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EXISTENCE RESULTS FOR NONLINEAR FIRST-ORDER PERIODIC BOUNDARY VALUE PROBLEM OF IMPULSIVE DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. In this paper, existence criteria of one solution to a nonlinear first-order periodic boundary value problem of impulsive dynamic equation on time scales are obtained by using the well-known Schaefer fixed-point theorem.

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1. Introduction

The theory of impulsive differential equations is emerging as an important area of investigation, since it is a lot richer than the corresponding theory of differential equations without impulse effects. Moreover, such equations may exhibit several real world phenomena in physics, biology, engineering, etc. (see [3, 4, 20]). At the same time, the boundary value problems for impulsive differential equations and impulsive difference equations have received much attention [2, 10, 16, 17, 21, 22, 24-27]. On the other hand, recently, the theory of dynamic equations on time scales has become a new important branch (See, for example, [1, 5, 6, 15, 18]). However, to the best of our knowledge, there is not much concerning for BVPs of impulsive dynamic equations on time scales [7, 8, 11-14, 23, 29].

Let $\mathbf{T}$ be a time scale, i.e., $\mathbf{T}$ is a nonempty closed subset of R. Let 0, T be points in $\mathbf{T}$, an interval $(0,T)_{\mathbf{T}}$ denoting time scales interval, that is, $(0,T)_{\mathbf{T}} := (0,T) \cap \mathbf{T}$. Other types of intervals are defined similarly.

In this paper, we are concerned with the existence of solutions for the following nonlinear first-order PBVP on time scale

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$$\begin{cases} x^{\triangle}(t)+p(t)x(\sigma(t))=f(t,x(\sigma(t))),\ t\in J:=[0,T]_{\mathbf{T}},\ t\neq t_k, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad k=1,2,\ldots,m, \\ x(t_k^+)-x(t_k^-)=I_k(x(t_k^-)),\ t=t_k,\ k=1,2,\ldots,m, \\ x(0)=x(\sigma(T)), \end{cases} \tag{1}$$

where $f: J\times R\to R$ is a given function, $I_k\in C(R,R)$, $p:[0,T]_{\mathbf{T}}\to(0,\infty)$ is right-dense continuous (that is $p\in\mathcal{R}^+$, where $\mathcal{R}^+$ will be defined in section 2), $t_k\in(0,T)_{\mathbf{T}}$, $0<t_1<\cdots<t_m<T$, and for each $k=1,2,\ldots,m$, $x(t_k^+)=\lim_{h\to 0^+}x(t_k+h)$ and $x(t_k^-)=\lim_{h\to 0^-}x(t_k+h)$ represent the right and left limits of $x(t)$ at $t=t_k$.

In [9], Cabada developed the method of lower and upper solutions coupled with the monotone iterative techniques to derive the existence of extremal solutions to the first-order PBVP of dynamic equations on time scales (in the one-dimensional case)

$$\begin{cases} u^{\triangle}(t)=f(t,u(t)),\ t=[a,b]_{\mathbf{T}}, \\ u(a)=u(\sigma(b)). \end{cases} \tag{2}$$

In [28], Sun and Li considered the existence of solutions to the following first-order PBVPs on time scales

$$\begin{cases} x^{\triangle}(t)+p(t)x(\sigma(t))=f(t,x(\sigma(t))),\ t=[0,T]_{\mathbf{T}}, \\ x(0)=x(\sigma(T)). \end{cases} \tag{3}$$

The main tool used in [28] is the well-known Schaefer fixed-point theorem [19]. In [29], the second author studied the problem (1). The existence of positive solutions to the problem (1) was obtained by means of the Guo-Krasnoselskii fixed point theorem. In this paper, we shall show that the PBVP (1) has at least one solution by means of the well-known Schaefer fixed-point theorem. Our results were motivated by the work [10].

In the remainder of this section, we state the well-know Schaefer fixed-point theorem [19].

Theorem 1. (Schaefer Fixed Point Theorem) *Let E be a normed linear space (possibly incomplete) and $\Phi: E\to E$ be a compact operator. Suppose that the set*

$$S=\{x\in E|x=\lambda\Phi(x),\ \text{some }\lambda\in(0,1)\}$$

is bounded. Then Φ has a fixed point in E.

2. Preliminaries

In this section, we state some fundamental definitions and results concerned time scales, so that the paper is self-contained. For more details, one can refer to [1, 5, 6, 15, 18].

Definition 1. Assume that $x:\mathbf{T}\to\mathbf{R}$ and fix $t\in\mathbf{T}$ (if $t=\sup\mathbf{T}$, we assume t is not left-scattered). Then x is called delta differentiable at $t\in\mathbf{T}$ if there exists a $\theta \in R$ such that for any given $\varepsilon > 0$, there is an open neighborhood U of t such that

$$|x(\sigma(t)) - x(s) - \theta\,|\sigma(t) - s|| \leq \varepsilon\,|\sigma(t) - s|\,,\ s \in U.$$

In this case, θ is called the delta derivative of x at $t \in \mathbf{T}$ and denote it by $\theta = x^{\triangle}(t)$. If $F^{\triangle}(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s)\triangle s = F(t) - F(a).$$

Definition 2. A function $f : \mathbf{T} \to \mathbf{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbf{T}$ and its left-sided limits exist at left-dense points in $\mathbf{T}$. The set of rd-continuous $f : \mathbf{T} \to \mathbf{R}$ will be denoted by C_{rd}.

Lemma 1. *If $f \in C_{rd}$ and $t \in \mathbf{T}^{\mathbf{k}}$, then*

$$\int_t^{\sigma(t)} f(s)\triangle s = \mu(t)f(t),$$

where $\mu(t) = \sigma(t) - t$ is the graininess function.

Lemma 2. *If $f^{\triangle} \geq 0$, then f is increasing.*

Lemma 3. *Assume that $f,\ g : \mathbf{T} \to \mathbf{R}$ are delta differentiable at t, then*

$$(fg)^{\triangle}(t) = f^{\triangle}(t)g(t) + f(\sigma(t))g^{\triangle}(t) = f(t)g^{\triangle}(t) + f^{\triangle}(t)g(\sigma(t)).$$

Definition 3. A function $p : \mathbf{T} \to \mathbf{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbf{T}^{\mathbf{k}}.$$

The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R}$.

Definition 4. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \quad \text{for all} \quad t \in \mathbf{T}\}$$

Definition 5. If $p \in \mathcal{R}$, then the delta exponential function is given by $e_p(t,s) = \exp\left(\int_s^t g(\tau)\triangle\tau\right)$, where

$$g(\tau) = \begin{cases} p(\tau), & \text{if } \mu(\tau) = 0, \\ \frac{1}{\mu(\tau)}Log(1 + p(\tau)\mu(\tau)), & \text{if } \mu(\tau) \neq 0, \end{cases}$$

here Log is the principal logarithm.

Lemma 4. *If $p \in \mathcal{R}$, then*

(1) $e_p(t,t) \equiv 1$;
(2) $e_p(t,s) = \frac{1}{e_p(s,t)}$;
(3) $e_p(t,u)e_p(u,s) = e_p(t,s)$;
(4) $e_p^{\triangle}(t,t_0) = p(t)e_p(t,t_0)$, *for $t \in \mathbf{T}^{\mathbf{k}}$ and $t_0 \in \mathbf{T}$.*

Lemma 5. *If $p \in \mathcal{R}^+$ and $t_0 \in \mathbf{T}$, then*

$$e_p(t, t_0) > 0 \text{ for all } t \in \mathbf{T}.$$

3. Main results

Throughout the rest of this paper, we always assume that the points of impulse t_k are right-dense for each $k = 1, 2, \dots, m$. Let

$PC = \{x \in [0, \sigma(T)]_{\mathbf{T}} \to R : x_k \in C(J_k, R),\ k = 1, 2, \dots, m$ and there exist $x(t_k^+)$ and $x(t_k^-)$ with $x(t_k^-) = x(t_k),\ k = 1, 2, \dots, m\}$,

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}]_{\mathbf{T}} \subset (0, \sigma(T)]_{\mathbf{T}}$, $k = 1, 2, \dots, m$ and $J_0 = [0, t_1]_{\mathbf{T}}$, $J_{m+1} = \sigma(T)$, $\forall x \in PC$, define the norm

$$\|x\|_{PC} = \max\left\{\|x_k\|_{J_k},\ k = 0, 1, \dots, m\right\},$$

obviously PC is a Banach space.

Definition 6. A function $x \in PC \cap C^1(J\backslash\{t_1, t_2, \dots, t_m\}, R)$ is said to be a solution of PBVP (1.1) if and only if x satisfies the dynamic equation

$$x^{\triangle}(t) + p(t)x(\sigma(t)) = f(t, x(\sigma(t))) \text{ everywhere on } J\backslash\{t_1, t_2, \dots, t_m\},$$

the impulsive conditions

$$x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)),\ k = 1, 2, \dots, m,$$

and the periodic boundary condition $x(0) = x(\sigma(T))$.

Lemma 6. ([29]) *Suppose $h : [0, T]_{\mathbf{T}} \to R$ is rd-continuous, then x is a solution of*

$$x(t) = \int_0^{\sigma(T)} G(t, s)h(s)\triangle s + \sum_{k=1}^{m} G(t, t_k)I_k(x(t_k)),\ t \in [0, \sigma(T)]_{\mathbf{T}},$$

where $G(t, s) = \begin{cases} \frac{e_p(s,t)e_p(\sigma(T),0)}{e_p(\sigma(T),0)-1}, & 0 \le s \le t \le \sigma(T), \\ \frac{e_p(s,t)}{e_p(\sigma(T),0)-1}, & 0 \le t < s \le \sigma(T), \end{cases}$ *if and only if x is a solution of the boundary value problem*

$$\begin{cases} x^{\triangle}(t) + p(t)x(\sigma(t)) = h(t),\ t \in J := [0, T]_{\mathbf{T}},\ t \ne t_k,\ k = 1, 2, \dots, m, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)),\ k = 1, 2, \dots, m, \\ x(0) = x(\sigma(T)). \end{cases}$$

Theorem 2. *Suppose*

(H_1) $f : J \times R \to R$ is continuous and there exist nonnegative constants α and K such that, for any $\lambda \in (0, 1)$, $\lambda\,|f(t, x)| \le \alpha[\lambda f(t, x) - p(t)x] + K$, $t \in J, x \in R$.

(H_2) $I_k : R \to R$ is continuous and there exist nonnegative constants β_k and N_k such that $|I_k(x)| \le \beta_k\,|x| + N_k$, for each $k = 1, 2, \dots, m$.

(H_3) $m\beta^* e_p(\sigma(T), 0) < e_p(\sigma(T), 0) - 1$, where $\beta^* = \max_{1 \le k \le m} \beta_k$.

Then the PBVP (1) has at least one solution.

Proof. Define an operator $\Phi : PC \to PC$ by

$$(\Phi x)(t) = \int_0^{\sigma(T)} G(t,s)f(s,x(\sigma(s)))\Delta s + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k)),\ t \in [0,\sigma(T)]_{\mathbf{T}}.$$

Note that $G^* = \sup_{t,s\in[0,\sigma(T)]_{\mathbf{T}}} |G(t,s)| \le \frac{e_p(\sigma(T),0)}{e_p(\sigma(T),0)-1}$. By Lemma 6, it is easy to see that fixed points of Φ are the solutions to the PBVP (1) .

First, we assert that Φ is continuous and completely continuous. The proof is divided into three steps.

Step 1: To show that $\Phi : PC \to PC$ is continuous, let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $\lim_{n\to\infty} x_n = x$ in PC. Then

$$\begin{aligned} & |(\Phi x_n)(t) - (\Phi x)(t)| \\ = & \left| \int_0^{\sigma(T)} G(t,s)\left[f(s,x_n(\sigma(s))) - f(s,x(\sigma(s)))\right]\Delta s \right. \\ & \left. + \sum_{k=1}^{m} G(t,t_k)\left[I_k(x_n(t_k)) - I_k(x(t_k))\right] \right| \\ \le & G^*\sigma(T)\cdot \|f(s,x_n(\sigma(s))) - f(s,x(\sigma(s)))\|_{\infty} + G^* m \cdot \Big\| I_k(x_n(t_k)) \\ & - I_k(x(t_k)) \Big\|_{\infty}^{*}. \end{aligned}$$

Since f, I_k are continuous, it follows that $\|\Phi x_n - \Phi x\|_{PC} \to 0$ $(n \to \infty)$. That is, $\Phi : PC \to PC$ is continuous.

Step 2: To show that Φ maps bounded sets into bounded sets in PC, let $B_l = \{x \in PC : \|x\|_{PC} \le l\}$, $M = \max_{t\in[0,T]_{\mathbf{T}},|x|\le l} |f(t,x)|$, and $N^* = \max_{1\le k\le m} N_k$.Then, for any $x \in B_l$, we have

$$\begin{aligned} |(\Phi x)(t)| &= \left| \int_0^{\sigma(T)} G(t,s)f(s,x(\sigma(s)))\Delta s + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k)) \right| \\ &\le G^*\left[\sigma(T)M + m(\beta^* l + N^*)\right]. \end{aligned}$$

which shows that $\Phi(B_l)$ is bounded.

Step 3: To show that Φ maps bounded sets into equicontinuous sets of PC, let t_1, $t_2 \in [0,\sigma(T)]_{\mathbf{T}}$, $x \in B_l$, then

$$\begin{aligned} |(\Phi x)(t_1) - (\Phi x)(t_2)| \le & \int_0^{\sigma(T)} |G(t_1,s) - G(t_2,s)|\,|f(s,x(\sigma(s)))|\,\Delta s \\ & + \sum_{k=1}^{m} |G(t_1,t_k) - G(t_2,t_k)|\,|I_k(x(t_k))|. \end{aligned}$$

The right-hand side tends to uniformly zero as $|t_1 - t_2| \to 0$.

Consequently, Steps 1-3 together with the Arzela-Ascoli Theorem show that $\Phi : PC \to PC$ is continuous and completely continuous.

Next, we assert that the set $S = \{x \in PC | x = \lambda\Phi(x),\ \text{some}\ \lambda \in (0,1)\}$ is bounded. Let $x \in S$. Then, there exists a $\lambda \in (0,1)$ such that $x = \lambda\Phi(x)$. So, from (H_1), (H_2) and (H_3), we have :

$$\begin{aligned}
|x(t)| &= |\lambda(\Phi x)(t)| \\
&= \lambda\left|\int_0^{\sigma(T)} G(t,s)f(s,x(\sigma(s)))\Delta s + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k))\right| \\
&\le G^*\cdot\int_0^{\sigma(T)} |\lambda f(s,x(\sigma(s)))|\,\Delta s + G^*\cdot\sum_{k=1}^{m}|I_k(x(t_k))| \\
&\le G^*\cdot\int_0^{\sigma(T)} \{\alpha[\lambda f(s,x(\sigma(s))) - p(s)x(\sigma(s))] + K\}\Delta s \\
&\qquad + G^*\cdot\sum_{k=1}^{m}[\beta_k\,|x(t_k)| + N_k] \\
&\le G^*\cdot\int_0^{\sigma(T)} [\alpha x^{\Delta} + K]\Delta s + G^* m(\beta^*\,\|x\| + N^*) \\
&= G^*\left[K\sigma(T) + m(\beta^*\,\|x\| + N^*)\right],\ t\in[0,\sigma(T)]_{\mathbf{T}},
\end{aligned}$$

which implies $\|x\|_{PC} \le \dfrac{e_p(\sigma(T),0)\,[K\sigma(T)+mN^*]}{e_p(\sigma(T),0)(1-m\beta^*)-1}$, i.e., $S=\{x\in PC|x=\lambda\Phi(x),\ \text{some } \lambda\in(0,1)\}$ is bounded. Thus, by the Theorem 1 we know that Φ has at least one fixed point, which is the desired solution of PBVP (1). □

Corollary 1. *Suppose*

(H_4) $f: J\times R \to R$ *is continuous and bounded.*

(H_5) $I_k: R\to R$, $k=1,2,\ldots,m$, *are continuous and bounded.*

Then the PBVP (1) has at least one solution.

4. Example

Example 1. Let $\mathbf{T}=[\mathbf{0},\mathbf{1}]\cup[\mathbf{2},\mathbf{3}]$. We consider the following PBVP on $\mathbf{T}$

$$\begin{cases} x^{\Delta}(t) + x(\sigma(t)) = f(t,x(\sigma(t))),\ t\in[0,3]_{\mathbf{T}},\ t\neq \frac12, \\ x\left(\frac12^+\right) - x\left(\frac12^-\right) = I(x(\frac12)), \\ x(0)=x(3), \end{cases} \tag{4}$$

where $p(t)\equiv 1$, $T=3$, $f(t,x(\sigma(t))) = t\arctan(x(\sigma(t)))^3$, and $I(x(\frac12)) = \frac{2e^2-1}{3e^2}x(\frac12)$. Then the PBVP (4) has at least one solution.

Proof. Since $p(t)\equiv 1$, $T=3$, and $\mathbf{T}=[\mathbf{0},\mathbf{1}]\cup[\mathbf{2},\mathbf{3}]$, it is easy to see that $e_p(\sigma(T),0)=2e^2$.

Choose $\alpha=0, K=\frac{3\pi}{2}, \beta^*=\frac{2e^2-1}{3e^2}$, $N^*=0$, then the conditions of Theorem 2 are satisfied. Thus, the PBVP (4) has at least one solution. □

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