3...\}. \end{cases}$

Hence for all y, $d(Tx,Ty) \leq x+y-1$ Again $d(x,y) = x+y$ so, $\Psi(d(x,y)) = (x+y)^2$ Therefore,

$$\begin{aligned} \Psi(d(Tx,Ty)) &\leq \Psi(x+y-1) = (x+y-1)^2 \\ &< (x+y+1)(x+y-1) = (x+y)^2 - 1 < (x+y)^2 - \frac{1}{2} \\ &= \Psi(d(x,y)) - \Phi(N(x,y)) \\ &\leq \Psi(M(x,y)) - \Phi(N(x,y)). \ (\text{Since} d(x,y) \leq M(x,y)) \end{aligned}$$

Case - III: $x = 2$. Then $y \in [0,1]$, $d(Tx,Ty) = 1 - (\ y - \frac{1}{2}y^2) \leq 1$ so that $\Psi(d(Tx,Ty)) \leq \Psi(1) = 1$. Again, $d(x,y) = 2+y$, so that $d(y,Tx) = 1-y$ and $d(x,Ty) = 2 + (\ y - \frac{1}{2}y^2)$

$N(x,y) = \min\{2+y, \frac{1}{2}(1-y+2+\ y - \frac{1}{2}y^2)\} = \frac{3}{2} - \frac{1}{2}y^2$ (for $y \in [0,1]$)

Therefore, $\Psi(M(x,y)) - \Phi(N(x,y)) \geq (2+y)^2 - \frac{1}{2} > 1 = \Psi(d(Tx,Ty))$.

Considering all the above cases we conclude that the conditions of theorem 2.1 remain valid for Φ, Ψ and T defined as in the above. It may be observed that $x = 0$ is the unique fixed point of T.

Theorem 2.3. *Let (X, d) be a complete metric space, and $T : X \to X$ be such that the following is satisfied for $x, y \in X$ with $x \neq y$,*

$$\Psi(d(Tx, Ty)) \leq \Psi(M(x, y)) - h(Q(x, y)) \tag{2.28}$$

where

$$M(x, y) = \max\{d(x, y), \frac{1}{2}(d(x, Tx) + d(y, Ty)), \frac{1}{2}(d(y, Tx) + d(x, Ty))\}, \tag{2.29}$$

$$Q(x, y) = \min\{d(x, y), \frac{1}{2}(d(x, Tx) + d(y, Ty)), \frac{1}{2}(d(y, Tx) + d(x, Ty))\}, \tag{2.30}$$

where $h : [0, \infty) \to [0, \infty)$ is such that $h(t) > 0$ for all $t > 0$, h is discontinuous at t =0 with $h(0) = 0$, and $\Psi : [0, \infty) \to [0, \infty)$ is an altering distance function. Further let for $x \in X$, there exists $N = N(x)$ such that $d(T^n x, T^{n+2}x) \geq 2d(T^{n+1}x, T^{n+2}x)$ for all $n > N$. Then T has a unique fixed point.

Proof. Starting with arbitrary $x_0 \in X$, we construct the sequence $\{x_n\}$ as in (2.4). Further we assume (2.5) for all $n \geq 0$, otherwise the fixed point of T automatically exists. Putting $x = x_n$ and $y = x_{n+1}$ in (2.28) for all n=0,1,2,3... we obtain

$$\Psi(d(x_{n+1}, x_{n+2})) \leq \Psi(M(x_n, x_{n+1})) - h(Q(x_n, x_{n+1})) \tag{2.31}$$

where

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), \frac{1}{2}(d(x_{n+1}, x_n) + d(x_{n+1}, x_{n+2})),$$
$$\frac{1}{2}(d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2}))\} \tag{2.32}$$

and $Q(x_n, x_{n+1}) = \min\{d(x_n, x_{n+1}), \frac{1}{2}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})),$

$$\frac{1}{2}(d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2}))\}. \tag{2.33}$$

If possible, let for some n, $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$. By the triangular inequality$0 < d(x_{n+1}, x_{n+2}) - d(x_n, x_{n+1}) \leq d(x_n, x_{n+2})$. Hence $N(x_n, x_{n+1}) > 0$. Then from (2.31), (2.32) and (2.33) we have by the property of h-function

$$\Psi(d(x_{n+1}, x_{n+2})) < \Psi(d(x_{n+1}, x_{n+2})).$$

which is a contradiction. Hence for all $n \geq 0$,

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}). \tag{2.34}$$

In view of (2.34) we obtain, for all $n \geq 0$,

$$M(x_n, x_{n+1}) = d(x_n, x_{n+1}), \quad Q(x_n, x_{n+1}) = \frac{1}{2}(d(x_n, x_{n+2})).$$

Using the above relations we have for all $n \geq 0$,

$$\Psi(d(x_{n+1}, x_{n+2})) < \Psi(d(x_n, x_{n+1})) - h(d(x_n, x_{n+2})). \tag{2.35}$$

Again (34) implies that the sequence $\{d(x_{n+1}, x_n)\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists $r \geq 0$ such that

$\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. As already observed in the proof of theorem (2.1) we have that $\lim_{n\to\infty} d(x_n, x_{n+2}) = 2r$.

Making $n \to \infty$ in (2.35) and using the above relation, by continuity Ψ-function and by h-function, we have $\Psi(r) \leq \Psi(r) - h(r)$, which by the property of Ψ-function and h-function implies that $r = 0$. Hence we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.36}$$

Next we prove that $\{x_n\}$ is a Cauchy sequence. If otherwise, we can have some $\epsilon > 0$ and corresponding sub sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that for every natural number k $n(k) > m(k) > k$ and

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon. \tag{2.37}$$

Corresponding to $m(k)$ we can choose $n(k)$ to be the least integer such that (2.37) is satisfied , so that we have

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{2.38}$$

From (2.37) $d(Tx_{m(k)-1}, Tx_{n(k)-1}) \neq 0$, hence $x_{m(k)-1} \neq x_{n(k)-1}$. Further proceeding identically way as in theorem (2.1), we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon, \tag{2.39}$$

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon, \tag{2.40}$$

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon, \tag{2.41}$$

and

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \epsilon. \tag{2.42}$$

Now putting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (2.28), (2.29) and (2.30) we get,

$$\Psi(d(x_{m(k)}, x_{n(k)})) \leq \Psi(M(x_{m(k)-1}, x_{n(k)-1})) - h(Q(x_{m(k)-1}, x_{n(k)-1})). \tag{2.43}$$

Now

$$M(x_{m(k)-1}, x_{n(k)-1}) = \max\{d(x_{m(k)-1}, x_{n(k)-1}), \frac{1}{2}(d(x_{m(k)-1}, x_{m(k)})$$

$$+d(x_{n(k)-1}, x_{n(k)})), \frac{1}{2}(d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)-1}, x_{m(k)}))\}. \tag{2.44}$$

$$Q(x_{m(k)-1}, x_{n(k)-1}) = \min\{d(x_{m(k)-1}, x_{n(k)-1}), \frac{1}{2}(d(x_{m(k)-1}, x_{m(k)})$$

$$+d(x_{n(k)-1}, x_{n(k)})), \frac{1}{2}(d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)-1}, x_{m(k)}))\}. \tag{2.45}$$

Making $k \to \infty$ and using (2.36), (2.39), (2.40) and (2.41), we obtain from (2.43), (2.44)and (2.45)

$$\underset{k\to\infty}{Lim} M(x_{m(k)-1}, x_{n(k)-1}) = \epsilon, \tag{2.46}$$

$$\underset{k\to\infty}{Lim} Q(x_{m(k)-1}, x_{n(k)-1}) = 0. \tag{2.47}$$

Further making $k \to \infty$ in (43), and using (2.46), (2.47), by continuity of Ψ we obtain

$$\Psi(\epsilon) \leq \Psi(\epsilon) - \underset{k\to\infty}{Lim}\ h(Q(x_{m(k)-1}, x_{n(k)-1})). \qquad (2.48)$$

By (2.48) and the fact that h has a discontinuity at $t = 0$ and $h(t) > 0$ for $t > 0$, we observe that the last term of the right hand side of the above inequality is non zero. Hence we arrive at a contradiction. Hence $\{x_n\}$ is a Cauchy sequence and therefore $\{x_n\}$ is convergent in the complete metric space X.

Let $x_n \to z$ as $n \to \infty$. By (2.36), there exists a subsequences $\{y_k\}$ of $\{x_n\}$ such that $z \neq y_k$ for all k. Substituting $x = y_k$ and $y = z$ in (2.28), (2.29) and (2.30) we obtain $\Psi(d(y_{k+1}, Tz)) \leq \Psi(M(y_k, z)) - h(Q(y_k, z))$ where,

$M(y_k, z) = \max\{d(y_k, z), \frac{1}{2}(d(y_k, y_{k+1}) + d(z, Tz)), \frac{1}{2}(d(y_k, Tz) + d(z, y_{k+1}))\}$,

$Q(y_k, z) = \min\{d(y_k, z), \frac{1}{2}(d(y_k, y_{k+1}) + d(z, Tz)), \frac{1}{2}(d(y_k, Tz) + d(z, y_{k+1}))\}$.

Making $k \to \infty$ in the above inequalities, we obtain $\Psi(d(z, Tz)) \leq \Psi(\frac{1}{2}(d(z, Tz)))$, which implies that $d(z, Tz) = 0$, that is, $z = Tz$. Hence z is a fixed point of T.

We next establish that the fixed point is unique. Let z_1 and z_2 be two fixed points of T and $z_1 \neq z_2$, then putting $x = z_1$ and $y = z_2$ in (2.28), (2.29) and (2.30) we obtain $\Psi(d(z_1, z_2)) \leq \Psi(d(z_1, z_2)) - h(d(z_1, z_2))$ which by the virtue of the property of Ψ-function and h functions implies $d(z_1, z_2) = 0$,that is $z_1 = z_2$. This completes the proof of the theorem.2.3.

Example 2.4. Let $Y = \{0, 1, 2, 3, 4, ...\}$ and

$$d(x,y) = \begin{cases} 1 & \text{if x,y} \in \{0,1\}, x \neq y \\ x + y, & \text{if at least one of x or y } \notin \{0,1\}, x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then Y being a closed subset of (S, d) of example (2.2) is a complete metric space. In fact S is a closed subset of the space X in example 2.2 which is a complement of (0,1) in X.

Let $T : X \to X$ be defined as

$$Tx = \begin{cases} x - 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Let, $\Psi : [0, \infty) \to [0, \infty)$ be defined as $\Psi(t) = t^2, \quad t \in [0, \infty)$

Let, $h : [0, \infty) \to [0, \infty)$ be defined as

$$h(t) = \begin{cases} \frac{1}{2}, & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases}$$

Then Theorem 2.3 is applicable to this example.

Remark. The difference between the contents of the two theorems is envisaged by the observation that the second theorem does not apply to the example associated with the first theorem.

References

1. Ya.I.Alber,S.Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert spaces*, in :I. Gohberg,Yu. Lyubich(Eds.), New Results in Operator Theory, in : Advances and Appl., **98**, Birkhuser, Basel, 1997, 7-22.

2. A. D. Arvanitakis, *A proof of the generalized Banach contraction conjecture*, Proc. Amer. Math. Soc., **131(12)** (2003), 3647-3656.
3. G.V.R.Babu, B.Lalitha and M.L.Sandhya, *Common fixed point theorems involving two generalized altering distance functions in four variables*, Proc. Jangjeon Math. Soc., **10(1)**(2007), 83-93.
4. D.W. Boyd, J.S.W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20** (1969), 458-464.
5. I . Beg, M. Abbas, *Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition*, Fixed Point Theory and Applications, 2006. Article ID 74503, 7 pages.
6. C. E. Chidume, H. Zegeye, S. J. Aneke, *Approximation of fixed points of weakly contractive nonself maps in Banach spaces*, J.Math. Anal. Appl., **270(1)** (2002), 189-199.
7. B.S.Choudhury, P.N. Dutta, *A unified fixed point result in metric spaces involving a two variable function*, Filomat, **14** (2000), 43-48.
8. B.S. Choudhury, *A common unique fixed point result in metric spaces involving generalized altering distances*, Mathematical Communications, **10** (2005), 105-110.
9. B.S. Choudhury, P.N. Dutta, *Common fixed points for fuzzy mappings using generalized altering distances*, Soochow J. Math., **31(1)** (2005), 71-81.
10. B.S. Choudhury, K.Das, *A new contraction principle in Menger spaces*, Acta Mathematica Sinica, **24(8)**(2008), 1379-1386.
11. B.S. Choudhury, K.Das, P.N. Dutta, *A fixed point result in Menger spaces using a real function*, Acta Math. Hungar., 2008, DOI: 10.1007/s10474-008-7242-3.
12. B.S. Choudhury, P.N. Dutta, A.kundu, Fixed point results using altering distance function and $\phi-$ function in metric spaces, IJMA **5**-8, (1-12), 2007, 281-287.
13. P.N. Dutta and B.S.Choudhury, *A Generalisation of Contraction Principle in Metric Spaces*, Fixed Point Theory and Applications, **2008**, (2008), Article ID 406368, 8 pages.
14. D.Ilic, V.Rakocevic, *Common fixed points for maps on cone metric space*, J.Math. Anal. Appl., **341** (2008), 876-882.
15. M.S, Khan, M. Swaleh, S. Sessa, *Fixed points theorems by altering distances between the points*, Bull. Austral. Math. Soc., **30** (1984), 1-9.
16. J. Merryfield, B. Rothschild, J. D. Stein, Jr., *An application of Ramsey's theorem to the Banach contraction principle*, Proceedings of the American Mathematical Society, **130 (4)** (2002), 927-933.
17. S.V.R. Naidu, *Some fixed point theorems in Metric spaces by altering distances*, Czechoslovak Mathematical Journal, **53**(1) (2003), 205-212.
18. B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Analysis: Theory,Methods and Application, **47(4)** (2001), 2683-2693.
19. K.P.R. Sastry, G.V.R. Babu, *Some fixed point theorems by altering distances between the points*, Ind. J. Pure. Appl. Math., **30(6)** (1999), 641-647.
20. K.P.R. Sastry, S.V.R.Naidu, G.V.R. Babu, G.A.Naidu, *Generalization of common fixed point theorems for weakly commuting map by altering distances*, Tamkang Jr. Math., **31(3)** (2000), 243-250.
21. Y.Song, *Coincidence points for noncommuting f-weakly contractive mappings*, Int. J. Comput. Appl. Math., (IJCAM), **2(1)** (2007), 51-57.
22. Q. Zhang and Y.Song, *Fixed point theory for generalized $\phi-$ weak contractions*, Applied Mathematics Letters, **22(1)** (2009), 75-78.

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