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STRONG CONVERGENCE OF A NEW ITERATIVE ALGORITHM FOR AVERAGED MAPPINGS IN HILBERT SPACES

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ABSTRACT. Let H be a real Hilbert space. Let $T : H \to H$ be an averaged mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real numbers in $(0,1)$. For given $x_0 \in H$, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = (1-\alpha_n)Tx_n, \quad n \geq 0.$$

Assume that the following control conditions hold:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then $\{x_n\}$ converges strongly to a fixed point of T.

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1. Introduction

Let H be a real Hilbert space, and C a closed convex subset of H. Recall that a mapping $S : C \to C$ is said to be non-expansive if

$$\|Sx - Sy\| \leq \|x - y\|,$$

for all $x, y \in C$. A mapping $T : C \to C$ is called an averaged mapping on C if there exists a non-expansive mapping $S : C \to C$ and a number $k \in (0,1)$ such that

$$T = (1-k)I + kS. \tag{1}$$

A point $x \in C$ is a fixed point of S provided $Sx = x$. Denote by $F(S)$ the set of fixed points of S; that is, $F(S) = \{x \in C : Sx = x\}$. If $F(S) \neq \emptyset$, then we obtain immediately that $F(T) = F(S)$.

It is clear that an averaged mapping is non-expansive, but not vice vera. An example of an averaged mapping is the metric projection from a Hilbert

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space onto a closed convex subset. It is known that averaged mappings are always asymptotically regular and their Picard iterates converge weakly. These properties make averaged mappings favorably useful in applications in e.g. inverse problems and image recovery (see [1-5]). In recent years, some iterative algorithms have been proposed for approximating fixed point of non-expansive mappings under some assumptions, please see [6-12]. You can find the related works in [13-26].

It is our purpose in this paper that we introduce a new iterative algorithm for averaged mappings. Furthermore, we prove that the proposed iterative algorithm converges strongly to a fixed point of an averaged mapping in Hilbert spaces.

2. Preliminaries

In this section, we collect the following well-known lemmas.

Lemma 2.1 *Let H be a real Hilbert space. Then there hold the following well-known results:*

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$ *for all* $x, y \in H$;
(ii) $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$ *for all* $x, y \in H$.

Lemma 2.2 (Demi-closed principle) *Let C be a nonempty closed convex of a real Hilbert space H. Let $S : C \to C$ be a non-expansive mapping. Then S is demi-closed on C, i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Sx_n \to 0$, then $x = Sx$.*

Lemma 2.3([6]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we will prove our main result.

Theorem 3.1 *Let H be a real Hilbert space. Let $T : H \to H$ be an averaged mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real numbers in $(0,1)$. For given $x_0 \in H$, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = (1-\alpha_n)Tx_n, \quad n \ge 0. \tag{2}$$

Assume that the following control conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. First, we prove that $\{x_n\}$ is bounded. Take $p \in F(T)$. From (2), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)Tx_n - p\| \\ &= \|(1-\alpha_n)(Tx_n - p) - \alpha_n p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|p\|.\end{aligned}$$

By induction, we can obtain

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \|p\|\}.$$

Hence, $\{x_n\}$ is bounded.

Since T is an averaged mapping, from (1), there exists a non-expansive mapping $S : H \to H$ and a constant $k \in (0,1)$ such that

$$T = (1-k)I + kS.$$

It is clear that $F(T) = F(S)$. Since S is non-expansive, hence we have

$$\begin{aligned}&\langle Sx - p, Sx - p\rangle \leq \langle x - p, x - p\rangle \\ \Rightarrow\ &\langle Sx - p, Sx - p\rangle \leq \langle x - p, x - Sx\rangle + \langle x - p, Sx - p\rangle \\ \Rightarrow\ &\langle Sx - p, Sx - x\rangle \leq \langle x - p, x - Sx\rangle \\ \Rightarrow\ &\langle Sx - x, Sx - x\rangle + \langle x - p, Sx - x\rangle \leq \langle x - p, x - Sx\rangle \\ \Rightarrow\ &\|Sx - x\|^2 \leq 2\langle x - p, x - Sx\rangle.\end{aligned} \tag{3}$$

Setting $y_n = Tx_n = (1-k)x_n + kSx_n, n \geq 0$. From (2), (3) and Lemma 2.1, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)y_n - p\|^2 \\ &= \|(1-\alpha_n)(y_n - p) - \alpha_n p\|^2 \\ &\leq (1-\alpha_n)^2\|y_n - p\|^2 - 2\alpha_n\langle p, x_{n+1} - p\rangle \\ &\leq \|x_n - p - k(x_n - Sx_n)\|^2 - 2\alpha_n\langle p, x_{n+1} - p\rangle \\ &= \|x_n - p\|^2 - 2k\langle x_n - Sx_n, x_n - p\rangle + k^2\|x_n - Sx_n\|^2 \\ &\quad - 2\alpha_n\langle p, x_{n+1} - p\rangle \\ &\leq \|x_n - p\|^2 - k\|x_n - Sx_n\|^2 + k^2\|x_n - Sx_n\|^2 \\ &\quad - 2\alpha_n\langle p, x_{n+1} - p\rangle \\ &= \|x_n - p\|^2 - k(1-k)\|x_n - Sx_n\|^2 - 2\alpha_n\langle p, x_{n+1} - p\rangle.\end{aligned} \tag{4}$$

Since $\{x_n\}$ is bounded, so there exists a constant $M \geq 0$ such that

$$-2\alpha_n\langle p, x_{n+1} - p\rangle \leq M\alpha_n \quad \text{for all} \quad n \geq 0.$$

Consequently, from (4), we get

$$\|x_{n+1} - p\|^2 - \|x_n - p\|^2 + k(1-k)\|x_n - Sx_n\|^2 \leq M\alpha_n. \tag{5}$$

Now we divide two cases to prove that $\{x_n\}$ converges strongly to p.

Case 1. Assume that the sequence $\{\|x_n - p\|\}$ is a monotonically decreasing sequence. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\|x_{n+1} - p\|^2 - \|x_n - p\|^2 \to 0,$$

this together with (i) and (5) imply that

$$\|x_n - Sx_n\| \to 0. \tag{6}$$

Since S is demi-closed (by Lemma 2.2), then it is easy to prove that $\{x_n\}$ converges weakly to a fixed point p of S and T.

Next, we prove that $\{x_n\}$ strongly converges to p. Indeed, from (4), we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1 - \alpha_n)\|y_n - p\|^2 - 2\alpha_n\langle p, x_{n+1} - p\rangle \\ &= (1 - \alpha_n)\|(1 - k)(x_n - p) + k(Sx_n - p)\|^2 - 2\alpha_n\langle p, x_{n+1} - p\rangle \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 - 2\alpha_n\langle p, x_{n+1} - p\rangle.\end{aligned} \tag{7}$$

It is clear that $\lim_{n\to\infty}\langle p, x_{n+1} - p\rangle = 0$. Hence, applying Lemma 2.3 to (7), we immediately deduce that $x_n \to p$.

Case 2. Assume that $\{\|x_n - p\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = \|x_n - p\|^2$ and let $\tau : N \to N$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in N : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. From (5), it is easy to see that

$$\|x_{\tau(n)} - Sx_{\tau(n)}\|^2 \leq \frac{M\alpha_{\tau(n)}}{k(1-k)} \to 0.$$

By the similar argument as that in Case 1, we conclude immediately that $x_{\tau(n)}$ weakly converges to p as $\tau(n) \to \infty$. At the same time, we note that, for all $n \geq n_0$,

$$0 \leq \|x_{\tau(n)+1} - p\|^2 - \|x_{\tau(n)} - p\|^2 \leq \alpha_{\tau(n)}[2\langle p - x_{\tau(n)+1}, p\rangle - \|x_{\tau(n)} - p\|^2],$$

which implies that

$$\|x_{\tau(n)} - p\|^2 \leq 2\langle p - x_{\tau(n)+1}, p\rangle.$$

Hence, we deduce that $\lim_{n\to\infty}\|x_{\tau(n)} - p\| = 0$. Therefore,

$$\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim_{n\to\infty}\Gamma_n = 0$, this is, $\{x_n\}$ converges strongly to p. This completes the proof. □

Remark 3.1 It is well-known that the Picard iteration has only weak convergence. However, our algorithm which is similar to the Picard iteration has strong convergence.

From the proof of Theorem 3.1, it is easy to prove the following corollary.

Corollary 3.1 *Let H be a real Hilbert space. Let $T : H \to H$ be a non-expansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real numbers in $(0, 1)$. Let $k \in (0, 1)$ be a constant. For given $x_0 \in H$, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = (1 - \alpha_n)[(1 - k)x_n + kTx_n], \quad n \geq 0.$$

Assume that the following control conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

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