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METHODS FOR ITERATIVE DISENTANGLING IN FEYNMAN'S OPERATIONAL CALCULI : THE CASE OF TIME DEPENDENT NONCOMMUTING OPERATORS

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ABSTRACT. The disentangling map from the commutative algebra to the noncommutative algebra of operators is the essential operation of Feynman's operational calculus for noncommuting operators. Thus formulas which simplify this operation are meaningful to the subject. In a recent paper the procedure for "methods for iterative disentangling" has been established in the setting of Feynman's operational calculus for time independent operators $A_1, \cdots, A_n$ and associated probability measures $\mu_1, \cdots, \mu_n$. The main purpose for this paper is to extend the procedure for methods for iterative disentangling to time dependent operators.

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1. Introduction

This paper extends the basic result of Jefferies, Johnson and Kim [7] on " methods for iterative disentangling " to the time-dependent and not necessarily probability measure setting. Feynman's operational calculus for noncommuting operators originated with Feynman's 1951 paper [2]. We are working here in the framework of the Feynman's operational calculus initiated by Jefferies and Johnson [3,4,5,6] and by the same authors and Nielsen [8].

Passing from probability measures to measures which are finite on any bounded interval of $\mathbb{R}$ is not difficult as discussed in [4], but time-dependence of the operators as in [8] yields a more complicated framework and so a somewhat more complicated proof than in [7]. However, the most essential ideas of the proof remain the same.

Let X be a separable Banach space over the complex numbers and let $\mathcal{L}(X)$ denote the space of bounded linear operators on X. Fix $T > 0$. For $i = 1, \cdots, n$

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let $A_i : [0,T] \to \mathcal{L}(X)$ be maps that are measurable in the sense that $A_i^{-1}(E)$ is a Borel set in [0,T] for any strong operator open set $E \subset \mathcal{L}(X)$. To each $A_i(\cdot)$ we associate a finite continuous Borel measure μ_i on $[0,T]$ and we require that, for each i,

$$r_i = \int_{[0,T]} ||A_i(s)||_{\mathcal{L}(X)} |\mu_i|(ds) < \infty.$$

For n positive numbers $r_1, \cdots, r_n$, let $\mathbb{A}(r_1, \cdots, r_n)$ be the space of complex-valued functions of n complex variables $f(z_1, \cdots, z_n)$, which are analytic at $(0, \cdots, 0)$, and are such that their power series expansion

$$f(z_1, \cdots, z_n) = \sum_{m_1, \cdots, m_n = 0}^{\infty} c_{m_1, \cdots, m_n} z_1^{m_1} \cdots z_n^{m_n}$$

converges absolutely, at least on the closed polydisk $|z_1| \le r_1, \cdots, |z_n| \le r_n$. Such functions are analytic at least in the open polydisk $|z_1| < r_1, \cdots, |z_n| < r_n$.

To the algebra $\mathbb{A}(r_1, \cdots, r_n)$ we associate as in [3] a disentangling algebra by replacing the z_i's with formal commuting objects $(A_i(\cdot), \mu_i\tilde{)}$, $i = 1, \cdots, n$. Rather than using the notation $(A_i(\cdot), \mu_i\tilde{)}$ below, we will often abbreviate to $A_i(\cdot\tilde{)}$. Consider the collection $\mathbb{D}\Big((A_1(\cdot), \mu_1\tilde{)}, \cdots, (A_n(\cdot), \mu_n\tilde{)}\Big)$ of all expressions of the form

$$f\Big(A_1(\cdot\tilde{)}, \cdots, A_n(\cdot\tilde{)}\Big) = \sum_{m_1, \cdots, m_n = 0}^{\infty} c_{m_1, \cdots, m_n} \Big(A_1(\cdot\tilde{)})^{m_1} \cdots (A_n(\cdot\tilde{)}\Big)^{m_n}$$

where $c_{m_1, \cdots, m_n} \in \mathbb{C}$ for all $m_1, \cdots, m_n = 0, 1, \cdots$, and

$$\begin{aligned} \Big\| f(A_1(\cdot\tilde{)}, \cdots, \quad A_n(\cdot\tilde{)})\Big\| &= \Big\| f(A_1(\cdot\tilde{)}, \cdots, A_n(\cdot\tilde{)})\Big\|_{\mathbb{D}(A_1(\cdot\tilde{)}, \cdots, A_n(\cdot\tilde{)})} \\ &= \sum_{m_1, \cdots, m_n = 0}^{\infty} |c_{m_1, \cdots, m_n}| r_1^{m_1} \cdots r_n^{m_n} < \infty. \end{aligned} \tag{1}$$

The function on $\mathbb{D}\Big((A_1(\cdot), \mu_1\tilde{)}, \cdots, (A_n(\cdot), \mu_n\tilde{)}\Big)$ defined by (1) makes $\mathbb{D}\Big((A_1(\cdot), \mu_1\tilde{)}, \cdots, (A_n(\cdot), \mu_n\tilde{)}\Big)$ into a commutative Banach algebra [8].

We refer to $\mathbb{D}\Big((A_1(\cdot), \mu_1\tilde{)}, \cdots, (A_n(\cdot), \mu_n\tilde{)}\Big)$ as the disentangling algebra associated with the n-tuple $\Big((A_1(\cdot), \mu_1\tilde{)}, \cdots, (A_n(\cdot), \mu_n\tilde{)}\Big)$.

We will often write $\mathbb{D}$ in place of $\mathbb{D}\Big(A_1(\cdot\tilde{)}, \cdots, A_n(\cdot\tilde{)}\Big)$ or $\mathbb{D}\Big((A_1(\cdot), \mu_1\tilde{)}, \cdots, (A_n(\cdot), \mu_n\tilde{)}\Big)$.

For $m = 0, 1, \cdots$, let S_m denote the set of all permutations of the integers $\{1, \cdots, m\}$, and given $\pi \in S_m$, we let

$$\Delta_m(\pi) = \Big\{(s_1, \cdots, s_m) \in [0,T]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < T\Big\}.$$

Now for nonnegative integers $m_1, \cdots, m_n$ and $m = m_1 + \cdots + m_n$, we define

$$C_i(s) = \begin{cases} A_1(s), & if\ i \in \{1, \cdots, m_1\} \\ A_2(s), & if\ i \in \{m_1+1, \cdots, m_1+m_2\} \\ \cdots & \\ A_n(s), & if\ i \in \{m_1+\cdots+m_{n-1}+1, \cdots, m\} \end{cases}$$

for $i = 1, \cdots, m$ and for all $0 \leq s \leq T$.

Definition 1. Let $P^{m_1,\cdots,m_n}(z_1, \cdots, z_n) = z_1^{m_1} \cdots z_n^{m_n}$. We define the disentangling map on this monomial by

$$\begin{aligned} &\mathcal{T}_{\mu_1,\cdots,\mu_n} P^{m_1,\cdots,m_n}\Big(\tilde{A}_1(\cdot), \cdots, \tilde{A}_n(\cdot)\Big) \\ &= \mathcal{T}_{\mu_1,\cdots,\mu_n}\Big((\tilde{A}_1(\cdot))^{m_1} \cdots (\tilde{A}_n(\cdot))^{m_n}\Big) \\ &= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \\ &(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \cdots, ds_m). \end{aligned}$$

Finally for $f \in \mathbb{D}\Big((A_1(\cdot), \tilde{\mu}_1), \cdots, (A_n(\cdot), \tilde{\mu}_n)\Big)$ given by

$$f\Big(\tilde{A}_1(\cdot), \cdots, \tilde{A}_n(\cdot)\Big) = \sum_{m_1,\cdots,m_n=0}^{\infty} c_{m_1,\cdots,m_n} \Big(\tilde{A}_1(\cdot)\Big)^{m_1} \cdots \Big(\tilde{A}_n(\cdot)\Big)^{m_n}$$

we set

$$\begin{aligned} &\mathcal{T}_{\mu_1,\cdots,\mu_n} f\Big(\tilde{A}_1(\cdot), \cdots, \tilde{A}_n(\cdot)\Big) \\ &:= \sum_{m_1,\cdots,m_n=0}^{\infty} c_{m_1,\cdots,m_n} \mathcal{T}_{\mu_1,\cdots,\mu_n} P^{m_1,\cdots,m_n}\Big(\tilde{A}_1(\cdot), \cdots, \tilde{A}_n(\cdot)\Big). \end{aligned}$$

We will often use the alternate notation indicated in the next two equalities :

$$P^{m_1,\cdots,m_n}_{\mu_1,\cdots,\mu_n}\Big(A_1(\cdot), \cdots, A_n(\cdot)\Big) = \mathcal{T}_{\mu_1,\cdots,\mu_n} P^{m_1,\cdots,m_n}\Big(\tilde{A}_1(\cdot), \cdots, \tilde{A}_n(\cdot)\Big)$$

and

$$f_{\mu_1,\cdots,\mu_n}\Big(A_1(\cdot), \cdots, A_n(\cdot)\Big) = \mathcal{T}_{\mu_1,\cdots,\mu_n} f\Big(\tilde{A}_1(\cdot), \cdots, \tilde{A}_n(\cdot)\Big).$$

2. Methods for iterative disentangling

Let d be a positive integer. For each $j = 1, \cdots, d$, let I_j be the nonempty subset of $I = \{1, \cdots, n\}$ such that $I_j = \{i_{j-1}+1, \cdots, i_j\}$ where $i_0 = 0$ and let $I_0 = I - (I_1 \cup \cdots \cup I_d)$. Now we introduce the abbreviated notation. We write

$$P^{m_i, i \in I}_{\mu_i, i \in I}\Big(A_i(\cdot), i \in I\Big) = P^{m_1,\cdots,m_n}_{\mu_1,\cdots,\mu_n}\Big(A_1(\cdot), \cdots, A_n(\cdot)\Big),$$

as well as

$$f\Big(A_i(\cdot\tilde{)}, i \in I\Big) = f\Big(A_1(\cdot\tilde{)}, \cdots, A_n(\cdot\tilde{)}\Big),$$
$$\mathbb{D}\Big(A_i(\cdot\tilde{)}, i \in I\Big) = \mathbb{D}\Big(A_1(\cdot\tilde{)}, \cdots, A_n(\cdot\tilde{)}\Big),$$
$$f(z_i, \in I) = f(z_1, \cdots, z_n),$$
$$\mathcal{T}_{\mu_i, i\in I} f(A_i(\cdot\tilde{)}) = \mathcal{T}_{\mu_1,\cdots,\mu_n} f\Big(A_1(\cdot\tilde{)}, \cdots, A_n(\cdot\tilde{)}\Big).$$

Similar interpretations are intended when we write

$$g(z_i, i \in I_j), \quad \mathcal{T}_{\nu_1,\cdots,\nu_k;\mu_i, i\in I_0}$$

and so on.

We begin with the case of monomials first.

Theorem 1. *Let $a_j, b_j, j = 1, \cdots, d$ be real numbers such that*

$$0 \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_d < b_d \le T.$$

Suppose that $\mu_i, i \in I_j$ have supports contained within $[a_j, b_j]$ for $j = 1, \cdots, d$. Let $\nu_j, j = 1, \cdots, d$, be any continuous probability measures having supports contained within $[a_j, b_j]$. Given nonnegative integers $m_1, \cdots, m_n$, let

$$K_j = P^{m_i, i\in I_j}_{\mu_i, i\in I_j}\Big(A_i(\cdot), i \in I_j\Big)$$

for $j = 1, \cdots, d$. Then we have

$$P^{m_1,\cdots,m_n}_{\mu_1,\cdots,\mu_n}\Big(A_1(\cdot), \cdots, A_n(\cdot)\Big) = P^{1,\cdots,1; m_i, i\in I_0}_{\nu_1,\cdots,\nu_d;\mu_i, i\in I_0}\Big(K_1, \cdots, K_d; A_i(\cdot), i \in I_0\Big). \tag{2}$$

Proof. Since the measures μ_1 are supported by $[a_1, b_1]$ and the measures $\mu_i, i \in I - I_1$ are supported by $[0, a_1] \cup [b_1, T]$, by applying Theorem 1 of [1], we obtain

$$P^{m_1,\cdots,m_n}_{\mu_1,\cdots,\mu_n}\Big(A_1(\cdot), \cdots, A_n(\cdot)\Big) = P^{1; m_i, i\in I-I_1}_{\nu_1;\mu_i, i\in I-I_1}\Big(K_1; A_i(\cdot), i \in I - I_1\Big).$$

Since the measures μ_2 are supported by $[a_2, b_2]$ and the measures $\nu_1, \mu_i,\ i \in I - (I_1 \cup I_2)$ are supported by $[0, a_2] \cup [b_2, T]$, by applying Theorem 1 of [1] again, we have

$$P^{m_1,\cdots,m_n}_{\mu_1,\cdots,\mu_n}\Big(A_1(\cdot), \cdots, A_n(\cdot)\Big) = P^{1,1; m_i, i\in I-(I_1\cup I_2)}_{\nu_1,\nu_2;\mu_i, i\in I-(I_1\cup I_2)}\Big(K_1, K_2; A_i(\cdot), i \in I - (I_1 \cup I_2)\Big).$$

Continuing this way through d steps, we arrive at (2). □

Corollary 1 follows immediately from the theorem just above.

Corollary 1. *Let $\mu_1, \cdots, \mu_n$ be given as in Theorem 1. Suppose $I_0 = \emptyset$, that is, $I = I_1 \cup \cdots \cup I_d$. For any nonnegative integers $m_1, \cdots, m_n$ we have*

$$P^{m_1,\cdots,m_n}_{\mu_1,\cdots,\mu_n}\Big(A_1(\cdot), \cdots, A_n(\cdot)\Big) = K_d \cdots K_1$$
$$= P^{m_i, i\in I_d}_{\mu_i, i\in I_d}\Big(A_i(\cdot), i \in I_d\Big) \cdots P^{m_i, i\in I_1}_{\mu_i, i\in I_1}\Big(A_i(\cdot), i \in I_1\Big).$$

Now we come to the theorem which allows us to iteratively disentangle a multilinear factor from $\mathcal{T}_{\mu_1,\cdots,\mu_n} f\Big(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim}\Big)$, where $f(z_1, \cdots, z_n)$ is an appropriately factorable analytic function of $z_1, \cdots, z_n$.

Theorem 2. *Let $\mu_1, \cdots, \mu_n$ and $\nu_1, \cdots, \nu_d$ be given as in Theorem 1. Assume that $g_j\Big(A_i(\cdot)^{\sim}, i \in I_j\Big) \in \mathbb{D}\Big(A_i(\cdot)^{\sim}, i \in I_j\Big)$ for $j = 1, \cdots, d$ and $h\Big(A_i(\cdot)^{\sim}, i \in I_0\Big) \in \mathbb{D}\Big(A_i(\cdot)^{\sim}, i \in I_0\Big)$. Let*

$$f(z_1, \cdots, z_n) = \left[\prod_{j=1}^{d} g_j(z_i, i \in I_j)\right] h(z_i, i \in I_0) \tag{3}$$

and let

$$K'_j := \mathcal{T}_{\mu_i, i\in I_j} g_j\Big(A_i(\cdot)^{\sim}, i \in I_j\Big)$$

for $j = 1, \cdots, d$. Then $f\Big(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim}\Big) \in \mathbb{D}\Big(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim}\Big)$ and

$$\mathcal{T}_{\mu_1,\cdots,\mu_n} f\Big(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim}\Big)$$
$$= \mathcal{T}_{\nu_1,\cdots,\nu_d;\mu_i, i\in I_0} F\Big(\tilde{K}'_1, \cdots, \tilde{K}'_d; A_i(\cdot)^{\sim}, i \in I_0\Big) \tag{4}$$

where $F\Big(w_1, \cdots, w_d; z_i, i \in I_0\Big) = w_1, \cdots, w_d h(z_i, i \in I_0)$.

Proof. Each $g_j, j = 1, \cdots, d$, belongs to the disentangling algebra $\mathbb{D}\Big(A_i\ (\cdot)^{\sim}, i \in I_j\Big)$ and h belongs to $\mathbb{D}(A_i(\cdot)^{\sim}, i \in I_0)$. Further, the union of the index sets, $I_0 \cup I_1 \cup \cdots \cup I_d$, is pairwise disjoint. Thus f, defined as the product (or elementary tensor) on the right-hand side of (3) is an element of $\mathbb{A}(r_1, \cdots, r_n)$, and so $f\Big(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim}\Big)$ belongs to $\mathbb{D}\Big(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim}\Big)$.

Since the measures $\mu_i, i \in I_1$ are supported by $[a_1, b_1]$ and the measure $\mu_i, i \in I - I_1$ are supported by $[0, a_1] \cup [b_1 T]$, by applying Theorem 3 of [1], we obtain

$$\mathcal{T}_{\mu_1,\cdots,\mu_n} f\Big(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim}\Big) = \mathcal{T}_{\nu_1;\mu_i, i\in I-I_1} F_1\Big(\tilde{K}'_1; A_i(\cdot)^{\sim}, i \in I - I_1\Big),$$

where

$$F_1(w_1; z_i, i \in I - I_1) = w_1 \left[\prod_{j=2}^{d} g_j(z_i, i \in I_j)\right] h(z_i, i \in I_0).$$

Since the measures $\mu_i, i \in I_2$ are supported by $[a_2, b_2]$ and the measures $\nu_1, \mu_i, i \in I-(I_1 \cup I_2)$ are supported by $[0, a_2] \cup [b_2, T]$, by applying Theorem 3 of [1] again, we have

$$\mathcal{T}_{\mu_1,\cdots,\mu_n} f\Big(A_1(\cdot\widetilde{)},\cdots,A_n(\cdot\widetilde{)}\Big)$$
$$= \mathcal{T}_{\nu_1,\nu_2;\mu_i,i\in I-(I_1\cup I_2)} F_2\Big(\tilde{K}_1', \tilde{K}_2'; A_i(\cdot\widetilde{)}, i \in I-(I_1\cup I_2)\Big)$$

where

$$F_2(w_1,w_2;z_i,i\in I-(I_1\cup I_2)) = w_1 w_2 \left[\prod_{j=3}^{d} g_j(z_i, i\in I_j)\right] h(z_i, i\in I_0).$$

After d steps, we arrive at the formula (4). □

Corollary 2. *Let $\mu_1,\cdots,\mu_n$ and $f(z_1,\cdots z_n)$ be given as in Theorem 2. Suppose $I = I_1\cup\cdots\cup I_d$. Then we have*

$$\mathcal{T}_{\mu_1,\cdots,\mu_n} f\Big(A_1(\cdot\widetilde{)},\cdots,A_n(\cdot\widetilde{)}\Big) = K_d'\cdots K_1'$$
$$= \mathcal{T}_{\mu_i,i\in I_d} g_d\Big(A_i(\cdot\widetilde{)}, i\in I_d\Big)\cdots\mathcal{T}_{\mu_i,i\in I_1} g_1\Big(A_i(\cdot\widetilde{)}, i\in I_1\Big).$$

Again we deal with the case of monomials first and then under the assumption that $I = I_1\cup I_1\cup\cdots\cup I_d$.

Theorem 3. *Let $a_j, b_j, j = 1,\cdots,d$ be a real numbers such that*

$$0 \le a_d \le \cdots \le a_2 \le a_1 < b_1 \le b_2 \le \cdots \le b_d \le T.$$

Suppose that $\mu_i, i\in I_j$ have supports contained within $[a_j, a_{j-1}]\cup[b_{j-1}, b_j]$ for $j = 1,\cdots,d$ where $a_0 = b_1$ and $b_0 = a_1$. Let η_j, $j = 1,\cdots,d$, be any continuous measures having supports contained within $[a_j, b_j]$. For given nonnegative integers $m_1,\cdots,m_n$, let

$$L_j = P^{1;m_i,i\in I_j}_{\eta_{j-1};\mu_i i\in I_j}\Big(L_{j-1}; A_i(\cdot), i\in I_j\Big) \tag{5}$$

for $j = 1,\cdots,d$ where L_0 is the identity operator and η_0 is any continuous probability measure having support contained within $[a_1, b_1]$. Then we have

$$P^{m_1,\cdots,m_n}_{\mu_1,\cdots,\mu_n}\Big(A_1(\cdot),\cdots,A_n(\cdot)\Big) = P^{1;m_i,i\in I_0}_{\eta_d;\mu_i,i\in I_0}\Big(L_d; A_i(\cdot), i\in I_0\Big). \tag{6}$$

Proof. Since the measures $\mu_i, i\in I_1$ are supported by $[a_1, b_1]$ and the measures $\mu_i, i\in I_2$ are supported by $[a_2, a_1]\cup[b_1, b_2]$, by applying Theorem 1 of [1], we obtain

$$P^{m_i,i\in I_1\cup I_2}_{\mu_i,i\in I_1\cup I_2}\Big(A_i(\cdot), i\in I_1\cup I_2\Big) = P^{1;m_i,i\in I_2}_{\eta_1;\mu_i,i\in I_2}\Big(L_1; A_i(\cdot), i\in I_2\Big)$$

which is the operator L_2 as defined in (5). Since the measures $\mu_i, i \in I_1 \cup I_2$ are supported by $[a_2, b_2]$ and the measures $\mu_i, i \in I_3$ are supported by $[a_3, a_2] \cup [b_2, b_3]$ by applying Theorem 1 of [1] again, we have

$$P^{m_i, i\in I_1\cup I_2\cup I_3}_{\mu_i, i\in I_1\cup I_2\cup I_3}\Big(A_i(\cdot), i \in I_1 \cup I_2 \cup I_3\Big) = P^{1; m_i, i\in I_3}_{\eta_2; \mu_i, i\in I_3}\Big(L_2; A_i(\cdot), i \in I_3\Big)$$

which is operator L_3. Continuing this way we obtain (6). □

Corollary 3. *Let $\mu_1, \cdots, \mu_n$ and $\eta_1, \cdots, \eta_d$ be given as in Theorem 3. Suppose $I = I_1 \cup I_1 \cup \cdots \cup I_d$. For any nonnegative integers $m_1, \cdots, m_n$ we have*

$$\begin{aligned} P^{m_1,\cdots,m_n}_{\mu_1,\cdots,\mu_n}\Big(A_1(\cdot), \cdots, A_n(\cdot)\Big) &= L_d \\ &= P^{1; m_i, i\in I_d}_{\eta_{d-1}; \mu_i, i\in I_d}\Big(L_{d-1}; A_i(\cdot), i \in I_d\Big) \end{aligned} \tag{7}$$

where L_{d-1} is given inductively by the formula (5). Equation (7) can be expressed more explicitly by the formula

$$\begin{aligned} &P^{m_1,\cdots,m_n}_{\mu_1,\cdots,\mu_n}\Big(A_1(\cdot), \cdots, A_n(\cdot)\Big) \\ &= P^{1; m_i, i\in I_d}_{\eta_{d-1}; \mu_i, i\in I_d}\Big(P^{1; m_i, i\in I_{d-1}}_{\eta_{d-2}; \mu_i, i\in I_{d-1}}(\cdots(P^{1; m_i, i\in I_2}_{\eta_1; \mu_i, i\in I_2}(P^{m_i, i\in I_1}_{\mu_i, i\in I_1} \\ &(A_i(\cdot), i \in I_1); A_i(\cdot), i \in I_2); \cdots); A_i(\cdot), i \in I_{d-1}); A_i(\cdot), i \in I_d\Big). \end{aligned}$$

Now we come to the second main result.

Theorem 4. *Let $\mu_1, \cdots, \mu_n$ and $\eta_1, \cdots, \eta_d$ be given as in Theorem 3 and let $f(z_1, \cdots, z_n)$ be given as in Theorem 2. For each $j = 1, \cdots, d$, let*

$$F_{j-1}(w_{j-1}; z_i, i \in I_j) = w_{j-1} g_j(z_i, i \in I_j) \tag{8}$$

and

$$L'_j = \mathcal{T}_{\eta_{j-1}; \mu_i, i\in I_j} F_{j-1}\Big(\tilde{L}'_{j-1}; A_i(\cdot)^{\sim}, i \in I_j\Big) \tag{9}$$

where $w_0 = 1$, L'_0 is the identity operator and η_0 is any continuous probability measure having support contained within $[a_1, b_1]$. Then

$$\mathcal{T}_{\mu_1,\cdots,\mu_n} f\Big(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim}\Big) = \mathcal{T}_{\eta_d; \mu_i, i\in I_0} F\Big(\tilde{L}'_d; A_i(\cdot)^{\sim}, i \in I_0\Big) \tag{10}$$

where $F(w_d; z_i, i \in I_0) = w_d h(z_i, i \in I_0)$.

Proof. Applying Theorem 3 of [1] to the function $(g_1 \otimes g_2)(z_i, i \in I_1 \cup I_2)$, we obtain

$$\mathcal{T}_{\mu_i, i\in I_1\cup I_2}\Big(g_1 \otimes g_2\Big)\Big(A_i(\cdot)^{\sim}, i \in I_1 \cup I_2\Big) = \mathcal{T}_{\eta_1; \mu_i, i\in I_2} F_1\Big(\tilde{L}'_1; A_i(\cdot)^{\sim}, i \in I_2\Big)$$

where F_1 and L'_1 are given by (8) and (9), respectively. Note that

$$L'_2 = \mathcal{T}_{\mu_i, i\in I_1\cup I_2}(g_1 \otimes g_2)\Big(A_i(\cdot)^{\sim}, i \in I_1 \cup I_2\Big)$$

and apply Theorem 3 of [1] again to the function $(g_1 \otimes g_2 \otimes g_3)(z_i, i \in I_1 \cup I_2 \cup I_3)$, Then we have

$$\mathcal{T}_{\mu_i, i \in I_1 \cup I_2 \cup I_3}(g_1 \otimes g_2 \otimes g_3)\Big(A_i(\cdot)^{\sim}, i \in I_1 \cup I_2 \cup I_3\Big)$$
$$= \mathcal{T}_{\eta_2; \mu_i, i \in I_3} F_2\Big(\tilde{L}_2'; A_i(\cdot)^{\sim}, i \in I_3\Big)$$

which is the operator L_3' as defined in (9). Continuing this way we obtain (10). □

Corollary 4. *Let* $\mu_1, \cdots, \mu_n, \eta_1, \cdots, \eta_d$ *and* $f(z_1, \cdots z_n)$ *be given as in Theorem 4. Suppose* $I = I_1 \cup \cdots \cup I_n$. *Then we have*

$$\mathcal{T}_{\mu_1, \cdots, \mu_n} f\Big(A_1(\cdot)^{\sim}, \cdots, A_n(\cdot)^{\sim}\Big) = L_d'$$
$$= \mathcal{T}_{\eta_{d-1}; \mu_i, i \in I_d} F_{d-1}\Big(\tilde{L}_{d-1}'; A_i(\cdot)^{\sim}, i \in I_d\Big)$$

where F_{d-1} *and* L_{d-1}' *are given inductively by the formulas (8) and (9) respectively.*

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