

ON THE PARAMETIC INTEREST OF THE BLACK-SCHOLES EQUATION

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ABSTRACT. We have discovered some parametrics λ in the Black-Scholes equation which depend on the interest rate r and the Volatility σ and later is named the parametic interest. On studying the parametic interest λ , we found that such λ gives the sufficient condition for the existence of solutions of the Black-Scholes equation which is either weak or strong solutions.

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1. Introduction

In financial mathematics, the famous equation named the Black-Scholes equation plays an important role in solving the option price of stocks. The Black-Scholes equation is given by

$$\frac{\partial}{\partial t}u(s, t) + rs\frac{\partial}{\partial s}u(s, t) + \frac{\sigma^2 s^2}{2} \frac{\partial^2}{\partial s^2}u(s, t) - ru(s, t) = 0 \quad (1)$$

with the terminal condition

$$u(s, T) = (s - p)^+ \quad (2)$$

for $0 \leq t \leq T$ where $u(s, t)$ is the option price at time t , r is the interest rate, s is the price of stock at time t , σ is the volatility of stock and p is the strike price.

They obtain the solution $u(s, t)$ of (1) that satisfies (2) of the form

$$u(s, t) = s\Phi\left(\frac{\ln\left(\frac{s}{p}\right) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}\right) - pe^{-r(T-t)}\Phi\left(\frac{\ln\left(\frac{s}{p}\right) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}\right) \quad (3)$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$, see [1, p. 91].

In this work, we study the solution of (1) in the other form. By changing the variable $R = \ln s$, $s \geq 1$. Then (1) is transformed to

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial R^2} + \left(r - \frac{\sigma^2}{2}\right) \frac{\partial V}{\partial R} - rV = 0 \quad (4)$$

where $V = V(R, t) = u(s, t)$.

The method of separation of variable and the Laplace transform are applied to solve for the solutions of (4). We obtained the such solution in the form

$$u(s, t, \lambda) = e^{\lambda t} \mathcal{L}^{-\infty}(\xi^\alpha) \quad (5)$$

where λ is the parametric interest, $\mathcal{L}^{-\infty}(\xi^\alpha)$ is the Inverse Laplace transform and

$$\alpha = \frac{-(3\sigma^2 - 2r) \pm \sqrt{(\sigma^2 + 2r)^2 - 8\sigma^2\lambda}}{2\sigma^2} \quad (6)$$

Next, we consider α as the following cases.

(i) Suppose $\alpha = m$ where m is some nonnegative integer, we obtained the solution in (5) as

$$u(s, t, \lambda) = e^{\lambda t} \delta^{(m)}(s) \quad (7)$$

where

$$\lambda = \lambda(r, \sigma) = (m+2)r - \frac{(m^2 + 3m + 2)}{2} \sigma^2 \quad (8)$$

is the parametric interest and $\delta^{(m)}(s)$ is the Dirac-delta distribution with m -derivative and $\delta^{(0)} = \delta$.

(ii) Suppose α is a negative real number, that is $\alpha < 0$, we obtained the solution in (5) as

$$u(s, t, \lambda) = \frac{e^{\lambda t} s^{-\alpha-1}}{\Gamma(-\alpha)} \quad (9)$$

where $\Gamma(-\alpha)$ is Gamma function of $-\alpha$.

In particular, if $\alpha = -n$ for some positive numbers n , then (9) reduces to

$$u(s, t, \lambda) = \frac{e^{\lambda t} s^{n-1}}{(n-1)!} \quad (10)$$

where

$$\lambda = \lambda(r, \sigma) = -\frac{\sigma^2}{2} n^2 + \left(\frac{3\sigma^2}{2} - r\right)n + 2r - \sigma^2. \quad (11)$$

If we compare the solution in (3) with the solution of (5), we see that the solution in (5) has given more details about the type of solutions which are both weak and strong solutions. The solution (7) is the weak solution of the Dirac-delta distribution and the solution of (9) and (10) are strong solutions or classical solution. We see that the solution in (7) which is the option price $u(s, t, \lambda)$ will not appear in the real world subject to the condition (8) of the parametric

interest. The solution in (9) or (10) is the strong solution happen in the real world subject to the parametric interest of (11). But the solution $u(s, t, \lambda)$ in (5) contains both weak solution of the form (7) and the strong solution of the form (9). Thus we can say that the solution (5) is much more general than (3).

2. Preliminaries

Before reaching the main results, the following definitions and the basic concepts are needed.

Definition 1. Given f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A and if there exists the real constant K , a and M such that $|f(t)| \leq Ke^{at}$ for $t \geq M$. Then the Laplace transform of $f(t)$, denoted by $\mathcal{L}f(t)$ is defined by

$$\mathcal{L}(t) = F(\xi) = \int_0^\infty e^{-\xi t} f(t) dt. \quad (12)$$

Lemma 1. Let $F(\xi)$ be the Laplace transform of $f(t)$. Then

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\xi) e^{\xi t} d\xi \quad (13)$$

is the Inverse Laplace transform of $F(\xi)$, denoted by

$$\mathcal{L}^{-1}F(\xi) = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\xi) e^{\xi t} d\xi$$

Proof. See [2, p. 216]. □

Lemma 2.

(i) $\mathcal{L}\delta(t) = 1$

(ii) $\mathcal{L}\delta^{(k)}(t) = \xi^k$ where $\delta^{(k)}$ is the Dirac-delta distribution with k -derivative and $\xi > 0$

(iii) $\mathcal{L}(t^p) = \frac{\Gamma(p+1)}{\xi^{p+1}}$ for $p > -1$ and $\xi > 0$. If p is a positive number n , then

$$\mathcal{L}(t^n) = \frac{n!}{\xi^{n+1}}, \quad \xi > 0$$

(iv) $\mathcal{L}[t^k f(t)] = (-1)^k F^{(k)}(\xi)$

(v) $\mathcal{L}[f^{(k)}(t)] = \xi^k F(\xi)$

Proof. See [2, p. 227-228]. □

3. Main results

Theorem 1. *Given the Black-Scholes equation*

$$\frac{\partial}{\partial t}u(s, t) + rs\frac{\partial}{\partial s}u(s, t) + \frac{\sigma^2 s^2}{2}\frac{\partial^2}{\partial s^2}u(s, t) - ru(s, t) = 0 \quad (14)$$

where $u(s, t)$ is the option price at time t with $0 \leq t \leq T$, s is the price of stock at time t , r is the interest rate and σ is the volatility of stock. Then we obtain

$$u(s, t, \lambda) = e^{\lambda t} \mathfrak{L}^{-1}(\xi^\alpha) \quad (15)$$

as the solution of (14) where λ is some parameters, $\mathfrak{L}^{-1}(\xi^\alpha)$ is the Inverse Laplace transform of ξ^α with

$$\alpha = \frac{(2r - 3\sigma^2) \pm \sqrt{(\sigma^2 + 2r)^2 - 8\sigma^2\lambda}}{2\sigma^2}. \quad (16)$$

In particular, if $\alpha = m$ where m is nonnegative integer, then (15) become $u(s, t, \lambda) = e^{\lambda t} \delta^{(m)}(s)$, $\delta^{(m)}$ is the Dirac-delta distribution with m -derivatives and $\delta^{(0)} = \delta$ and $\lambda = (m+2)r - \frac{(m^2 + 3m + 2)}{2}\sigma^2$ by (16). If α is negative real number, that is $\alpha < 0$, then (15) become

$$u(s, t, \lambda) = \frac{e^{\lambda t} s^{-\alpha-1}}{\Gamma(-\alpha)} \quad (17)$$

where $\Gamma(-\alpha)$ is the Gamma function.

In particular, if α is negative integer and suppose $\alpha = -n$, then (17) reduces to

$$u(s, t, \lambda) = \frac{e^{\lambda t} s^{n-1}}{\Gamma(n)} = \frac{e^{\lambda t} s^{n-1}}{(n-1)!} \quad (18)$$

and $\lambda = -\frac{\sigma^2}{2}n^2 + (\frac{3\sigma^2}{2} - r)n + 2r - \sigma^2$ by (16).

Proof. By changing the variable $R = \ln s$, $s \geq 1$. Then (14) is transformed to

$$\frac{\partial}{\partial t}V(R, t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial R^2}V(R, t) + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial}{\partial R}V(R, t) - rV(R, t) = 0 \quad (19)$$

where $V(R, t) = u(s, t)$. By method of separation of variable, let $V(R, t) = X(R)U(t)$, $\frac{\partial V}{\partial t} = X(R)U'(t)$, $\frac{\partial V}{\partial R} = X'(R)U(t)$ and $\frac{\partial^2 V}{\partial R^2} = X''(R)U(t)$. Substitute into (19),

$$X(R)U'(t) + \frac{\sigma^2}{2}X''(R)U(t) + \left(r - \frac{\sigma^2}{2}\right)X'(R)U(t) - rX(R)U(t) = 0$$

or

$$\frac{U'(t)}{U(t)} + \frac{\sigma^2}{2}\frac{X''(R)}{X(R)} + \left(r - \frac{\sigma^2}{2}\right)\frac{X'(R)}{X(R)} - r = 0.$$

Let

$$\frac{U'(t)}{U(t)} = -\frac{\sigma^2}{2} \frac{X''(R)}{X(R)} - \left(r - \frac{\sigma^2}{2}\right) \frac{X'(R)}{X(R)} + r = \lambda,$$

where λ is a parameter.

Now $\frac{U'(t)}{U(t)} = \lambda$, thus $U(t) = ce^{\lambda t}$. For simplicity, choose $c = 1$, thus $U(t) = e^{\lambda t}$. Now

$$\frac{\sigma^2}{2} \frac{X''(R)}{X(R)} + \left(r - \frac{\sigma^2}{2}\right) \frac{X'(R)}{X(R)} - r + \lambda = 0$$

or

$$\sigma^2 X''(R) + (2r - \sigma^2)X'(R) - (2r - 2\lambda)X(R) = 0.$$

Put $R = \ln s$ and let $X(R) = X(\ln s) = Y(s)$. Then

$$X'(R) = s \frac{dY(s)}{ds} \text{ and } X''(R) = s^2 \frac{d^2Y(s)}{ds^2} + s \frac{dY(s)}{ds}.$$

Thus

$$\sigma^2 \left[s^2 \frac{d^2Y(s)}{ds^2} + s \frac{dY(s)}{ds} \right] + (2r - \sigma^2)s \frac{dY(s)}{ds} - (2r - 2\lambda)Y(s) = 0$$

or

$$\sigma^2 s^2 Y''(s) + 2rsY'(s) - (2r - 2\lambda)Y(s) = 0. \quad (20)$$

The equation (20) is the Euler's equation of order 2. Now take the Laplace transform to (20) and use (iv) and (v) of Lemma 2, see [3].

$$\sigma^2 \frac{d^2}{ds^2} [\xi^2 Y(\xi)] + (-1)2r \frac{d}{ds} [\xi Y(\xi)] - (2r - 2\lambda)Y(\xi) = 0$$

or

$$\sigma^2 \xi^2 Y''(\xi) + (4\sigma^2 - 2r)\xi Y'(\xi) + (2\sigma^2 - 4r + 2\lambda)Y(\xi) = 0.$$

That is the Euler's equation.

Let $Y(\xi) = \xi^\alpha$ and substitute into such equation,

$$[\sigma^2 \alpha(\alpha - 1) + (4\sigma^2 - 2r)\alpha + (2\sigma^2 - 4r + 2\lambda)]\xi^\alpha = 0.$$

Thus

$$\sigma^2 \alpha(\alpha - 1) + (4\sigma^2 - 2r)\alpha + (2\sigma^2 - 4r + 2\lambda) = 0$$

or

$$\sigma^2 \alpha^2 + (3\sigma^2 - 2r)\alpha + (2\sigma^2 - 4r + 2\lambda) = 0. \quad (21)$$

Thus

$$\alpha = \frac{(2r - 3\sigma^2) \pm \sqrt{(\sigma^2 + 2r)^2 - 8\sigma^2 \lambda}}{2\sigma^2}. \quad (22)$$

We obtain the solution $Y(\xi) = \xi^\alpha$ or $Y(s) = \mathfrak{L}^{-1}Y(\xi) = \mathfrak{L}^{-1}(\xi^\alpha)$.

Now $V(R, t) = X(R)U(t)$ or $u(s, t) = Y(s)U(t) = e^{\lambda t} \mathfrak{L}^{-1}(\xi^\alpha)$. Thus the option price

$$u(s, t, \lambda) = e^{\lambda t} \mathfrak{L}^{-1}(\xi^\alpha). \quad (23)$$

Now we consider, from (22), the case α is real root. That is $\lambda \leq \frac{(\sigma^2 + 2r)^2}{8\sigma^2}$ in (22).

(i) If $\alpha < 0$, then from (23) and (iii) of Lemma 2, we obtain

$$u(s, t, \lambda) = \frac{e^{\lambda t} s^{-\alpha-1}}{\Gamma(-\alpha)} \quad (24)$$

which is a strong or classical solution.

(ii) If $\alpha \geq 0$ and α is some integer m . By applying Lemma 2, (i) and (ii) to (22), we obtain

$$u(s, t, \lambda) = e^{\lambda t} \delta^{(m)}(s) \quad (25)$$

where $\delta^{(m)}(s)$ is the Dirac-delta distribution of m -derivatives and $\delta^{(0)} = \delta$, thus for $\alpha = m$ in (21), we obtain

$$\lambda = (m+2)r - \frac{(m^2 + 3m + 2)}{2}\sigma^2, \quad m = 0, 1, 2, \dots \quad (26)$$

We see that the solution $u(s, t, \lambda)$ in (25) and the value of λ in (26) is a weak solution. We can not compute in classical solution or ordinary solution. Thus the option price $u(s, t, \lambda)$ in (25) will not appear subject to the condition of λ in (26). \square

Corollary 1. *The solution $u(s, t, \lambda)$ in (17) of the Theorem reduces to*

$$u(s, t, \lambda) = \frac{e^{\lambda t} s^{n-1}}{(n-1)!} \quad (27)$$

for $\alpha = -n$ where n is some positive integer and

$$\lambda = -\frac{\sigma^2}{2}n^2 + \left(\frac{3\sigma^2}{2} - r\right)n + 2r - \sigma^2. \quad (28)$$

Proof. Put $\alpha = -n$ in (23), we obtain

$$u(s, t, \lambda) = \frac{e^{\lambda t} s^{n-1}}{\Gamma(n)} = \frac{e^{\lambda t} s^{n-1}}{(n-1)!}$$

and from (21) with $\alpha = -n$, we also obtain the parametric interest

$$\lambda = -\frac{\sigma^2}{2}n^2 + \left(\frac{3\sigma^2}{2} - r\right)n + 2r - \sigma^2.$$

\square

4. Conclusion

The solution $u(s, t, \lambda) = e^{\lambda t} \delta^{(m)}(s)$ where $\lambda = (m+2)r - \frac{(m^2 + 3m + 2)}{2}\sigma^2$ for some nonnegative integer m is the weak solution of Delta-distribution. The option price $u(s, t, \lambda)$ disappear subject to the condition of such λ .

The solution $u(s, t, \lambda) = \frac{e^{\lambda t} s^{n-1}}{(n-1)!}$ where $\lambda = -\frac{\sigma^2}{2}n^2 + (\frac{3\sigma^2}{2} - r)n + 2r - \sigma^2$ for some positive integer n is the strong solution or classical solution of the price of stock s .

In particular if $n = 1$, then $\lambda = r$. Thus we obtained $u(s, t, r) = \frac{e^{rt} s^0}{0!} = e^{rt}$. This means the option price $u(s, t, \lambda)$ is equal to the value of 1 dollar put in the Bank with the interest rate r at the time t . If $n = 2$, then $\lambda = 0$, we obtained $u(s, t, 0) = \frac{e^{0t} s^{2-1}}{(2-1)!} = s$. This means the option price $u(s, t, \lambda)$ is equal to the price s of the stock at any time t .

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