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EXISTENCE OF SPANNING 3-TREES IN A 3-CONNECTED LOCALLY FINITE VAP-FREE PLANE GRAPH

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ABSTRACT. In this paper we prove the existence of spanning 3-trees in a 3-connected infinite locally finite VAP-free plane graph. Together with the results of Barnette and the author, this yields that every finite or infinite 3-connected locally finite VAP-free plane graph contains a spanning 3-tree.

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1. Introduction

Notation and terminology not defined in this paper may be found in [3] or [9]. A *spanning subgraph* H of a graph G is a subgraph which contains all vertices of G. If a spanning subgraph T of G is a tree, then we say that T is a *spanning tree* in G. For a positive integer k, a spanning tree T is a *k-tree*, if $d_T(x) \leq k$ for all $x \in V(T)$.

Many problems in graph theory have quite simple solutions in the finite case whereas in the infinite case the solution may be extremely complicated or the problem may even remain a conjecture. Such a problem is often solved by finding a way to decompose the whole graph into smaller fragments that preserve some specific properties of the original graph and are such that a solution of the problem for the fragments gives rise to a solution for the whole graph (for example, see [2] or [4]).

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In this article, we study the existence of a spanning 3-tree in 3-connected plane graphs. In finite case, as a classical result, Barnette [1] made the following remark.

Theorem A (Barnette). *Every circuit graph contains a spanning 3-tree.*

It may be noted that a circuit graph G is a 2-connected plane graph such that $G \cup (\partial G \times \{v\})$ is 3-connected for a further vertex v; or equivalently, for every vertex cut S of G with $|S| = 2$, every component of $G - S$ contains a vertex of ∂G. (Further equivalent forms for such a graph can be found in [5] or [7].)

Barnette's theorem was slightly improved by Jung [6] by showing the following result: For a circuit graph G and for arbitrary given $u, v \in \partial G$ (or $u, v, w \in \partial G$), there exists a spanning 3-tree T with $d_T(u) = 1$ and $d_T(v) \leq 2$ (or $d_T(u) \leq 2$, $d_T(v) \leq 2$ and $d_T(w) \leq 2$, respectively). Using these results, Jung [6] extended the theorem of Barnette into the 3LV-graphs. As introduced in [9], it may be noted that a 3LV-graph is a 3-connected infinite, locally finite plane graph which contains no vertex-accumulation point (=VAP) and no unbounded faces.

Theorem B (Jung). *In every 3LV-graph there exists a spanning 3-tree.*

To extend Theorem B to general 3-connected locally finite VAP-free plane graphs, it is necessary to show the existence of such a tree in LV-graphs. From the point of view we in this paper prove that in every LV-graph one can find a spanning 3-tree; namely

Theorem C. *Every LV-graph contains a spanning 3-tree.*

Let G be a 3-connected locally finite VAP-free plane graph. If G is finite, then we have a spanning 3-tree in G by Theorem A, since G is in particular a circuit graph. If G is infinite, then G is either an LV-graph or a 3LV-graph; i.e., G either contains an unbounded face or does not contain such a face, respectively. For the former case, the existence of a spanning 3-tree in G follows from Theorem C; on the other hand, for the latter case we can also obtain a 3-tree in such a graph by Theorem B. Thus we proved the following main result.

Corollary. *In every 3-connected locally finite VAP-free plane graph there exists a spanning 3-tree.*

2. Terminology and preliminaries

In order that the present paper be more self-contained, we include some terminology concerning the structure of LV-graphs (following Jung [9]).

Let G be an infinite connected plane graph. A finite set of unbounded separating paths $\mathcal{P} = \{P_1, \ldots, P_n\}$ in G will be called a *semicycle* if there exist connected subgraphs $G_0, G_1, \ldots, G_n$ of G such that

[S1] $G = \bigcup_{i=0}^{n} G_i, \quad G_0 \cap G_i = P_i$ for all $i \in \{1, \ldots, n\}$

and $G_i \cap G_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, and

[S2] G_0 is finite, but G_i $(i = 1, \ldots, n)$ are infinite.

In this case, the finite subgraph G_0 of G is called the *center* of the semicycle $\mathcal{P}$, which will be denoted by $C(\mathcal{P})$. A semicycle $\mathcal{P}$ is *induced* if all paths in $\mathcal{P}$ are induced. Two semicycles $\mathcal{P}$ and $\mathcal{P}'$ are *disjoint* if $V(\mathcal{P}) \cap V(\mathcal{P}') = \emptyset$; for convenience, the set of vertices $V(\mathcal{P})$ (respectively, the set of edges $E(\mathcal{P})$) of $\mathcal{P}$ will be understood to be the union of all vertices (respectively, edges) of the paths in $\mathcal{P}$.

Let $\mathcal{P}$ and $\mathcal{P}'$ be disjoint semicycles with $\mathcal{P} \subseteq C(\mathcal{P}')$ in a connected plane graph G. A *$(\mathcal{P}, \mathcal{P}')$-semiring* in G is a subgraph of G consisting of not only the cycles in $\mathcal{P}$ and $\mathcal{P}'$ but also all vertices and edges lying between $\mathcal{P}$ and $\mathcal{P}'$. *Bridges* of a $(\mathcal{P}, \mathcal{P}')$-semiring $\mathcal{R}$ are defined by the bridges of $\mathcal{P} \cup \mathcal{P}'$ in $\mathcal{R}$. For $k \in \{0, 1, 2, \ldots\}$, a bridge B of $\mathcal{R}$ is *of type k* if $|V(B) \cap V(\mathcal{P}')| = k$.

A $(\mathcal{P}, \mathcal{P}')$-semiring $\mathcal{R}$ is said to be *tight* if it satisfies following conditions:

[T1] $\mathcal{P}$ and $\mathcal{P}'$ are induced.

[T2] For each infinite component H of $G - C(\mathcal{P})$, there exists exactly one path P in $\mathcal{P}'$ such that the endvertices of P are adjacent to the endvertices of the foot of H.

[T3] Each bridge of $\mathcal{R}$ is of type ≤ 2.

[T4] If B is a bridge of type 2, then the two vertices of B of attachment on $\mathcal{P}'$ are adjacent in G.

Recall that an LV-graph is an infinite locally finite 3-connected VAP-free plane graph containing unbounded faces. Jung gave a so-called 'structure theorem' for LV-graphs as follows: Let G be an LV-graph and let $\mathcal{P}_0$ be an induced semicycle in G. Then there exists an infinite sequence of pairwise disjoint induced semicycles $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots)$ such that

(1) $\mathcal{P}_j \subseteq C(\mathcal{P}_{j+1})$ for all $j \in \{0, 1, 2, \ldots\}$,

(2) $(\mathcal{P}_j, \mathcal{P}_{j+1})$-semiring is tight, for all $j \in \{0, 1, 2, \ldots\}$, and

(3) $G = \bigcup_{j=0}^{\infty} C(\mathcal{P}_j)$.

In [9], Jung gave some important results for solving the problem in this paper, as a preparatory work. In order that the present paper be more self-contained, we include the results from his substantial paper.

(2.1) *Let B be a bridge of type 0 of a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph, and let x_0 and $\overline{x}$ be the first and the last vertex of attachment of B on $\mathcal{P}$, respectively. Then there exists a spanning 3-forest F in B such that:*

(1) *Each component of F contains exactly one vertex of attachment of B on $\mathcal{P}$,*

(2) *$d_F(x) \leq 1$ for each vertex x of attachment of B on $\mathcal{P}$, and*

(3) *$d_F(x_0) = d_F(\overline{x}) = 0$.*

(2.2.1) *Let B be a bridge of type 1 of a tight $(\mathcal{P},\mathcal{P}')$-semiring in an LV-graph, and let x_0 be the first vertex of attachment of B on $\mathcal{P}$. Let further $V(B)\cap V(\mathcal{P}')=\{y\}$. Then there exists a spanning 3-forest F in B such that:*

(1) *Each component of F contains exactly one vertex of attachment of B on $\mathcal{P}\cup\mathcal{P}'$,*
(2) *$d_F(x_0)=0$ and $d_F(x)\leq 1$ for each vertex x of attachment of B on $\mathcal{P}$, and*
(3) *$d_F(y)=0$.*

(2.2.2) *Let B, x_0 and y as in the Proposition (2.2.1) be given. Further let $\overline{x}$ be the last vertex of attachment of B on $\mathcal{P}$. Then there exists a spanning 3-forest F in B such that:*

(1) *One component of F contains the vertices $\overline{x}$ and y, and each of the remaining components of F contains exactly one vertex of attachment of $\mathcal{P}$.*
(2) *$d_F(x_0)=0$, $d_F(\overline{x})=1$ and $d_F(x)\leq 1$ for each vertex x of attachment of B on $\mathcal{P}$,*
(3) *$d_F(y)=0$.*

(2.3.1) *Let B be a bridge of type 2 of a tight $(\mathcal{P},\mathcal{P}')$-semiring in an LV-graph. Let x_0 be the first vertex of attachment of B on $\mathcal{P}$ and let $\{y_1,y_2\}=V(B)\cap V(\mathcal{P}')$. Then there exists a spanning 3-forest F in B such that:*

(1) *A component T of F contains both y_1 and y_2, but it does not contain a vertex of attachment of B on $\mathcal{P}$.*
(2) *Each component of $F-T$ contains exactly one vertex of attachment of B on $\mathcal{P}$,*
(3) *$d_F(x_0)=0$ and $d_F(x)\leq 1$ for each vertex x of attachment of B on $\mathcal{P}$, and*
(4) *$d_F(y_1)=d_F(y_2)=1$.*

(2.3.2) *Let B, x_0, y_1 and y_2 as in the Proposition (2.3.1) be given. Further assume $|V(B)\cap V(\mathcal{P})|\geq 2$. Then there exists a spanning 3-forest F in B such that:*

(1) *A component T of F contains y_2, but it does not contain a vertex of attachment of B on $\mathcal{P}$.*
(2) *Each component of $F-T$ contains exactly one vertex of attachment of B on $\mathcal{P}$, and moreover one of them contains the vertex y_1.*
(3) *$d_F(x_0)=0$ and $d_F(x)\leq 1$ for each vertex x of attachment of B on $\mathcal{P}$, and*
(4) *$d_F(y_1)=1$ and $d_F(y_2)\leq 1$.*

(2.3.3) *Let B, x_0, y_1 and y_2 as in the Proposition (2.3.1) be given. Further assume $V(B) \cap V(\mathcal{P}) = \{x_0\}$. Then there exists a spanning 3-forest F in B which contains exactly two components T_1 and T_2, such that:*

(1) *$V(T_1) = \{y_1\}$ and $x_0, y_2 \in V(T_2)$.*
(2) *$d_F(x_0) = d_F(y_2) = 1$.*

3. Spanning 3-forests in a bridge

In this section we give several properties concerning the existence of a spanning 3-forest in a bridge of a semiring, which are similar to those described in the preceding section. The object here is to construct a spanning 3-forest satisfying certain desired conditions. The proofs are essentially the same as those of [9], though the required conditions are partially changed, and therefore we only present the main results without proofs.

(3.1.1) *Let B be a bridge of type 1 of a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph, and let x_0 $\overline{x}$ be the first and the last vertex of attachment of B on $\mathcal{P}$, respectively. Let further $V(B) \cap V(\mathcal{P}') = \{y\}$. Then there exist a spanning 3-forest F in B and a component T of F such that:*

(1) *$V(T) \cap V(\mathcal{P} \cup \mathcal{P}') = \{x_0, \overline{x}\}$, and each of the remaining components of F contains exactly one vertex of attachment of $(\mathcal{P} \cup \mathcal{P}') - \{x_0, \overline{x}\}$.*
(2) *$d_F(x_0) = 0$, $d_F(\overline{x}) = 1$ and $d_F(x) \leq 1$ for each vertex x of attachment of B on $\mathcal{P}$,*
(3) *$d_F(y) = 0$.*

(3.1.2) *Let B, x_0, $\overline{x}$ and y as in the Proposition (3.1.1) be given. Then there exist a spanning 3-forest F in B and a component T of F such that:*

(1) *$V(T) \cap V(\mathcal{P} \cup \mathcal{P}') = \{x_0, \overline{x}, y\}$, and each of the remaining components of F contains exactly one vertex of attachment of $(\mathcal{P} \cup \mathcal{P}') - \{x_0, \overline{x}, y\}$.*
(2) *$d_F(x_0) = d_F(\overline{x}) = 1$ and $d_F(x) \leq 1$ for each vertex x of attachment of B on $\mathcal{P}$,*
(3) *$d_F(y) = 1$.*

(3.2.1) *Let B be a bridge of type 2 of a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph. Let further $|V(B) \cap V(\mathcal{P})| \geq 2$ and $V(B) \cap V(\mathcal{P}') = \{y_1, y_2\}$. Finally set x_0 and $\overline{x}$ the first and the last vertex of attachment of B on $\mathcal{P}$, respectively. Then there exist a spanning 3-forest F in B and two components T_1 and T_2 of F such that:*

(1) *$V(T_1) \cap V(\mathcal{P} \cup \mathcal{P}') = \{x_0, \overline{x}\}$ and $V(T_2) \cap V(\mathcal{P} \cup \mathcal{P}') = \{y_1, y_2\}$.*
(2) *Each of the remaining components of F contains exactly one vertex of attachment of $(\mathcal{P}) - \{x_0, \overline{x}\}$.*
(3) *$d_F(x_0) = d_F(\overline{x}) = 1$ and $d_F(x) \leq 1$ for each vertex x of attachment of B on $\mathcal{P}$,*

(4) $d_F(y_1) = d_F(y_2) = 1$.

(3.2.2) *Let B, x_0, $\overline{x}$, y_1 and y_2 as in the Proposition (3.2.1) be given, and let further $|V(B) \cap V(\mathcal{P})| \geq 2$. Then there exist a spanning 3-forest F in B and two components T_1 and T_2 of F such that:*

(1) $V(T_1) \cap V(\mathcal{P} \cup \mathcal{P}') = \{x_0, \overline{x}, y_1\}$ *and* $V(T_2) \cap V(\mathcal{P} \cup \mathcal{P}') = \{y_2\}$.
(2) *Each of the remaining components of F contains exactly one vertex of attachment of* $(\mathcal{P}) - \{x_0, \overline{x}\}$.
(3) $d_F(x_0) = d_F(\overline{x}) = 1$ *and* $d_F(x) \leq 1$ *for each vertex x of attachment of B on $\mathcal{P}$,*
(4) $d_F(y_1) = 1$ *and* $d_F(y_2) \leq 1$.

(3.3) *Let B be a bridge of type 0 of a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph, and let x_0 and $\overline{x}$ be the first and the last vertex of attachment of B on $\mathcal{P}$, respectively. Then there exists a spanning 3-forest F in B and a component T of F such that:*

(1) $V(T) \cap V(\mathcal{P} \cup \mathcal{P}') = \{x_0, \overline{x}\}$, *and each of the remaining components of F contains exactly one vertex of attachment of* $(\mathcal{P} \cup \mathcal{P}') - \{x_0, \overline{x}\}$.
(2) $d_F(x_0) = d_F(\overline{x}) = 1$ $d_F(x) \leq 1$ *for each vertex x of attachment of B on $\mathcal{P}$.*

(3.4.1) *Let B be a bridge of type 2 of a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph. Let further x_0 and $\overline{x}$ be the first and the last vertex of attachment of B on $\mathcal{P}$, and set $\{y_1, y_2\} = V(B) \cap V(\mathcal{P}')$. Then there exists a spanning 3-forest F in B such that:*

(1) *A component T of F contains both $\overline{x}$ and y_2, but it does not contain a vertex of attachment of B on* $\mathcal{P} \cup \mathcal{P}' - \{\overline{x}, y_2\}$.
(2) *Each component of $F - T$ contains exactly one vertex of attachment of B on $\mathcal{P} \cup \mathcal{P}'$,*
(3) $d_F(x_0) = 0$ *and* $d_F(x) \leq 1$ *for each vertex x of attachment of B on $\mathcal{P}$, and*
(4) $d_F(y_1) \leq 1$ *and* $d_F(y_2) = 1$.

(3.4.2) *Let B, x_0, $\overline{x}$, y_1 and y_2 as in the Proposition (3.4.1) be given. Then there exists a spanning 3-forest F in B such that:*

(1) *A component T of F contains the vertices x_0, $\overline{x}$ and y_2, but it does not contain a vertex of attachment of B on $\mathcal{P} \cup \mathcal{P}'$.*
(2) *Each component of $F - T$ contains exactly one vertex of attachment of B on $\mathcal{P} \cup \mathcal{P}'$.*
(3) $d_F(x_0) = d_F(\overline{x}) = 1$ *and* $d_F(x) \leq 1$ *for each vertex x of attachment of B on $\mathcal{P}$, and*
(4) $d_F(y_1) \leq 1$ *and* $d_F(y_2) = 1$.

4. Main tools

Let $\mathcal{R}$ be a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph G and let P be a separating path in $\mathcal{R}$ with $H \cup K = G$ and $H \cap K = P$. Further let H be infinite and $H^{(1)}, \dots, H^{(r)}$ be infinite components of $H - P$ Since P is induced and unbounded, there exist uniquely determined separating paths $P^{(1)}, \dots, P^{(r)}$ of $\mathcal{R}'$ satisfying the properties (see Proposition 3.1 in [8]):

(1) Each of the endvertices of $P^{(i)}$ is adjacent to an endvertex of the foot of $H^{(i)}$ on P $(i = 1, \dots, r)$.

(2) For each bridge connecting P with $\bigcup_{i=1}^{r} P^{(i)}$ there exists an index $j \in \{1, \dots, r\}$ such that all vertices of attachment of B lie on $P \cup P^{(j)}$, and $|V(B) \cap V(P^{(j)})| \leq 2$.

(3) If $V(B) \cap V(P^{(i)}) = \{z, z'\}$ for a bridge B connecting P with $\bigcup_{i=1}^{r} P^{(i)}$, $z \neq z'$, it must hold $zz' \in E(P^{(i)})$.

Set $W^{(1)}, \dots, W^{(r)}$ the feet of $P^{(1)}, \dots, P^{(r)}$ each of which contains an infinite component of $H - P$. Then we can easily see that $W^{(1)}, \dots, W^{(r)}$ are edge-disjoint.

For $j = 1, \dots, r$, let us denote the endvertices of $W^{(j)}$ by x_j and $\bar{x}_j$, and those of $P^{(j)}$ by y_j and $\bar{y}_j$, in the clockwise order. Then we see that

$$P^{(j)} \cup W^{(j)} \cup \{x_j y_j, \bar{x}_j \bar{y}_j\}$$

forms a cycle, which will be denoted by $C^{(j)}$; in particular this cycle is induced since $P^{(j)}$ and $W^{(j)}$ are induced.

We will say that the subgraph, denoted by $L^{(j)}$, of G induced by the vertices not only on $C^{(j)}$ but also in the interior of the cycle a *cell* of P (with respect to $P^{(j)}$), and the bridges of $\mathcal{R}$ which lie in the interior of $C^{(j)}$ the *inner bridges* in $L^{(j)}$. Clearly the path P contains exactly r cells, namely $L^{(1)}, \dots, L^{(r)}$.

Remark. For $j = 1, \dots, r$, we can in similar way obtain an edge-disjoint feet $W_1^{(j)}, \dots, W_{n_j}^{(j)}$ on $P^{(j)}$, since $P^{(j)}$ is a separating path of G. We claim that there exists at least one inner bridge (say B) in $L^{(j)}$ of type 1 or 2 with

$$V(B) \cap V(W_1^{(j)} - \bar{z}_j) \neq \emptyset \quad \text{and} \quad E(B) \cap E(W^{(j)} - x_j) \neq \emptyset,$$

where $\bar{z}_j$ is the endvertex of $W_1^{(j)}$ on $W^{(j)}$.

To see this, let us denote z_j the another endvertex of $W_1^{(j)}$, and suppose to the contrary that the assertion is false. If $y_j = z_j$, then G is separated by the vertices x_j and $\bar{z}_j$, which contradicts the 3-connectedness of G. On the other hand, if $y_j \neq z_j$, G is in this case separated by $\bar{z}_j$ and the vertex adjacent to z_j on $P^{(j)}$, a contradiction.

To describe our main results in this paper, we need to introduce some terminology. Let T be a finite tree. We may say that a sequence of vertices $(u_1, \dots, u_s)$ with $u_i \in V(T)$ and $u_i \neq u_j$ $(i \neq j)$ *lies on a path in this order*, if $P_i \subseteq P_j$ for $i \leq j$ and $i, j \in \{1, \dots, s\}$, where P_i (and P_j) is the u_1, u_i-path (u_1, u_j-path, respectively) on T. In this case, the edge incident to u_i on the $u_i u_{i+1}$-path is called *path-incident* with u_i in T (with respect to $(u_1, \dots, u_s)$). In Definition 4.1 below, the same notation as the arguments above are used.

Definition 4.1. An induced semicycle $\mathcal{P}$ in an LV-graph satisfies the hypothesis (†) (with $T_\mathcal{P}$), if there exists a spanning 3-tree $T_\mathcal{P}$ in $C(\mathcal{P})$ such that each separating path P in $\mathcal{P}$ satisfies one of the following 3 properties [V1]–[V3]:

[V1] $d_{T_\mathcal{P}}(x) \leq 2$ for all $x \in V(P)$.

[V2] There exists exactly one vertex $\tilde{w}_P$ on the $u, \bar{x}_1$-subpath of P with $\tilde{w}_P \neq \bar{x}_1$, such that
- (a) $d_{T_\mathcal{P}}(\tilde{w}_P) = 3$ and $d_{T_\mathcal{P}}(x) \leq 2$ for all $x \in V(P) \setminus \{\tilde{w}_P\}$.
- (b) If $V(\widetilde{W}) =: \{\tilde{w}_P = w_1, \dots, w_t = u\}$ $(t \geq 1)$ for the $\tilde{w}_P, u$-subpath $\widetilde{W}$ of P, then the sequence $(w_1, \dots, w_t)$ in $T_\mathcal{P}$ lies on a path in this order.

[V3] There exists exactly one edge $\tilde{e}_P = \tilde{u}_P \tilde{v}_P$ on the $u, \bar{x}_1$-subpath of P, such that
- (a) $\tilde{e}_P \in E(T_\mathcal{P})$.
- (b) $d_{T_\mathcal{P}}(\tilde{u}_P) = d_{T_\mathcal{P}}(\tilde{v}_P) = 3$ and $d_{T_\mathcal{P}}(x) \leq 2$ for all $x \in V(P) \setminus \{\tilde{u}_P, \tilde{v}_P\}$.
- (c) If $V(\widetilde{W}) =: \{\tilde{v}_P = w_1, \tilde{u}_P = w_2, \dots, w_t = u\}$ $(t \geq 2)$ for the $\tilde{v}_P, u$-subpath $\widetilde{W}$ of P, then the sequence $(w_1, \dots, w_t)$ in $T_\mathcal{P}$ lies on a path in this order.

We will call the vertex $\tilde{w}_P$ in case [V2] a *3-vertex* of P, and the edge in case [V3] a *3-edge* of P.

Remark. From the definition above we easily verify the following:

(1) Every subsequence of a sequence of vertices lying on a path lies on the same path.

(2) If an induced semicycle $\mathcal{P}$ satisfies the hypothesis (†) with a spanning 3-tree and P is an element of $\mathcal{P}$, then the vertex $\tilde{w}_P$ in case [V2] (or the edge $\tilde{e}_P$ in case [V3]) lies on the foot $W^{(1)}$ or on the u, x_1-subpath of P; i.e., neither the vertex $\tilde{w}_P$ nor the edge $\tilde{e}_P$ lies on the feet $W^{(2)}, \dots, W^{(r)}$.

5. Proof of the main theorem

For a given tight $(\mathcal{P}, \mathcal{P}')$-semiring, we assume that the semicycle $\mathcal{P}$ satisfies the hypothesis (†) with a spanning 3-tree $T_\mathcal{P}$ in $C(\mathcal{P})$. Let P be a separating path in $\mathcal{P}$ and further set $L^{(1)}, \dots, L^{(r)}$ the cells of P. Then, from the hypothesis

(†) for the semicycle $\mathcal{P}$, P satisfies one of the properties [V1]–[V3]. We will now construct a spanning 3-tree $T_{\mathcal{P}'}$ in $C(\mathcal{P}')$ which satisfies the hypothesis (†) with $T_{\mathcal{P}'}$. To do this, we first construct a spanning 3-forest for each bridge B of type 0. Such a 3-forest, denoted by F_B, in B can be obtained from , which satisfies the assertions in the theorem. Then we set

$$F_0 := \cup\{F_B \mid B \text{ is a bridge of type } 0\}$$

(I) P satisfies [V1]

For $i = 1, \ldots, r$, consider the cell $L^{(i)}$. Since P fulfills the property [V1], it contains neither 3-vertices nor 3-edges; i.e., $d_{T_{\mathcal{P}}}(x) \leq 2$ for all $x \in V(W^{(i)})$. We choose a bridge (say B_0) of type 1 or 2 in $L^{(i)}$. Set further

$$V(P^{(i)}) =: \{y_i = v_0, v_1, \ldots, v_s = \bar{y}_i\},$$

and for each $j \in \{1, \ldots, s\}$

$$Y_j := \begin{cases} \emptyset, & \text{if } v_j, v_{j+1} \in V(B) \text{ for a bridge } B \text{ in } L^{(i)} \\ \{v_j v_{j+1}\}, & \text{otherwise} \end{cases}$$

For each bridge B of type 1 (or of type 2, respectively) ($\neq B_0$) in $L^{(i)}$, use (2.2.1) (or (2.3.1), respectively) to obtain a spanning 3-forest F_B in B satisfying the conditions in the lemma.

Now consider the bridge B_0. If B_0 is trivial, then set $F_{B_0} := B_0$. On the other hand, if B_0 is of type 2 and $|V(B_0) \cap V(W^{(i)})| = 1$, then, by using (2.2.2), we may obtain a spanning 3-forest F_{B_0} in B_0. We finally consider the case that $|V(B_0) \cap V(W^{(i)})| \geq 2$. If B_0 is of type 1, we use (3.4.1) to obtain a spanning 3-forest F_{B_0} in B_0 satisfying the properties in the lemma. But, if B_0 is of type 2, then using (2.3.3) we also have a spanning 3-forest F_{B_0} in B_0. Thus in any case we obtain a spanning 3-forest F_{B_0} in B_0.

Then, by denoting $\{\tilde{u}_{P^{(i)}}, \tilde{v}_{P^{(i)}}\} := V(B_0) \cap V(P^{(i)})$, we finally set

$$F^{(i)} := \begin{cases} \left[\bigcup_{\substack{B\,:\,\text{bridge} \\ \text{in } L^{(i)}}} F_B\right] \cup \left[\bigcup_{j=1}^{s} Yj\right], & \text{if } B_0 \text{ is of type 1.} \\ \left[\bigcup_{\substack{B\,:\,\text{bridge} \\ \text{in } L^{(i)}}} F_B\right] \cup \left[\bigcup_{j=1}^{s} Yj\right] \cup \{\tilde{u}_{P^{(i)}} \tilde{v}_{P^{(i)}}\}, & \text{if } B_0 \text{ is of type 2.} \end{cases}$$

and for each separating path P in $\mathcal{P}$ satisfying [V1]

$$T_P := T_{\mathcal{P}} \cup \left[\bigcup_{i=1}^{r} F^{(i)}\right] \cup F_0$$

Proposition 5.1. *The constructed T_P is a spanning 3-tree in*

$$H_P := C(\mathcal{P}) \cup \left[\bigcup_{i=1}^{r} L^{(i)}\right] \cup [\cup\{B \mid B \text{ is a bridge of type } 0\}]$$

such that each of the separating paths $P^{(i)}$ $(i = 1, \cdots, r)$ satisfies one of $[V1]$ – –$[V3]$ *with respect to T_P.*

Proof. From our construction we easily see that T_P is connected and contains no cycles, and it follows that T_P is a tree. For $i \in \{1, \ldots, r\}$, since F_B is a spanning subgraph of a bridge B in $L^{(i)}$, we have $V(F^{(i)}) = V(L^{(i)})$, and therefore T_P is a spanning tree in H_P. To verify that T_P is a 3-tree, consider the bridges B in $L^{(i)}$. If B is of type 0 (or of type 1 or 2, respectively), then we see

$$d_{F_B}(x_0) = d_{F_B}(\bar{x}) = 0,$$
$$(\text{or } \; d_{F_B}(x_0) = 0 \quad \text{and} \quad d_{F_B}(\bar{x}) \leq 1, \text{ respectively}),$$

and thus $d_{F^{(i)}}(x) \leq 1$ for all $x \in V(W^{(i)})$. But, since $d_{T_{\mathcal{P}}}(x) \leq 2$ from the assumption, we conclude that $d_{T_P}(x) \leq 2 + 1 = 3$ for all $x \in V(W^{(i)})$. Since for the remaining vertices z in H_P we can obviously have $d_{T_P}(z) \leq 3$, we have shown that T_P is a 3-tree in H_P.

Now consider the bridge B_0. By noting that B_0 is of type 1 or 2, we first consider the former case. Let $\tilde{w}_{P^{(i)}}$ be the first vertex of attachment of B_0 on $P^{(i)}$. From the choice of B_0 we see that the vertex $\tilde{w}_{P^{(i)}}$ lies on the first component of $G - P^{(i)}$ in the natural order. Then, if $\tilde{w}_{P^{(i)}} = y_i$, $P^{(i)}$ satisfies the condition [V1]. On the other hand, if $\tilde{w}_{P^{(i)}} \neq y_i$, then $P^{(i)}$ satisfies in this case the condition [V2] (with the 3-vertex $\tilde{w}_{P^{(i)}}$). Now consider the case that B_0 is of type 2 with the vertices $\tilde{u}_{P^{(i)}}$ and $\tilde{v}_{P^{(i)}}$ of attachment on $P^{(i)}$. If $d_{T_P}(\tilde{u}_{P^{(i)}}) = 3$, then we use similar arguments to verify that $P^{(i)}$ satisfies the property [V3] (with the 3-edge $\tilde{u}_{P^{(i)}}\tilde{v}_{P^{(i)}}$), since the edge is contained in T_P. For the remaining cases we can obviously see that $P^{(i)}$ satisfies the properties [V1] or [V2] (with the 3-vertex $\tilde{v}_{P^{(i)}}$). The fact that the sequence on $\tilde{w}_{P^{(i)}}, y_i$-subpath (or $\tilde{v}_{P^{(i)}}, y_i$-subpath) of $P^{(i)}$ lies on a path in this order follows from the construction. □

(II) P satisfies [V2]

For a bridge B in $L^{(i)}$ we may denote the $x_0, \overline{x}$-path on P by P_B, and set $V_B := V(P_B) \setminus \{x_0\}$, where x_0 and $\overline{x}$ are the first and the last vertex (in the clockwise order) of attachment of B on P, respectively. In particular, if $x_0 = \overline{x}$, we simply set $P_B = \{\overline{x}\}$ and $V_B = \emptyset$.

(1) The cell $L^{(i)}$ $(i = 2, \dots, r)$

From the hypothesis and the Remark in the preceding section we have $d_{T_P}(x) \leq 2$ for all $x \in V(W^{(i)})$. In this case, using the argument similar to the case (I), we can obtain a 3-forest $F^{(i)}$, which covers all vertices of $L^{(i)}$.

(2) The cell $L^{(1)}$

Let $\tilde{w}_P$ the 3-vertex of P and let B_0 and Y_j $(j = 1, \dots, s)$ be the form described in Case (I). Set further

$$\Gamma := \left\{B \mid B \text{ is a bridge in } L^{(1)} \text{ with } |V(B) \cap V(W^{(1)})| \geq 2\right\}$$

and $\widetilde{W}$ the $u, \tilde{w}_P$-subpath of P. We define a subset Γ' of Γ holding the following property:

$$B \in \Gamma' \quad \text{if and only if} \quad V_B \cap V(\widetilde{W}) \neq \emptyset$$

Finally we set

$$\Delta := \left\{B \mid B \text{ is a bridge in } L^{(1)} \text{ with } \tilde{w}_P \in V_B\right\}$$

If $\Delta = \emptyset$, applying the similar process as in (I), we obtain a spanning 3-forest $F^{(1)}$ in $L^{(1)}$.

Now we assume that $\Delta \neq \emptyset$. From the definition of $\tilde{w}_P$, for each $B \in \Delta$ there exists exactly one bridge $\widetilde{B} \in \Delta$ such that all vertices of $B - P_B$ are contained in a facial cycle of $\widetilde{B}$. If $\widetilde{B}$ is of type 0, then we use (2.1) (in the case $B_0 \in \Gamma'$) or (3.3) (in the case $B_0 \in \Gamma$) to obtain a 3-forest $F_{\widetilde{B}}$ in $\widetilde{B}$. To investigate the remaining bridges (of type 1 or 2) we classify in two cases.

Case 1: B is a bridge of type 1 or 2 with $B \neq B_0$.

We first consider the case $B \notin \Gamma$. If B is trivial, then set $F_B = \emptyset$. Otherwise (i.e., B is of type 2 with $|V(B) \cap V(W^{(i)})| \geq 2$) we use (2.3.1) to obtain a 3-forest F_B in B satisfying the properties in the lemma. Now consider the case $B \in \Gamma$. In the case $B \notin \Gamma'$, there exists a 3-forest F_B in B by (2.2.1) or (2.3.1) if B is of type 1 or 2, respectively. On the other hand, if $B \in \Gamma'$, using (3.1.1) or (3.2.1) we also have a 3-forest F_B in B.

Case 2: The bridge B_0.

First consider the case that B_0 is of type 1. If B_0 is trivial, then we set $F_{B_0} := B_0$. Now let B_0 is nontrivial. If $V_{B_0} \cap V(\widetilde{W}) = \emptyset$ (i.e., $\widetilde{B} \notin \Gamma'$), then we use (2.2.2) to obtain a 3-forest F_{B_0}. On the other hand, if $V_{B_0} \cap V(\widetilde{W}) \neq \emptyset$, we also have such a 3-forest F_{B_0} by (3.1.2).

We now investigate the case that B_0 is of type 2. If $B_0 \notin \Gamma$, then we in similar way have a spanning 3-forest F_{B_0} in B_0. Otherwise, by (3.4.1) (or (3.4.2), respectively) there exists a spanning 3-forest F_{B_0} in B_0 in the case $x_0 \notin \widetilde{W}$ (or $x_0 \in \widetilde{W}$, respectively), where x_0 is the first vertex of attachment of B_0 on $W^{(i)}$ in the clockwise order.

Combining the results in Case 1 and Case 2 we finally define:

$$F^{(i)} := \begin{cases} \left[\bigcup_{\substack{\mathrm{B\,:\,bridge} \\ \mathrm{in}\ L^{(i)}}} F_B\right] \cup \left[\bigcup_{j=1}^{s} Yj\right], & \text{if } B_0 \text{ is of type 1.} \\ \left[\bigcup_{\substack{\mathrm{B\,:\,bridge} \\ \mathrm{in}\ L^{(i)}}} F_B\right] \cup \left[\bigcup_{j=1}^{s} Yj\right] \cup \{\tilde{u}_{P^{(i)}}\tilde{v}_{P^{(i)}}\}, & \text{if } B_0 \text{ is of type 2.} \end{cases}$$

where $\{\tilde{u}_{P^{(i)}}, \tilde{v}_{P^{(i)}}\} := V(B_0) \cap V(P^{(i)})$ and Y_j $(j = 1, \dots, s)$ are defined in (I).

To define a spanning 3-tree T_P by summing up the constructed 3-forests $F^{(i)}$ $(i = 1, \dots, r)$ in each cell $L^{(i)}$, we need to define a set of edges $E_1 \subsetneq E(T_{\mathcal{P}})$ as follows:

Let $\widetilde{W}$ be the $\tilde{w}_P, u$-subpath on $W^{(1)}$, and set

$$\Gamma'' := \begin{cases} \Gamma' \cup \{\widetilde{B}, B_0\}, & \text{if } V(B_0) \cap V(W^{(1)}) =: \{\overline{x}\} \text{ and } \overline{x} \in \widetilde{W} \\ \Gamma' \cup \{\widetilde{B}\}, & \text{otherwise} \end{cases}$$

For each bridge B in $\Gamma' \cup \{\widetilde{B}\}$, we set

$$\overline{x}_B := \begin{cases} \text{the first vertex of } B \text{ on } W^{(1)}, & \text{if } \tilde{w}_P \notin V_B \\ \tilde{w}_P, & \text{otherwise} \end{cases}$$

If $B_0 \in \Gamma''$ and $\{\overline{x}\} = V(B) \cap V(W^{(1)})$, then we set $\overline{x}_{B_0} = \overline{x}$. Then we obviously have $\overline{x}_B \neq \overline{x}_{B'}$, for bridges B and B' with $B \neq B'$. Since the sequence of vertices on $\widetilde{W}$ lies on a path in this order (by [V2]), there exists an edge (say e_B) which is path-incident to $\overline{x}_B$ in $T_{\mathcal{P}}$ for each $B \in \Gamma''$. We set then $E_1 := \{e_B \mid B \in \Gamma''\}$.

By means of the set of edges E_1, we finally set

$$T_P := \left[T_{\mathcal{P}} \cup \left(\bigcup_{i=1}^{r} F^{(i)}\right) \cup \{x_1 y_1\} \cup F_0\right] - E_1$$

in case of $B_0 \in \Gamma''$ and $|V(B_0) \cap V(W^{(1)})| = 1$, and otherwise set

$$T_P := \left[T_{\mathcal{P}} \cup \left(\bigcup_{i=1}^{r} F^{(i)}\right) \cup F_0\right] - E_1.$$

Proposition 5.2. *The constructed T_P is a spanning 3-tree in*

$$H_P := C(\mathcal{P}) \cup \left[\bigcup_{i=1}^{r} L^{(i)}\right] \cup [\cup\{B \mid B \text{ is a bridge of type } 0\}]$$

such that each of the separating paths $P^{(i)}$ $(i = 1, \cdots, r)$ satisfies one of $[V1]$ – $-[V3]$ with respect to T_P.

Proof. For each cell $L^{(i)}$ $(i = 2, \ldots, r)$ we adapt the arguments similar to those in the proof of Proposition 5.1. We now consider the cell $L^{(1)}$. If $\Delta = \emptyset$, then we also use the method similar to that in the case (I) to obtain a spanning 3-tree T_P in H_P satisfying one of the conditions [V1]–[V3].

Now assume that $\Delta \neq \emptyset$. First, we investigate the graph $H := F^{(1)} \cup T_{\mathcal{P}}$. Set $B \in \Gamma' \cup \{\widetilde{B}\}$ with the first vertex x_B and the last vertex $\overline{x}_B$ on P, in the clockwise order. Since we have used the results in (3.1.1), (3.1.2), (3.2.1), (3.2.1) or (3.4.2) in the construction of F_B, we can conclude that $d_H(\overline{x}_B) \leq 4$ (or $d_H(\tilde{w}_P) \leq 4$, if $B = \widetilde{B}$) and there exists a $x_B, \widetilde{x}_B$-path in F_B.

On the other hand, since $T_{\mathcal{P}}$ is a tree with $x_B, \widetilde{x}_B \in V(T_{\mathcal{P}})$, we can also have a $x_B, \widetilde{x}_B$-path in $T_{\mathcal{P}}$. But, since the two $x_B, \widetilde{x}_B$-paths are disjoint (except for the vertices x_B and $\widetilde{x}_B$), we obtain a cycle (say C_B) in H containing x_B and $\widetilde{x}_B$. From the fact that the sequence of vertices of $\tilde{w}_P, u$-path on P lies on a path, it follows that $e_B \in E(C_B)$. For two bridges B, B' with $B \neq B'$, we can also have $|V(C_B) \cap V(C_{B'})| \leq 1$, and therefore we conclude that $H - e_B$ is connected and further $d_{H-e_B}(\overline{x}_B) \leq 3$ and $d_{H-e_B}(\tilde{w}_P) \leq 3$. If $B_0 \in \Gamma_1$ with a vertex $\overline{x}_{B_0}$ of attachment on P, then there also exists a cycle C_{B_0} in H with $x_1y_1, e_{B_0} \in E(C_{B_0})$, such that $H - e_{B_0}$ is a connected subgraph of H. Since $x_1 \neq \overline{x}_{B_0}$ from the choice of B_0, we can in similar way show that the vertex $\overline{x}_{B_0}$ has the degree at most 3 in $H - e_{B_0}$, which follows that $H - K_1$ is a spanning 3-tree in $C(\mathcal{P}) \cup L^{(1)}$. Thus we have shown that T_P is a spanning 3-tree in H_P.

It remains to prove that $P^{(i)}$ satisfies one of the conditions [V1]–[V3]. But, by using similar method in Proposition 5.1, we can without difficulty verify the assertion, and thus we omit to describe it. Note that, in this case, $P^{(i)}$ satisfies [V1] or [V2] if B_0 is of type 1, and [V1]–[V3] if B_0 is of type 2. □

(III) P satisfies [V3]

First we consider the cells $L^{(i)}$ $(i = 2, \ldots, r)$. From the condition [V3] and the Remark in the preceding section, we have $d_{T_{\mathcal{P}}}(x) \leq 2$ for each $x \in V(W^{(i)}) \backslash \{x_2\}$, where x_2 is the first vertex of $W^{(2)}$ in the clockwise order. (It may noted that it is possible for the vertex x_2 to be incident to the 3-edge $\tilde{e}_P$.) In this case we use the case (I) to obtain a 3-forest $F^{(i)}$ in $W^{(i)}$.

Now consider the cell $L^{(1)}$. Let $\tilde{e}_P = \tilde{u}_P\tilde{v}_P$ be the given 3-edge of P and $\widetilde{W}$ the $\tilde{v}_P, u$-subpath of P with $\tilde{e}_P \in E(\widetilde{W})$. Set further

$$\Delta := \{B \mid B \text{ is a bridge in } L^{(i)} \text{ with } \tilde{v}_P \in V_B\}.$$

Recall that $V_B = V(P_B)\backslash\{x_0\}$, where P_B is the $x_0, \overline{x}$-path on P.

If $\Delta = \emptyset$, we apply the process similar to that in the case (I) to obtain a 3-forest $F^{(1)}$ in $W^{(1)}$. We now consider the case $\Delta \neq \emptyset$. For each $B \in \Delta$, let us denote $\widetilde{B} \in \Delta$ the bridge in Δ, in a facial cycle of which all vertices of $B - P_B$ are contained. Let further B_0 be the bridge introduced in (I). By setting Γ and Γ' as in the case (II), we use the same arguments similar to the case (II) to obtain a 3-forest F_B, for each $B \in \Gamma$ (including $\widetilde{B}$ and B_0). If we set Y_j $(j = 1, \ldots, s)$ as defined in the case (I), we finally define

$$F^{(i)} := \begin{cases} \left[\bigcup_{\substack{B \,:\, \text{bridge} \\ \text{in } L^{(i)}}} F_B\right] \cup \left[\bigcup_{j=1}^{s} Yj\right], & \text{if } B_0 \text{ is of type 1} \\ \left[\bigcup_{\substack{B \,:\, \text{bridge} \\ \text{in } L^{(i)}}} F_B\right] \cup \left[\bigcup_{j=1}^{s} Yj\right] \cup \{\tilde{u}_{P^{(i)}}\tilde{v}_{P^{(i)}}\}, & \text{if } B_0 \text{ is of type 2} \end{cases}$$

where $\{\tilde{u}_{P^{(i)}}, \tilde{v}_{P^{(i)}}\} := V(B_0) \cap V(P^{(i)})$ and Y_j $(j = 1, \ldots, s)$ are defined in (I).

Now we choose a set of edges $E_1 \subseteq E(T_{\mathcal{P}})$ as in the case (II), by replacing $\tilde{v}_P$ by $\tilde{w}_P$. It may be noted that it is possible to be $\tilde{v}_P = \overline{x}_1$. Then, since $\tilde{e}_P \in E(T_{\mathcal{P}})$ by [V3], it must be hold $\tilde{e}_P \in E_1$. We finally set

$$T_P := \left[T_{\mathcal{P}} \cup \left(\bigcup_{i=1}^{r} F^{(i)}\right) \cup \{x_1y_1\} \cup F_0\right] - E_1,$$

if $B_0 \in \Gamma''$ and $|V(B_0) \cap V(W^{(1)})| = 1$, and otherwise

$$T_P := \left[T_{\mathcal{P}} \cup \left(\bigcup_{i=1}^{r} F^{(i)}\right) \cup F_0\right] - E_1.$$

Proposition 5.3. *The constructed T_P is a spanning 3-tree in*

$$H_P := C(\mathcal{P}) \cup \left[\bigcup_{i=1}^{r} L^{(i)}\right] \cup [\cup\{B \mid B \text{ is a bridge of type } 0\}]$$

such that each of the separating paths $P^{(i)}$ $(i = 1, \cdots, r)$ satisfies one of $[V1]$ – –$[V3]$ with respect to T_P.

Proof. Using an argument similar to the vertex $\tilde{w}_P$ in the case (II), we can also obtain $d_{T_{\mathcal{P}}} \leq 3$. Since $\tilde{e}_P \in K_1$, it follows that $\tilde{e}_P \in E(T_P)$, which implies $d_{T_{\mathcal{P}}} \leq 3$. The remaining assertions can be proved by the analogous arguments as in Proposition 5.2. □

Now we summarize Proposition 5.1, 5.2 and 5.3.

Theorem 5.4. *Let $\mathcal{R}$ be a tight $(\mathcal{P}, \mathcal{P}')$-semiring in an LV-graph, such that the semicycle $\mathcal{P}$ satisfies the hypothesis (†) with a spanning 3-tree $T_\mathcal{P}$ in $Z(\mathcal{P})$. Then*

$$T_{\mathcal{P}'} := \cup\{T_P \mid P \text{ is a separating path in } P\}$$

is a spanning 3-tree in $Z(\mathcal{P}')$ with $T_\mathcal{P} \subseteq T_{\mathcal{P}'}$ such that the semicycle $\mathcal{P}'$ satisfies the hypothesis (†) with $T_{\mathcal{P}'}$.

Proof. The fact that $T_{\mathcal{P}'}$ is a spanning 3-tree in $Z(\mathcal{P}')$ follows from Proposition 5.1, 5.2 and 5.3, by considering the uniqueness of $\mathcal{P}'$. Let P' be a separating path in $\mathcal{P}'$. By the structure properties in Section 2 we have a separating path P in $\mathcal{P}$ such that each of the endvertices of P' is adjacent to one of the endvertices of P. Since $\mathcal{R}$ is tight and P satisfies one of [V1]–[V3], it follows that P' also satisfies one of [V1]–[V3], and consequently $\mathcal{P}'$ satisfies the hypothesis (†) with $T_{\mathcal{P}'}$. □

As seen in the structure properties in section 2 for an LV-graph G and for an arbitrary given induced semicycle $\mathcal{P}_0$ in G, there exists an infinite sequence of pairwise disjoint induced semicycles $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \dots)$, whose union covers all vertices of G. To use this property in this article, we shall need to define an induced semicycle $\mathcal{P}_0$ as a 'starting' semicycle. To do this, let us choose an arbitrary edge $e_0 = x_0y_0$ incident to an unbounded face of G. Then we clearly have the unique facial cycle (say C_0) containing the edge e_0. We may denote P_0 the x_0, y_0-path on C_0 which does not contain the edge e_0. Then it is not hard to see $|V(P_0)| \geq 3$, and moreover, since G is 3-connected, no vertex of $V(P_0) \setminus \{x_0, y_0\}$ is incident to an unbounded face, which implies that P_0 is a separating path in G. Also, from the same reason, the path P_0 in particular is induced. By setting $\mathcal{P}_0 := \{P_0\}$, we obtain an induced semicycle with $C(\mathcal{P}_0) = C_0$. We can now prove the main result in this paper.

Proof of Theorem C. Let G be an LV-graph and let $\mathcal{P}_0$ is an induced 'starting' semicycle in G obtained from the method above. Then, by the structure theorem in Section 2, there exists an infinite sequence of pairwise disjoint induced semicycles $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \dots)$ such that

(1) $\mathcal{P}_j \subset C(\mathcal{P}_{j+1})$ for all $j \in \{0, 1, 2, \dots\}$,
(2) $(\mathcal{P}_j, \mathcal{P}_{j+1})$-semiring is tight, for all $j \in \{0, 1, 2, \dots\}$; and
(3) $G = \bigcup_{j=0}^{\infty} C(\mathcal{P}_j)$.

Obviously $T_0 := P_0$ is a spanning 3-tree in $C_0 = C(\mathcal{P}_0)$ satisfying the property [V1], and thus $\mathcal{P}_0$ fulfills the hypothesis (†) with T_0.

Now assume that, for $j \geq 1$, a spanning 3-tree T_j in $C(\mathcal{P}_j)$ is constructed, such that $\mathcal{P}_j$ satisfies the hypothesis (†) with T_j. Then, by Proposition 5.1, 5.2 and 5.3 and the fact that the $(\mathcal{P}_j, \mathcal{P}_{j+1})$-semiring is tight, we again obtain a spanning 3-tree T_{j+1} in $C(\mathcal{P}_{j+1})$ with the corresponding properties. Therefore we have a sequence of 3-trees $(T_0, T_1, T_2, \dots)$ in G with

$$T_j \subset T_{j+1} \quad \text{and} \quad V(T_j) = V(C(\mathcal{P}_j)) \quad \text{for all} \quad j \in \{0, 1, 2, \dots\}$$

By setting $T := \bigcup_{j=0}^{\infty} T_j$, we get a spanning 3-tree in G, and therefore our proof is complete. □

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