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EXISTENCE RESULTS FOR THIRD-ORDER THREE-POINT BOUNDARY VALUE PROBLEM ON TIME SCALES

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ABSTRACT. In this paper, a third-order three-point boundary value problem on time scales is considered. We establish criteria for the existence of a solution and a positive solution by using the Leray-Schauder fixed point theorem. An example is also given to illustrate our results.

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1. Introduction

As it is well known, the theory of time scales was introduced by Stefan Hilger in his PhD thesis in 1988[1], a time scale $\mathbf{T}$ is a nonempty closed subset of R. We make the blanket assumption that 0, T are points in $\mathbf{T}$. By an interval (0,T), we always mean the intersection of the real interval (0,T) with the given time scale; that is (0, T)$\cap\mathbf{T}$.

In recent years, there is much attention paid to the existence of positive solution for second-order three-point boundary value problem on time scales, see[2-13] and references therein, for example, Sun[8] considered the following third-order two-point boundary value problem on time scales:

$$u^{\Delta\Delta\Delta}(t) + f(t, u(t), u^{\Delta\Delta}(t)) = 0, \ t \in [a, \sigma(b)],$$

$$u(a) = A, \quad u(\sigma^b) = B, \quad u^{\Delta\Delta}(a) = C,$$

where $a, b \in T$ and $a < b$. Some existence criteria of solution and positive solution are established by using Leray-Schauder fixed point theorem.

However, to the best of our knowledge, there are not many results concerning three-point boundary value problem of third-order on time scales.

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In this paper, we consider the following third-order three-point boundary value problem on general time scales $\mathbf{T}$:

$$u^{\Delta\Delta\Delta}(t) = f(t, u(t), u^{\Delta\Delta}(t)), \quad t \in [0, \rho(1)], \tag{1}$$

$$u(0) = 0, \quad u(\eta) = u(\sigma^2(1)), \quad u^{\Delta\Delta}(0) = 0. \tag{2}$$

The purpose of this paper is to establish some existence criteria of solution and positive solution for the BVP (1) and (2) by using the Leray-Schauder fixed point theorem. The method of this paper is motivated by [13].

2. Preliminaries and lemmas

For convenience, we list the following definitions which can be found in [1,3,5-10].

Definition 1. A time scale $\mathbf{T}$ is a nonempty closed subset of real numbers R. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and backward jump operator ρ, respectively,

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T},$$

$$\rho(r) = \sup\{\tau \in \mathbf{T} \mid \tau < r\} \in \mathbf{T}.$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense.

Definition 2. Fix $t \in \mathbf{T}$. Let $f : \mathbf{T} \longrightarrow R$. the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|,$$

for all $s \in U$. Define $f^{\Delta^n}(t)$ to be the delta derivative of $f^{\Delta^{n-1}}(t)$, i.e., $f^{\Delta^n}(t) = (f^{\Delta^{n-1}}(t))^{\Delta}$. If $F^{\Delta}(t) = f(t)$, then define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

Definition 3. If $x \in C_{rd}$, and $t \in \mathbf{T}^k$, then

$$\int_a^t x(s)\Delta s = \mu(t)x(t),$$

where $\mu(t) = \sigma(t) - t$ is the graininess function.

Lemma 1. Assume $G(t, s)$ is the Green's function of

$$\begin{cases} -u^{\Delta\Delta} = 0, \quad t \in [0, 1], \\ u(0) = 0, \quad u(\eta) = u(\sigma^2(1)), \end{cases}$$

then

$$G(t,s)=\begin{cases} t, & t\le s\le \eta,\\ \sigma(s), & \sigma(s)\le t \ \ and\ \ s\le\eta,\\ \frac{\sigma^2(1)-\sigma(s)}{\sigma^2(1)-\eta}t, & \sigma(\eta)\le s \ \ and\ \ t\le s,\\ \sigma(s)-t+\frac{\sigma^2(1)-\sigma(s)}{\sigma^2(1)-\eta}t, & \sigma(\eta)\le s\le\sigma(s)\le t.\end{cases}$$

3. Main results

Now, for convenience, we introduce the following notations. We denote

$$R^+=[0,+\infty),\quad R^-=(-\infty,0],\quad h=\max_{t\in[0,\sigma^2(1)]}\int_0^{\sigma^2(1)}|G(t,s)|\Delta s.$$

Our main results are the following.

Theorem 1. *Suppose that $f:[0,\rho(1)]\times R\times R\longrightarrow R$ is continuous and there exist $c>0$ and $0<k\le 1/2h$ such that*

$$\max\{|f(t,u,v)|: t\in[0,\rho(1)],\ |u|\le c,\ |v|\le kc\}\le kc.$$

Then the BVP (1) and (2) has at least one solution u^* satisfying

$$|u^*(t)|\le c,\quad t\in[0,\sigma^2(1)],\quad and|(u^*)^{\Delta\Delta}(t)|\le kc,\quad t\in[0,1].$$

Proof. Let

$$C_1=\{u|u:[0,\sigma^2(1)]\longrightarrow R\ \ is\ \ continuous\}$$

and

$$C_2=\{v|v:[0,1]\longrightarrow R\ \ is\ \ continuous\}$$

be equipped with their norms

$$|u|_1=\max_{t\in[0,\sigma^2(1)]}|u(t)|, and\ \ |v|_2=\max_{t\in[0,1]}|v(t)|,$$

respectively, and $E=C_1\times C_2$, for any $(u,v)\in E$, its norm

$$\|(u,v)\|=\max\left\{|u|_1,\frac{1}{k}|v|_2\right\},$$

then E is a Banach space. Furthermore, it is easy to know that system (1)-(2) is equivalent to the following system of integral equations:

$$u(t)=\int_0^{\sigma^2(1)}G(t,s)v(s)\Delta s,\quad t\in[0,\sigma^2(1)], \tag{3}$$

$$v(t)=-\int_0^t f(t,u(s),v(s))\Delta s,\quad t\in[0,1]. \tag{4}$$

Define an operator $\phi:E\longrightarrow E:$

$$\phi(u,v)=(\phi_1(u,v),\phi_2(u,v)).$$

where

$$\phi_1(u,v)(t) = \int_0^{\sigma^2(1)} G(t,s)v(s)\Delta s, \quad t \in [0, \sigma^2(1)],$$

$$\phi_2(u,v)(t) = -\int_0^t f(t, u(s), v(s))\Delta s, \quad t \in [0,1].$$

Then system (3) and (4), and so the BVP (1) and (2) is equivalent to the fixed point equation

$$\phi(u,v) = (u,v), \quad (u,v) \in E.$$

Moreover, it is easy to see that $\phi : E \longrightarrow E$ is completely continuous. Let

$$E_c = \{(u,v) \in E : \|(u,v)\| \leq c\}.$$

Then E_c is a closed convex subset of E.

Suppose that $(u,v) \in E_c$, then $|u|_1 \leq c$ and $|v|_2 \leq kc$. So,

$$|u(t)| \leq c, \quad t \in [0, \sigma^2(1)], \tag{5}$$

and

$$|v(t)| \leq kc, \quad t \in [0,1], \tag{6}$$

which means that

$$|f(t,u,v)| \leq kc, \quad t \in [0, \rho(1)]. \tag{7}$$

From (6), Lemma 1 and $0 < k \leq 1/2h$, we have

$$\begin{aligned} |\phi_1(u,v)|_1 &= \max_{t\in[0,\sigma^2(1)]} |\textstyle\int_0^{\sigma^2(1)} G(t,s)v(s)\Delta s| \\ &\leq \max_{t\in[0,\sigma^2(1)]} \textstyle\int_0^{\sigma^2(1)} |G(t,s)v(s)|\Delta s \\ &\leq kc \max_{t\in[0,\sigma^2(1)]} \textstyle\int_0^{\sigma^2(1)} |G(t,s)|\Delta s \\ &= kch \leq c. \end{aligned} \tag{8}$$

On the other hand, it follows from (7) and Lemma 1 that

$$\begin{aligned} |\phi_2(u,v)|_2 &= \max_{t\in[0,1]} |-\textstyle\int_0^t f(t,u(s),v(s))\Delta s| \\ &\leq \max_{t\in[0,1]} \textstyle\int_0^t |f(t,u(t),v(t))|\Delta s \\ &\leq \textstyle\int_0^1 |f(s,u(s),v(s))|\Delta s \\ &\leq kc. \end{aligned} \tag{9}$$

In view of (8) and (9), we know that

$$\|\phi(u,v)\| = \max\left\{|\phi_1(u,v)|_1, \frac{1}{k}|\phi_2(u,v)|_2\right\} \leq c,$$

which shows that $\phi : E_c \longrightarrow E_c$. Then, it follows from the Leray-Schauder fixed point theorem that ϕ has a fixed point $(u^*, v^*) \in E_c$. In other words, the BVP (1) and (2) has one solution $u^* \in C_1$, which satisfies

$$|u^*|_1 \le c, \quad |(u^*)^{\Delta\Delta}|_2 \le kc.$$

□

Theorem 2. *Suppose that $f : [0, \rho(1)] \times R^+ \times R^- \longrightarrow R^+$ is continuous and there exist $c > 0$ and $0 < k \le 1/2h$ such that*

$$\max\{|f(t,u,v)| : t \in [0,\rho(1)],\ 0 \le u \le c,\ -kc \le v \le 0\} \le kc.$$

Then the BVP (1) and (2) has at least one solution u^* satisfying

$$0 \le u^*(t) \le c, \quad t \in [0, \sigma^2(1)], \quad \text{and} - kc \le (u^*)^{\Delta\Delta}(t) \le 0, \quad t \in [0,1].$$

Proof. Let

$$f_1(t,u,v) = \begin{cases} f(t,u,v), & (t,u,v) \in [0,\rho(1)] \times R^+ \times R^-, \\ f(t,u,0), & (t,u,v) \in [0,\rho(1)] \times R^+ \times R^+, \end{cases}$$

and

$$f_2(t,u,v) = \begin{cases} f_1(t,u,v), & (t,u,v) \in [0,\rho(1)] \times R^+ \times R, \\ f_1(t,0,v), & (t,u,v) \in [0,\rho(1)] \times R^- \times R. \end{cases}$$

Then $f_2 : [0,\rho(1)] \times R \times R \longrightarrow R^+$ is continuous and

$$\begin{aligned} &\max\{|f_2(t,u,v)| : t \in [0,\rho(1)],\ |u| \le c,\ |v| \le kc\} \\ &= \max\{|f(t,u,v)| : t \in [0,\rho(1)],\ 0 \le u \le c,\ -kc \le v \le 0\} \\ &\le kc. \end{aligned}$$

Consider the BVP

$$u^{\Delta\Delta\Delta}(t) = f_2(t, u(t), u^{\Delta\Delta}(t)), \quad t \in [0,\rho(1)], \tag{10}$$

$$u(0) = 0, \quad u(\eta) = u(\sigma^2(1)), \quad u^{\Delta\Delta}(0) = 0. \tag{11}$$

By Theorem 1, we know that the BVP (10) and (11) has one solution u^* satisfying

$$|u^*(t)| \le c, \quad t \in [0,\sigma^2(1)], \quad |(u^*)^{\Delta\Delta}(t)| \le kc, \quad t \in [0,1].$$

In view of $f_2(t, u^*(t), (u^*)^{\Delta\Delta}(t)) \ge 0, \quad t \in [0,\rho(1)]$ and Lemma 1, we get

$$(u^*)^{\Delta\Delta}(t) = -\int_0^t f_2(s, u^*(s), (u^*)^{\Delta\Delta}(s))\Delta s \le 0, \quad t \in [0,1].$$

This means that u^* is a nonnegative concave function on $[0,\sigma^2(1)]$. So,

$$f_2(t, u^*(t), (u^*)^{\Delta\Delta}(t)) = f(t, u^*(t), (u^*)^{\Delta\Delta}(t)), \quad t \in [0,\rho(1)].$$

Therefore, u^* is a solution of the BVP(1),(2) and satisfies

$$0 \le u^*(t) \le c, \quad t \in [0,\sigma^2(1)], \quad -kc \le (u^*)^{\Delta\Delta}(t) \le 0, \quad t \in [0,1].$$

□

Corollary. *Suppose that all the conditions of Theorem 2 are fulfilled. If* $f(t, 0, 0)$ *is not identically zero for* $t \in [0, \rho(1)]$, *then the* $BVP(1)$and (2) *has one positive solution.*

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