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NOTE ON CONNECTED (g, f)-FACTORS OF GRAPHS

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ABSTRACT. In this note we present a short proof of the following result by Zhou, Liu and Xu. Let G be a graph of order n, and let a and b be two integers with $1 \leq a < b$ and $b \geq 3$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$ and $f(V(G)) - V(G)$ even. If $n \geq \frac{(a+b-1)^2+1}{a}$ and $\delta(G) \geq \frac{(b-1)n}{a+b-1}$, then G has a connected (g, f)-factor.

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1. Introduction

All graphs considered in this paper will be finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex x is denoted by $d_G(x)$. Set $\delta(G) = \min\{d_G(x)|x \in V(G)\}$, the minimum degree of G. For any $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. Then a (g, f)-factor of G is a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. If F is connected, we call it a connected (g, f)-factor. For convenience, we write $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$, $f(S) = \sum_{x \in S} f(x)$ and $g(T) = \sum_{x \in T} g(x)$.

Many authors have investigated factors [1–5], connected factors [6–8], and factorizations [9,10]. Zhou, Liu and Xu gave the result about connected (g, f)-factors by the minimum degree and the order of a graph G [11]. In this note we present a short proof of the following result [11].

Theorem 1. *Let G be a graph of order n, and let a and b be two integers with $1 \leq a < b$ and $b \geq 3$. Let g and f be two integer-valued functions defined on*

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$V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$ and $f(V(G)) - V(G)$ even. If

$$n \geq \frac{(a+b-1)^2+1}{a}$$

and

$$\delta(G) \geq \frac{(b-1)n}{a+b-1},$$

then G has a connected (g,f)-factor.

The short proof of Theorem 1 relies heavily on the following results.

Theorem 2. [12] *Let G be a graph of order $n \geq 3$. If the minimum degree of G is at least $\frac{n}{2}$, then G has a Hamiltonian cycle.*

Theorem 3. [13] *Let G be a graph of order n and let a and b be integers with $1 \leq a \leq b$. Let h be an integer-valued function defined on $V(G)$ such that $a \leq h(x) \leq b$ for each $x \in V(G)$ and $h(V(G)) \equiv 0$ (mod 2). If*

$$n > \frac{(a+b)(a+b-3)}{a}$$

and

$$\delta(G) \geq \frac{bn}{a+b},$$

then G has an h-factor.

Theorem 4. [8] *Let G be a graph, and let g and f be two positive integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x) \leq d_G(x)$ for each $x \in V(G)$. If G has both a (g,f)-factor and a Hamiltonian path, then G contains a connected $(g, f+1)$-factor.*

2. The proof of theorem 1

We now prove Theorem 1. Let G be a graph which satisfies the conditions of Theorem 1, and $\delta(G)$ the minimum degree of G. By $n \geq \frac{(a+b-1)^2+1}{a} > 3$, $\delta(G) \geq \frac{(b-1)n}{a+b-1} \geq \frac{n}{2}$ and by Theorem 2, G has a Hamiltonian cycle.

Define a function $h : V(G) \to Z$ as $h(x) = g(x)$ for any $x \in V(G)$ if $g(V(G)) \equiv 0$ (mod 2); otherwise $h(x) = g(x)$ for any $x \in V(G) \setminus \{v\}$ and $h(v) = g(v) + 1$, where v is any vertex in $V(G)$ with $g(v) < f(v) - 1$. Note that such a vertex v exists because if $g(x) = f(x) - 1$ for any $x \in V(G)$, then $1 \equiv g(V(G)) = f(V(G)) - |V(G)|$ (mod 2), which contradicts the assumption of this theorem.

Then G has an h-factor since G satisfies all the conditions of Theorem 3. In fact, $a \leq h(x) \leq b-1$ for each $x \in V(G)$, $h(V(G)) \equiv 0$ (mod 2),

$$n \geq \frac{(a+b-1)^2+1}{a} > \frac{(a+(b-1))(a+(b-1)-3)}{a},$$

and

$$\delta(G) \geq \frac{(b-1)n}{a+b-1}.$$

Since G has both a Hamiltonian cycle and an h-factor, i.e. a $(g, f-1)$-factor, by Theorem 4, G has a connected (g, f)-factor. This completes the proof of Theorem 1. $\square$

Remark. Let us show that the condition $\delta(G) \geq \frac{(b-1)n}{a+b-1}$ in Theorem 1 cannot be replaced by $\delta(G) \geq \frac{(b-1)n}{a+b-1} - 1$. Let $a = b-1 \geq 1, t \geq 2$ be three integers, $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, and $G = K_{at+1} \cup K_{(b-1)t+1}$. Then we have $n = |V(G)| = (at+1) + ((b-1)t+1) = 2at+2 > \frac{(a+b-1)^2+1}{a}$, $f(V(G)) - |V(G)|$ even and $\delta(G) = at = \frac{n-2}{2} = \frac{n}{2} - 1 = \frac{(b-1)n}{a+b-1} - 1$. Obviously, G satisfies all the conditions of Theorem 1 excepting that $\delta(G) \geq \frac{(b-1)n}{a+b-1}$ and G is a disconnected graph. Thus, G hasn't a connected (g, f)-factor. In the above sense, the result in Theorem 1 is best possible.

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References

1. J. R. Correa and M. Matamala, *Some remarks about factors of graphs*, Journal of Graph Theory **57**(2008), 265–274.
2. S. Zhou, *Independence number, Connectivity and (a, b, k)-critical graphs*, Discrete Mathematics **309(12)**(2009), 4144–4148.
3. S. Zhou and J. Jiang, *Notes on the binding numbers for (a, b, k)-critical graphs*, Bulletin of the Australian Mathematical Society **76(2)**(2007), 307–314.
4. S. Zhou and Y. Xu, *Neighborhoods of independent sets for (a, b, k)-critical graphs*, Bulletin of the Australian Mathematical Society **77(2)**(2008), 277–283.
5. S. Zhou and X. Xue, *(g, f)-Factors of graphs with prescribed properties*, Journal of Systems Science and Mathematical Sciences **28(2)** (2008), 193–196.
6. H. Matsuda, *On 2-edge-connected $[a, b]$-factors of graphs with Ore-type condition*, Discrete Mathematics **296**(2005), 225–234.
7. M. Kouider and P. D. Vestergaard, *Connected factors in graphs - a survey*, Graphs and Combinatorics **21**(2005), 1–26.
8. G. Li, Y. Xu, C. Chen and Z. Liu, *On connected $(g, f+1)$-factors in graphs*, Combinatorica **25(4)**(2005), 393–405.
9. G. Liu, B. Zhu, *Some problems on factorizations with constraints in bipartite graphs*, Discrete Applied Mathematics **128**(2003), 421–434.
10. S. Zhou and M. Zong, *Randomly orthogonal factorizations of $(0, mf-(m-1)r)$-graphs*, Journal of the Korean Mathematical Society **45(6)**(2008), 1613–1622.
11. S. Zhou, H. Liu and Y. Xu, *A degree condition for graphs to have connected (g, f)-factors*, Bulletin of the Iranian Mathematical Society **35(1)**(2009), 199–209.
12. G. A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. **2**(1952), 69–81.
13. P. Katerinis and N. Tsikopoulos, *Minimum degree and F-factors in graphs*, New Zealand J. Math. **29**(2000), 33–40.

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