J. Appl. Math. & Informatics Vol. **28**(2010), No. 3 - 4, pp. 883 - 892
Website: http://www.kcam.biz

OSCILLATORY PROPERTY OF SOLUTIONS FOR A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH PERTURBATION

QUANXIN ZHANG*, FANG QIU AND LI GAO

ABSTRACT. This paper is concerned with oscillation property of solutions of a class of second order nonlinear differential equations with perturbation. Four new theorems of oscillation property are established. These results develop and generalize the known results. Among these theorems, two theorems in the front develop the results by Yan J(Proc Amer Math Soc, 1986, 98: 276-282), and the last two theorems in this paper are completely new for the second order linear differential equations.

AMS Mathematics Subject Classification : 34C10, 34C15
Key word and phrases : Nonlinear; differential equations with perturbation; oscillation criterion.

1. Introduction

As is well known, the comparison and separation theory of zero distribution for second order homogeneous linear differential equations established by G.Sturm lay foundation of oscillation theory for differential equations. During one and a half century, oscillation theory of differential equations has developed quickly and played an important role in qualitative theory of differential equations and theory of boundary values problem. The study of oscillation theory plays a important role in physical sciences and technology; for example, the oscillation of building and machine, electromagnetic vibration in radio technology and optical science, self-excited vibration in control system, sound vibration, beam vibration in synchrotron accelerator, the vibration sparked for burning rocket engine, the complicated oscillation in chemical reaction, etc. All the different phenomena can be unified into oscillation theory through an oscillation equation. There are many books on the oscillation theory, we choose to refer to [1,2]. Firstly, we give the oscillatory definition of the solution of a differential equation.

Received October 3, 2009. Revised October 16, 2009. Accepted November 17, 2009.
*Corresponding author.

Definition 1. $x(t)$ is a solution of a differential equation, if $x(t)$ is not the eventually zero solutions, and existing of a sequence $\{t_i\}, \lim_{i\to\infty} t_i = \infty$, such that $x(t_i) = 0$. Such a solution is said to be oscillatory and otherwise it is said to be nonoscillatory. A nonoscillatory solution $x(t)$ is called weakly oscillatory if $x'(t)$ changes sign for large arbitrary t. See [1].

Definition 2. An equation is called oscillatory if all its solutions are oscillatory.

The oscillatory theory of second order nonlinear differential equations have been widely applied in research of a lossless high-speed computer network and physical sciences. In this paper, we study the oscillatory behavior of solutions for a class of second order nonlinear differential equation with perturbation

$$(a(t)\psi(x(t))x^{'}(t))^{'} + Q(t, x(t)) = P(t, x(t), x^{'}(t)), \quad ^{'} = \frac{d}{dt}. \tag{1}$$

Where we let

(A_1) $a : [t_0, +\infty) \to \mathrm{R}$ $(\mathrm{R} = (-\infty, +\infty))$ is positive continuously differentiable function;

(A_2) $\psi : \mathrm{R} \to \mathrm{R}$ is continuously differentiable function and $\psi(u) > 0$ for $u \neq 0$;

(A_3) $Q : [t_0, +\infty) \times R \to R$ is continuous, and there exists continuous function $q(t)$ and continuously differentiable function $f(x)$, where $q : [t_0, +\infty) \to R$, $q(t) \neq 0$, i.e., exist t_k ,$t_k \to +\infty$ such that $q(t_k) \neq 0$, $f : R \to R$, $uf(u) > 0$ for $u \neq 0$ and $f^{'}(u) > 0$, such that $\dfrac{Q(t, x)}{f(x)} \geq q(t)$ for $x \neq 0$;

(A_4) $P : [t_0, +\infty) \times R^2 \to R$ is continuous and there exists $p(t) : [t_0, +\infty) \to R$ which is continuous, such that $x(t)P(t, x(t), x^{'}(t)) \leq x(t)p(t)x^{'}(t)$ for $x \neq 0$.

It is easy to see that Eq.(1) can be transformed into

$$(a(t)\psi(x(t))x^{'}(t))^{'} + p(t)x^{'}(t) + q(t)f(x(t)) = 0, \tag{E_1}$$

if $Q(t, x(t)) = q(t)f(x(t)), P(t, x(t), x^{'}(t)) = -p(t)x^{'}(t)$. In (E_1), if $a(t) = 1, \psi(u) = 1$, the Eq.(E_1) can be transformed into

$$x''(t) + p(t)x^{'}(t) + q(t)f(x(t)) = 0; \tag{E_2}$$

In (E_1), if $\psi(u) = 1, f(x) = x$, then the Eq.(E_1) transformed into

$$(a(t)x^{'}(t))^{'} + p(t)x^{'}(t) + q(t)x(t) = 0. \tag{E_3}$$

In Eq.(E_2), if $p(t) = 0, f(x) = x$, the Eq.(E_2) is simplified to

$$x''(t) + q(t)x(t) = 0. \tag{E_4}$$

The equation (E_4) has many oscillation criterion. One of the most well-known criterion is Wintner's oscillation criterion [3]. It states that the linear equation (E_4) is oscillatory if

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(x)\mathrm{d}x\mathrm{d}s = \infty.$$

In 1986, J.Yan [4] extended and improved the criterion to the equation (E_3). The recent paper by Cakmak [5] discusses the oscillation criterion of the equation (E_2) with damping, which extended and improved the Wintner's result. Among the other papers in the study of equation (E_1) , we choose to refer to [6-11]. This article is based on [12] and continues to discuss the oscillatory behavior of (1) by using the generalized Riccati transformation and the integral averaging technique. We established four new oscillation criterion on certain conditions. Among these theorems, two theorems in the front develop the results in [4], and the last two theorems in this paper are completely new for the second order linear differential equation (E_3).

2. Main results

Theorem 1. *Let* $0 < c_1 \le \varphi(x) \le c_2, f'(x) \ge k > 0, x \ne 0$. *If there exsit continuously differentiable function* $\rho(t) : [t_0, +\infty) \to (0, +\infty)$ *and a constant* α, *such that*

$$\limsup_{t\to+\infty} t^{-\alpha} \int_{t_0}^{t} \Big\{ (t-\tau)^{\alpha}\rho(\tau)[q(\tau) + \frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}] - \frac{1}{4}[(t-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)}$$
$$+\alpha\rho(\tau) - (t-\tau)\rho'(\tau)]^2(t-\tau)^{\alpha-2}\frac{c_2a(\tau)}{k\rho(\tau)} \Big\} d\tau = +\infty, \qquad (2)$$

then (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq.(1) and $x(t) \ne 0$ for $t \ge t_0$. Consider the function

$$W(t) = \frac{a(t)\psi(x(t))x'(t)}{f(x(t))}, \quad t \ge t_0.$$

Then it follows from (1) that

$$\begin{aligned} W'(t) &= \frac{-Q(t,x(t)) + P(t,x(t),x'(t))}{f(x(t))} - a(t)\psi(x(t))f'(x(t))\frac{x'^2(t)}{f^2(x(t))} \\ &\le -q(t) + \frac{p(t)x'(t)}{f(x(t))} - a(t)\psi(x(t))f'(x(t))\frac{x'^2(t)}{f^2(x(t))} \\ &\le -q(t) - \frac{1}{\psi(x(t))}[\frac{k}{a(t)}W^2(t) - \frac{p(t)}{a(t)}W(t)] \\ &= -q(t) + \frac{1}{\psi(x(t))}\frac{p^2(t)}{4ka(t)} - \frac{1}{\psi(x(t))}\left[\sqrt{\frac{k}{a(t)}}W(t) - \frac{p(t)}{2\sqrt{ka(t)}}\right]^2 \\ &\le -q(t) + \frac{p^2(t)}{4kc_1a(t)} - \frac{1}{c_2}\left[\sqrt{\frac{k}{a(t)}}W(t) - \frac{p(t)}{2\sqrt{ka(t)}}\right]^2 \\ &= -q(t) - \frac{(c_1-c_2)p^2(t)}{4kc_1c_2a(t)} - \frac{1}{c_2}\left[\frac{k}{a(t)}W^2(t) - \frac{p(t)}{a(t)}W(t)\right]. \end{aligned}$$

Consequently, for $t \geq s \geq t_0$, we obtain

$$\int_s^t (t-\tau)^\alpha \rho(\tau) W'(\tau) \mathrm{d}\tau \leq -\int_s^t (t-\tau)^\alpha \rho(\tau)[q(\tau) + \frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}]\mathrm{d}\tau - \frac{1}{c_2}\int_s^t (t-\tau)^\alpha \rho(\tau)[\frac{k}{a(t)}W^2(\tau) - \frac{p(\tau)}{a(\tau)}W(\tau)]\mathrm{d}\tau.$$

Now, since

$$\int_s^t (t-\tau)^\alpha \rho(\tau) W'(\tau) \mathrm{d}\tau = \alpha \int_s^t (t-\tau)^{\alpha-1}\rho(\tau)W(\tau)\mathrm{d}\tau - \int_s^t (t-\tau)^\alpha \rho'(\tau)W(\tau)\mathrm{d}\tau - W(s)(t-s)^\alpha \rho(s),$$

it follows that

$$\begin{aligned}
&\int_s^t (t-\tau)^\alpha \rho(\tau)[q(\tau) + \frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}]\mathrm{d}\tau \\
\leq\ & (t-s)^\alpha \rho(s)W(s) - \frac{k}{c_2}\int_s^t \frac{(t-\tau)^\alpha \rho(\tau)W^2(\tau)}{a(\tau)}\mathrm{d}\tau \\
&-\int_s^t [(t-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)} + \alpha\rho(\tau) - (t-\tau)\rho'(\tau)](t-\tau)^{\alpha-1}W(\tau)\mathrm{d}\tau, \quad (3)
\end{aligned}$$

where $t \geq s \geq t_0$. Therefore

$$\begin{aligned}
&\int_s^t \Big\{ (t-\tau)^\alpha \rho(\tau)[q(\tau) + \frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}] - \frac{1}{4}[(t-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)} \\
&+\alpha\rho(\tau) - (t-\tau)\rho'(\tau)]^2 (t-\tau)^{\alpha-2}\frac{c_2a(\tau)}{k\rho(\tau)} \Big\}\mathrm{d}\tau \\
\leq\ & (t-s)^\alpha \rho(s)W(s) - \int_s^t \Big\{ (t-\tau)^{\frac{\alpha}{2}}[\frac{k\rho(\tau)}{c_2a(\tau)}]^{\frac{1}{2}}W(\tau) + \frac{1}{2}[(t-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)} + \alpha\rho(\tau) \\
&-(t-\tau)\rho'(\tau)](t-\tau)^{\frac{\alpha-2}{2}}[\frac{c_2a(\tau)}{k\rho(\tau)}]^{\frac{1}{2}} \Big\}^2 \mathrm{d}\tau \\
\leq\ & (t-s)^\alpha \rho(s)W(s), \quad t \geq s \geq t_0. \quad (4)
\end{aligned}$$

Divide two side of (4) separately by t^α. Further,we obtain

$$\limsup_{t\to+\infty} t^{-\alpha}\int_{t_0}^t (t-\tau)^\alpha \rho(\tau)\left[q(\tau) + \frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}\right]\mathrm{d}\tau \leq \limsup_{t\to+\infty} t^{-\alpha}(t-s)^\alpha\rho(s)W(s),$$

$t \geq s \geq t_0$. which contradicts the condition(2). The proof of Theorem 1 is complete. □

Corollary. *In Theorem 1,the condition (2) can be replaced by the below conditions:*

$$\text{(i)} \quad \limsup_{t\to+\infty} t^{-\alpha}\int_{t_0}^t (t-\tau)^\alpha \rho(\tau)[q(\tau) + \frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}]\mathrm{d}\tau = +\infty,$$

(ii) $$\lim_{t\to+\infty} t^{-\alpha}\int_{t_0}^{t}[(t-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)}+\alpha\rho(\tau)-(t-\tau)\rho'(\tau)]^2 \times(t-\tau)^{\alpha-2}\frac{c_2a(\tau)}{k\rho(\tau)}\mathrm{d}\tau<+\infty.$$

Remark 1. Theorem 1 above develops Theorem 1 in [4].

Theorem 2. *Suppose that* $0<c_1\le\psi(x)\le c_2, f'(x)\ge k>0, x\ne 0$, *and there exist continuously differentiable function* $\rho(t):[t_0,+\infty)\to(0,+\infty)$ *and the constant* $\alpha\in(1,+\infty)$, *such that*

$$\limsup_{t\to+\infty} t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}\rho(\tau)[q(\tau)+\frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}]\mathrm{d}\tau<+\infty. \tag{5}$$

In addition, if there exists continuous function $\varphi(t):[t_0,+\infty)\to \mathrm{R}$ *such that*

$$\liminf_{t\to+\infty} t^{-\alpha}\int_{s}^{t}\Big\{(t-\tau)^{\alpha}\rho(\tau)\Big[q(\tau)+\frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}\Big]-\frac{1}{4}\Big[(t-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)} +\alpha\rho(\tau)-(t-\tau)\rho'(\tau)\Big]^2(t-\tau)^{\alpha-2}\frac{c_2a(\tau)}{k\rho(\tau)}\Big\}\mathrm{d}\tau\ge\varphi(s), \tag{6}$$

and the below (7) holds,

$$\lim_{t\to+\infty} t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}\frac{\varphi_+^2(\tau)}{\rho(\tau)a(\tau)}\mathrm{d}\tau=+\infty, \tag{7}$$

where $\varphi_+(t)=\max\{\varphi(t),0\}$, *then (1) is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1) and $x(t)\ne 0$ for $t\ge t_0$. Consider the function

$$W(t)=\frac{a(t)\psi(x(t))x'(t)}{f(x(t))},\quad t\ge t_0.$$

Proceeding as in the proof of Theorem 1, we obtain (4). Divide two side of (4) separately by t^{α}. Further,we get

$$\liminf_{t\to+\infty} t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}\rho(\tau)\Big[q(\tau)+\frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}\Big]\mathrm{d}\tau\le\liminf_{t\to+\infty} t^{-\alpha}(t-s)^{\alpha}\rho(s)W(s),$$

$t\ge s\ge t_0$, with condition (6), it follows that $\varphi(s)\le\rho(s)W(s)$, for $s\ge t_0$. Therefore, we obtain

$$\varphi_+^2(s)\le\rho^2(s)W^2(s). \tag{8}$$

Define functions $u(t)$ and $v(t)$ by

$$u(t)=t^{-\alpha}\int_{t_0}^{t}\Big[(t-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)}+\alpha\rho(\tau)-(t-\tau)\rho'(\tau)\Big](t-\tau)^{\alpha-1}W(\tau)\mathrm{d}\tau, t>t_0;$$

$$v(t)=t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}\rho(\tau)\frac{kW^2(\tau)}{c_2a(\tau)}\mathrm{d}\tau, t>t_0.$$

Then, it follows from (3) that

$$u(t)+v(t) \le t^{-\alpha}(t-t_0)^{\alpha}\rho(t_0)W(t_0)-t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}\rho(\tau)[q(\tau)+\frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}]\mathrm{d}\tau. \tag{9}$$

Moreover, from the condition (6), we have

$$\liminf_{t\to+\infty} t^{-\alpha}\int_{s}^{t}(t-\tau)^{\alpha}\rho(\tau)\Big[q(\tau)+\frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}\Big]\mathrm{d}\tau \ge \varphi(s), s\ge t_0 \tag{10}$$

and

$$\begin{aligned}&\limsup_{t\to+\infty} t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}\rho(\tau)\Big[q(\tau)+\frac{(c_1-c_2)p^2(\tau)}{4kc_1c_2a(\tau)}\Big]\mathrm{d}\tau\\ &\quad-\liminf_{t\to+\infty}\frac{1}{4}t^{-\alpha}\int_{t_0}^{t}\Big[(t-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)}\\ &\quad+\alpha\rho(\tau)-(t-\tau)\rho'(\tau)\Big]^2(t-\tau)^{\alpha-2}\frac{c_2a(\tau)}{k\rho(\tau)}\mathrm{d}\tau\ge\varphi(t_0),\end{aligned} \tag{11}$$

(5) and (11) illustrate that there exists a sequence

$$\{t_n\}_1^{\infty}, t_n > t_0, n=1,2,3,..., \lim_{n\to+\infty} t_n = +\infty, \tag{12}$$

such that

$$\begin{aligned}\lim_{n\to+\infty}\frac{1}{4}t_n^{-\alpha}\int_{t_0}^{t_n}&[(t_n-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)}+\alpha\rho(\tau)-(t_n-\tau)\rho'(\tau)]^2\\ &\times(t_n-\tau)^{\alpha-2}\frac{c_2a(\tau)}{k\rho(\tau)}\mathrm{d}\tau<+\infty.\end{aligned} \tag{13}$$

Then, as $t\to+\infty$, it follows from (9) and (10) that

$$\limsup_{t\to+\infty}\{u(t)+v(t)\}\le\rho(t_0)W(t_0)-\varphi(t_0)=\beta. \tag{14}$$

Hence, for sufficiently large n , we have

$$u(t_n)+v(t_n)<\beta. \tag{15}$$

Because $v(t) = \int_{t_0}^{t}(1-\frac{\tau}{t})^{\alpha}\rho(\tau)\frac{kW^2(\tau)}{c_2a(\tau)}\mathrm{d}\tau > 0$, $t > t_0$ is increasing, we can get $\lim_{t\to+\infty} v(t) = C$, where $C = +\infty$ or C is a positive constant. Suppose that $C=+\infty$, then $\lim_{n\to+\infty} v(t_n) = +\infty$, and it follows from (15) that

$$\lim_{n\to+\infty} u(t_n) = -\infty. \tag{16}$$

According to (15) and (16), we get $\frac{u(t_n)}{v(t_n)}+1<\varepsilon$, where ε is a constant and $0<\varepsilon<1$, i.e., for sufficiently large t_n, we have

$$\frac{u(t_n)}{v(t_n)}<\varepsilon-1<0. \tag{17}$$

On the other hand, it follows from Schwarz's inequality that

$$
\begin{aligned}
0 &\le t_n^{-2\alpha}\Big(\int_{t_0}^{t_n}\Big[(t_n-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)}+\alpha\rho(\tau)-(t_n-\tau)\rho'(\tau)\Big](t_n-\tau)^{\alpha-1}W(\tau)\mathrm{d}\tau\Big)^2\\
&\le \left(t_n^{-\alpha}\int_{t_0}^{t_n}\Big[(t_n-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)}+\alpha\rho(\tau)-(t_n-\tau)\rho'(\tau)\Big]^2(t_n-\tau)^{\alpha-2}\frac{c_2a(\tau)}{k\rho(\tau)}\mathrm{d}\tau\right)\\
&\qquad\times\left(t_n^{-\alpha}\int_{t_0}^{t_n}(t_n-\tau)^{\alpha}\frac{k\rho(\tau)}{c_2a(\tau)}W^2(\tau)\mathrm{d}\tau\right).
\end{aligned}
$$

Thus

$$
0\le\frac{u^2(t_n)}{v(t_n)}\le t_n^{-\alpha}\int_{t_0}^{t_n}[(t_n-\tau)\frac{\rho(\tau)p(\tau)}{c_2a(\tau)}+\alpha\rho(\tau)-(t_n-\tau)\rho'(\tau)]^2(t_n-\tau)^{\alpha-2}\frac{c_2a(\tau)}{k\rho(\tau)}\mathrm{d}\tau.
$$

From (13), we obtain

$$
0\le\lim_{n\to+\infty}\frac{u^2(t_n)}{v(t_n)}<+\infty,
$$

which contradicts the (16) and (17) . If $\lim_{t\to+\infty}v(t)=C<+\infty$, it follows from (8) that

$$
\begin{aligned}
\lim_{t\to+\infty}\frac{k}{c_2}t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}\frac{\varphi_+^2(\tau)}{\rho(\tau)a(\tau)}\mathrm{d}\tau &\le \lim_{t\to+\infty}\frac{k}{c_2}t^{-\alpha}\int_{t_0}^{t}(t-\tau)^{\alpha}\rho(\tau)\frac{W^2(\tau)}{a(\tau)}\mathrm{d}\tau\\
&= C<+\infty,
\end{aligned}
$$

which contradicts the condition (7). The proof of Theorem 2 is complete. □

Remark 2. Theorem 2 above develops Theorem 2 in [4].

Theorem 3. *Let* $\psi(x)f'(x)\ge k>0, x\ne 0$, *and*

$$
0<\int_0^{\varepsilon}\frac{\psi(u)}{f(u)}\mathrm{d}u<+\infty,\quad \int_{-\varepsilon}^{0}\frac{\psi(u)}{f(u)}\mathrm{d}u>-\infty. \tag{18}
$$

If there exists differentiable function $\rho(t):[t_0,+\infty)\to(0,+\infty)$, *such that*

$$
(a(t)\rho(t))'\ge 0,\quad \limsup_{t\to+\infty}\frac{a(t)\rho(t)}{\int_{t_0}^{t}\rho(s)\mathrm{d}s}<+\infty, \tag{19}
$$

and

$$
\lim_{t\to+\infty}\frac{1}{\int_{t_0}^{t}\rho(s)\mathrm{d}s}\int_{t_0}^{t}\rho(s)\int_{t_0}^{s}[q(\tau)-\frac{p^2(\tau)}{4ka(\tau)}]\mathrm{d}\tau\mathrm{d}s=+\infty, \tag{20}
$$

then (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1) and $x(t)>0$ for $t\ge t_1\ge t_0$. Consider the function

$$
W(t)=\frac{a(t)\psi(x(t))x'(t)}{f(x(t))},\quad t\ge t_1.
$$

Then it follows from (1) that

$$\begin{aligned} W'(t) &= \frac{-Q(t,x(t))+P(t,x(t),x'(t))}{f(x(t))} - a(t)\psi(x(t))f'(x(t))\frac{x'^2(t)}{f^2(x(t))} \\ &\le -q(t) + \frac{p(t)x'(t)}{f(x(t))} - a(t)\psi(x(t))f'(x(t))\frac{x'^2(t)}{f^2(x(t))} \\ &= -q(t) + \frac{p^2(t)}{4a(t)\psi(x(t))f'(x(t))} - \Big[\sqrt{a(t)\psi(x(t))f'(x(t))}\frac{x'(t)}{f(x(t))} \\ &\quad - \frac{p(t)}{2\sqrt{a(t)\psi(x(t))f'(x(t))}}\Big]^2 \le -q(t) + \frac{p^2(t)}{4ka(t)}. \end{aligned}$$

Integrating above inequality from t_1 to t $(t \ge t_1)$, we get

$$\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} + \int_{t_1}^{t}\Big[q(s) - \frac{p^2(s)}{4ka(s)}\Big]\mathrm{d}s \le C, \tag{21}$$

where $C = \frac{a(t_1)\psi(x(t_1))x'(t_1)}{f(x(t_1))}$. Multiplying the two side of (21) by $\rho(t)$ and integrating, we have

$$\int_{t_1}^{t} a(s)\rho(s)\frac{\psi(x(s))x'(s)}{f(x(s))}\mathrm{d}s + \int_{t_1}^{t}\rho(s)\int_{t_1}^{s}[q(\tau) - \frac{p^2(\tau)}{4ka(\tau)}]\mathrm{d}\tau\mathrm{d}s \le C\int_{t_1}^{t}\rho(s)\mathrm{d}s.$$

Hence,

$$\begin{aligned} &(a(t)\rho(t))\int_0^{x(t)}\frac{\psi(u)}{f(u)}\mathrm{d}u - \int_{t_1}^{t}(a(s)\rho(s))'\int_0^{x(s)}\frac{\psi(u)}{f(u)}\mathrm{d}u\mathrm{d}s \\ &\quad + \int_{t_1}^{t}\rho(s)\int_{t_1}^{s}[q(\tau) - \frac{p^2(\tau)}{4ka(\tau)}]\mathrm{d}\tau\mathrm{d}s \\ &\le C\int_{t_1}^{t}\rho(s)\mathrm{d}s + a(t_1)\rho(t_1)\int_0^{x(t_1)}\frac{\psi(u)}{f(u)}\mathrm{d}u. \end{aligned} \tag{22}$$

Define $G(t)$ by $G(t) = \int_0^{x(t)}\frac{\psi(u)}{f(u)}\mathrm{d}u$. From two front terms of (22) , it follows that there exists a sequence $\{T_n\}_1^{+\infty}$, such that

$$a(T_n)\rho(T_n)G(T_n) - \int_{t_1}^{T_n}(a(s)\rho(s))'G(s)\mathrm{d}s > 0;$$

or there exists $T \ge t_1$, such that

$$a(t)\rho(t)G(t) - \int_{t_1}^{t}(a(s)\rho(s))'G(s)\mathrm{d}s \le 0 \tag{23}$$

for $t \ge T$. Consider the first case. Divide two side of (22) by $\int_{t_1}^{t}\rho(s)\mathrm{d}s$ and replace t with T_n. From conditions (18) and (20), we get a contradiction. Consider the second case . Let $H(t) = \int_{t_1}^{t}(a(s)\rho(s))'G(s)\mathrm{d}s$, then $H(t)$ is nonnegative and not decreasing. From (23) ,we obtain that $H(t) > 0$ for $t \geq T$ where T is the same as the above, and $a(t)\rho(t)H'(t) \leq (a(t)\rho(t))'H(t)$, thus $(\frac{a(t)\rho(t)}{H(t)})' \geq 0$. Consequently,

$$\frac{a(t)\rho(t)}{H(t)} \geq \frac{a(T)\rho(T)}{H(T)}.$$

for$t \geq T$. Further,according to (23) ,we have

$$a(t)\rho(t)G(t) \leq H(t) \leq H(T)\frac{a(t)\rho(t)}{a(T)\rho(T)}.$$

It concludes that $G(t)$ is upper bounded , so we mark it with k_1. Noticing $(a(t)\rho(t))' \geq 0 (t \geq t_0)$ with (22) ,we have $\int_{t_1}^{t} \rho(s) \int_{t_1}^{s} [q(\tau) - \frac{p^2(\tau)}{4ka(\tau)}]\mathrm{d}\tau\mathrm{d}s$ $\leq C\int_{t_1}^{t} \rho(s)\mathrm{d}s + a(t_1)\rho(t_1)\int_{0}^{x(t_1)} \frac{\psi(u)}{f(u)}\mathrm{d}u + k_1 a(t)\rho(t)$. Divide two side of the above inequality by $\int_{t_1}^{t} \rho(s)\mathrm{d}s$ and calculate the upper limits for $t \to +\infty$. According to (18), (19),(20)it follows a contradiction. The proof for the case $x(t) < 0$ for $t \geq t_1$ is similar. Then the proof of Theorem 3 is complete. □

Theorem 4. *Let $\psi(x)f'(x) \geq k > 0, x \neq 0$. Suppose (18) holds, and for every constant M we have*

$$\int_{t_0}^{+\infty} \frac{M}{a(s)}\mathrm{d}s - \int_{t_0}^{+\infty} \frac{1}{a(s)} \int_{t_0}^{s} [q(\tau) - \frac{p^2(\tau)}{4ka(\tau)}]\mathrm{d}\tau\mathrm{d}s = -\infty, \tag{24}$$

then (1) is oscillatory.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of Eq.(1) and $x(t) > 0$ for $t \geq t_1 \geq t_0$. Consider the function

$$W(t) = \frac{a(t)\psi(x(t))x'(t)}{f(x(t))}, \quad t \geq t_1.$$

Proceeding as in the proof of the above Theorem 3 ,we obtain (21) , i.e. ,

$$\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} \leq M - \int_{t_1}^{t} [q(s) - \frac{p^2(s)}{4ka(s)}]\mathrm{d}s,$$

where $M = \frac{a(t_1)\psi(x(t_1))x'(t_1)}{f(x(t_1))}$. Dividing two side of the above inequality by $a(t)$ and integrating , we get

$$\int_{t_1}^{t} \frac{\psi(x(s))x'(s)}{f(x(s))}\mathrm{d}s \leq \int_{t_1}^{t} \frac{M}{a(s)}\mathrm{d}s - \int_{t_1}^{t} \frac{1}{a(s)} \int_{t_1}^{s} [q(\tau) - \frac{p^2(\tau)}{4ka(\tau)}]\mathrm{d}\tau\mathrm{d}s.$$

From condition (24), letting $t \to +\infty$ gives

$$I(t) = \int_{t_1}^{t} \frac{\psi(x(s))x'(s)}{f(x(s))}\mathrm{d}s = \int_{x(t_1)}^{x(t)} \frac{\psi(u)}{f(u)}\mathrm{d}u \to -\infty.$$

On the other hand , if $x(t) \geq x(t_1)$ for sufficiently large T, then $I(t) \geq 0$, which follows a contradiction ; if $x(t) < x(t_1)$, it follows from (18) that

$$I(t) = -\int_{x(t)}^{x(t_1)} \frac{\psi(u)}{f(u)} \mathrm{d}u \geq -\int_{0}^{x(t_1)} \frac{\psi(u)}{f(u)} \mathrm{d}u > -\infty,$$

which is another contradiction. For the case $x(t) < 0$ for $t \geq t_1$, the proof is similar. Then the proof of Theorem 4 is complete. □

References

1. Ladde G.S., Lakshmikantham V. and Zhang B.G.: *Oscillation theory of differential equations with deviating arguments.* Marcel Dekker, New York(1987).
2. Yan J.R.: *Oscillation Theory of ordinary differential equations.* Shanxi education press, Taiyuan (1992). (in Chinese)
3. Wintner A.: *A criterion of oscillatory stability,* Quart. Appl. Math., **7**, 115-117(1949).
4. Yan J.R.: *Oscillation theorems for second order linear differential equations with damping,* Proc. Amer. Math. Soc., **98**, 276-282(1986).
5. Cakmak D.: *Oscillation criteria for nonlinear second order differential equations with damping,* Ukrainian Mathematical Journal, **60**(5), 799-809(2008).
6. Rogovchenko Yu.V.: *On oscillation of a second order nonlinear delay differential equation,* Funkcial. Ekvac., **3**, 1-29(2000).
7. Ohriska J.: *Oscillation of second order linear delay differential equations,* Cent. Eur. J. Math., **6**(3), 439-452(2008).
8. Zhang Q.X., Yan J.R.: *Oscillatory behavior of a second order nonlinear differential equation with damping,* Journal of Systems Science and Mathematical Science, **24**(3), 296-302(2004). (in Chinese)
9. Yan J.R., Zhang Q.X.: *Oscillatory theorems for second order nonlinear differential equations with damping,* Journal of Systems Science and Mathematical Science, **13**(3), 276-278(1993). (in Chinese)
10. Cecchi M., Marini M.: *Oscillatory and nonoscillatory behavior of a second order functional differential equation,* Rocky Mount. J. Math., **22**, 1259-1276(1992).
11. Philos C.G., Purnaras I.K.: *On the oscillation of second order nonlinear differential equations,* Arch. Math., **59**, 260-271(1992).
12. Zhang Q.X., Wang L.: *Oscillatory behavior of solutions for a class of second order nonlinear differential equation with perturbation,* Acta Appl. Math., doi: 10.1007/s10440-009-9483-8 (2009).

Department of Mathematics and Information Science, Binzhou University, Shandong 256603, People's Republic of China.
E-mail address: 3314744@163.com(Q.Zhang)