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OPTIMAL INVESTMENT FOR THE INSURER IN THE LEVY MARKET UNDER THE MEAN-VARIANCE CRITERION †

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ABSTRACT. In this paper we apply the martingale approach, which has been widely used in mathematical finance, to investigate the optimal investment problem for an insurer under the criterion of mean-variance. When the risk and security assets are described by the Lévy processes, the closed form solutions to the maximization problem are obtained. The mean-variance efficient strategies and frontier are also given.

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1. Introduction

The problem of optimal investment for a general insurer has attracted more and more attention since the work of Browne [1] where the claim process is approximated by a drifted Brownian motion and the stock price process was modeled by a geometric Brownian motion. Browne [1] maximized the expected constant absolute risk aversion(CARA) from the terminal wealth. Browne [1] also showed that the target of minimizing the ruin probability and the target of maximizing the exponential utility of the reserve (at a future time) produce the same type of strategy when the interest rate of risk-free asset is zero. The strategy is to invest a fixed amount of money in the risky asset.

Hipp and Plum [2] used the classical Cramér-Lundberg model to describe the surplus of the insurance company and assumed that the insurer have an option of investing part of his or her reserve in the risky asset (there is no risk-less asset), which follows a geometric Brownian motion with the target of minimizing ruin probability. Explicit solutions can be obtained in the case of exponential distributed claim-size. The set-up is adopted by most of the works on this subject

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since 2000. Later, there are some papers considering the similar optimal investment problem, such as, Liu and Yang [4], Yang and Zhang [5], Wang [6], Wang et al. [7] and etc.

In Liu and Yang [4], the model in Hipp and Plum [2] incorporate a non-zero risk-free interest rate is reconsidered. In this case, closed-form solution can not be obtained. But, they provided numerical results for optimal policy for maximizing survival probability under different claim-size distribution assumptions. By the approach of stochastic dynamic programming (the HJB equation), Yang and Zhang [5] used a jump-diffusion to model the risk process and considerd the portfolio problems to optimize various objective functions. In particular, they obtained a closed-form solution to maximize the expected CARA utility. See also Wang [6] via another approach when the claim process is supposed to be a pure jump process and the insurer have the option of investing in multiple risky assets.

The mean-variance criteria to portfolio selection problem was firstly proposed by Markowitz [8]. Later on, it has become one of the foundations of modern finance theory. See Merton [9] for the single-period, Li and Ng [10] for the discrete-time, multi-period model, Zhou and Li [13] for the continuous-time, Zhou and Yin [14] and etc. Recently, Wang et al. [7] pointed out that the mean-variance criteria was also of interest for an insurers' optimal investment problem using a martingale approach.

The martingale approach has been widely used in mathematical finance to investigate the optimal investment problem. See Kramkov and Schachermayer [11] for a detailed description. Recently, Lévy process has attracted more and more attention in the financial modeling, see Cont and Tankov [12].

This paper is a continuation of Wang et al. [7] and the model used in this paper is essential due to Wang et al. [7]. The risk process is modeled by a Lévy process adn the insurer has option to invest in two tradeable assets whose prices are described by the Lévy process. The optimal strategies are obtained explicitly under the mean-variance criterion via the martingale approach. It is remarkable that the optimal strategies depend on the jumps of the stock price process.

The remainder of this paper is organized as follows. In Section 2, we give the mathematical description of the investment problem. In Section 3, we first obtain strategies under the mean-variance and then work out the mean-variance efficient strategies.

2. Preliminaries

The following assumptions for the continuous-time security market model are made:

(1) continuous trading is allowed;

(2) no transaction cost or tax is involved in trading; and

(3) all assets are divisible, that is, fractional units of assets can be traded.

In Wang et al. [7], the insurer can invest in a security market described by the standard Black-Scholes model, where there are two tradeable assets: a bond and a stock. The price process $B(t)$ of the bond follows

$$dB(t) = rB(t)dt,$$

where r is the risk-free interest rate. The price process $S(t)$ of the stock satisfies the following stochastic differential equation (SDE):

$$dS(t) = bS(t)dt + \sigma S(t)dW_t^{(1)},$$

where b is the appreciation rate, $\sigma > 0$ is the volatility, $T > 0$ is a fixed and finite time horizon, and $W_t^{(1)}$ is a 1-dimensional standard Brownian motion defined on a filtered complete probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t\ P\}$. In this paper, we generalize the above price process model to a stochastic cash flow, which is still denoted by $S(t)$ and satisfies

$$dS(t) = bS(t)dt + \sigma S(t)dW_t^{(1)} + \sigma S(t)dL_t^{(1)}, \tag{1}$$

where $L^{(1)}$ is a 1-dimensional compensated pure jump Lévy process defined on

a filtered complete probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t\ P\}$. In Yang and Zhang [5], the risk process R_t of the insurer is modeled by

$$dR_t = \alpha dt + \beta d\overline{W}(t) - d\left(\sum_{i=1}^{N(t)} Y_i\right), \tag{2}$$

where α and β are constants, $\overline{W}$ is another 1-dimensional standard Brownian motion, and $\sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process defined on $\{\Omega, \mathcal{F}, \mathcal{F}_t\ P\}$. $N(t)$ is a homogeneous Poisson process with intensity λ and represents the number of claims occurring in time interval $[0, T]$. Y_i is the size of the i-th claim. Thus the compound Poisson process $\sum_{i=1}^{N(t)} Y_i$ represents the cumulative amount of claims in time interval $[0, T]$. The claims' sizes $Y = \{Y_i, i \geq 1\}$ are assumed to be an i.i.d. sequence with a common cumulative distribution function F satisfying $F(0) = 0$ and $\int_{\mathbb{R}} x^2 dF(x) < \infty$. The diffusion term $\overline{W}(t)$ stands for the uncertainty associated with the surplus of the insurer at time t. $\left(W^{(1)}, \overline{W}\right)$ is a 2-dimensional Brownian motion such that the correlation coefficient of the components is ρ. $\overline{W}$ can be written as $\overline{W}(t) = \rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(2)}$, where $W^{(2)}$ is another 1-dimensional standard Brownian motion independent of $W^{(1)}$. Furthermore, it is usually assumed that $W^{(1)}$, and $Y = \{Y_i, i \geq 1\}$ are mutually independent. If $L_t^{(2)}$ denotes the compensated compound Poisson process $\sum_{i=1}^{N(t)} Y_i - \lambda m_F t$, where m_F is the mean of F , then the risk process (2) can be written as

$$dR_t = (\alpha - \lambda m_F)\, dt + \beta \rho dW_t^{(1)} + \beta\sqrt{1-\rho^2} dW_t^{(2)} - dL_t^{(2)}. \tag{3}$$

As in Wang et al. [7], we will use the following risk process which is still denoted by R_t and satisfies

$$dR_t = cdt + \beta \rho dW_t^{(1)} + \beta\sqrt{1-\rho^2} dW_t^{(2)} - dL_t^{(2)}, \tag{4}$$

where c and β are two constants, $W^{(1)}$ and $W^{(2)}$ are two independent 1-dimensional standard Brownian motions, and $L_t^{(2)}$ is a 1-dimensional compensated pure jump Lévy process. They are all defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. $\mathcal{F}_t$ is the usual augmentation of the natural filtration generated by $W^{(1)}, W^{(2)}, L^{(1)}, L^{(2)}$ with $\mathcal{F} = \mathcal{F}_T$, and

$W^{(1)}, W^{(2)}, L^{(1)}, L^{(2)}$ are mutually independent. For $i = 1, 2$, let μ_i denote the jump measure of $L^{(i)}$ and ν_i denote the dual predictable projection of μ_i which has the form $\nu_i(dt, dx) = dt \times m_i(dx)$ with $m_i(\{0\}) = 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)m_i(dx) < \infty$. Obviously, Equation (4) is the generalization of Eqaution(2). Throughout this paper, we make the following assumption on the Lévy measure m_i to avoid some tedious technical arguments:

(**A:**) $\int_{\mathbb{R}} x^2 m_i(dx) < \infty$.

It is well known that, under assumption (**A:**), $L^{(i)}$ is square-integrable and has the following Lévy decomposition (see, e.g., Cont and Tankov [12]):

$$L_t^{(i)} = \int_0^t \int_{\mathbb{R}} x \left(\mu_i(ds, dx) - \nu_i(ds, dx)\right). \tag{5}$$

Remark 1. For risk process model (2), $m_2(dx) = \lambda F(dx)$.

The insurer is allowed to invest in stock as well as in bond. A trading strategy of the insurer can be modeled by an $\mathcal{F}_t$-predictable process $\pi = (\pi_t)$, where π_t represents the dollar amount invested in the stock at time t.

Definition 1. A trading strategy π is called admissible if $E\left[\int_0^T \pi_t^2 dt\right] < \infty$.

The set of all admissible trading strategies is denoted by Π. Corresponding to an admissible trading strategy π and an initial capital x_0, the wealth process $X^{x_0,\pi}$ of the insurer follows the dynamics

$$\begin{cases} dX_t^{x_0,\pi} &= \pi_t \left(bdt + \sigma dW_t^{(1)} + \sigma dL_t^{(1)}\right) + (X_t^{x_0,\pi} - \pi_t)\, rdt + cdt + \beta\rho dW_t^{(1)} \\ & \qquad + \beta\sqrt{1-\rho^2}dW_t^{(2)} - dL_t^{(2)}, \\ X_0^{x_0,\pi} &= x_0, \end{cases}$$

that is

$$\begin{aligned} X_t^{x_0,\pi} =& e^{rt}(x_0 + \int_0^t ((b-r)\pi_s + c)\, e^{-rs} ds + \int_0^t (\sigma\pi_s + \beta\rho)e^{-rs} dW_s^{(1)} \\ &+ \int_0^t \beta\sqrt{1-\rho^2}e^{-rs}dW_s^{(2)} + \int_0^t \sigma\pi_s e^{-rs} dL_s^{(1)} - \int_0^t e^{-rs} dL_s^{(2)}), \\ =& e^{rt}x_0 + \frac{c\left(e^{rt}-1\right)}{r} + (b-r)\int_0^t \pi_s e^{r(t-s)} ds + \int_0^t (\sigma\pi_s + \beta\rho)e^{r(t-s)} dW_s^{(1)} \\ &+ \int_0^t \beta\sqrt{1-\rho^2}e^{r(t-s)}dW_s^{(2)} + \int_0^t \sigma\pi_s e^{r(t-s)} dL_s^{(1)} - \int_0^t e^{r(t-s)} dL_s^{(2)}. \end{aligned} \tag{6}$$

Suppose that the insurer has a utility function U of the terminal wealth, then the aim of the insurer is to

$$Maximize \quad E\left[U\left(X_T^{x_0,\pi}\right)\right], \quad \pi \in \Pi. \tag{7}$$

The utility function U is strictly concave and continuously differentiable on $\mathbb{R}$. Since U is strictly concave, there exists at most unique optimal terminal wealth for the company. The following Proposition is from Wang et al. [7], Proposition 2.1.

Proposition 1. *If there exists a strategy $\pi^* \in \Pi$ such that*

$$E\left[U'\left(X_T^{x_0,\pi^*}\right)X_T^{x_0,\pi}\right] \quad \text{is constant over} \quad \pi \in \Pi. \tag{8}$$

then π^* is the optimal trading strategy.

Remark 2. Such sufficient conditions for the optimal strategies are well known in the martingale approach to the optimal investment, cf. Karatzas et al. [15], among others.

In the following sections, we will apply the previous proposition to work out the optimal strategies explicitly for the commonly used quadratic utility.

To conclude this section, we introduce some notations and a martingale representation theorem that will be used in the next section.

Some notations:

• $\mathcal{P}$: the predictable σ-algebra on $\Omega \times [0,T]$, which is generated by all left-continuous and $\mathcal{F}_t$-adapted processes

• $\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R})$

• $L(\mathcal{P})$: the set of all $\mathcal{F}_t$-predictable, $\mathbb{R}$-valued processes θ_1 such that $\int_0^T |\theta_1(t)|^2 dt < \infty$ a.s.

• $L^2(\mathcal{P})$: the set of all $\mathcal{F}_t$-predictable, $\mathbb{R}$-valued processes θ_1 such that $E\left[\int_0^T |\theta_1(t)|^2 dt\right] < \infty$ a.s.

• $L(\widetilde{\mathcal{P}})$: the set of all $\widetilde{\mathcal{P}}$-measurable, $\mathbb{R}$-valued functions θ_3 defined on $\Omega \times [0,T] \times \mathbb{R}$ such that $\sqrt{\sum_{0<s\le t} |\theta_3(s,\triangle L_s)|^2 1_{(\triangle L_s \neq 0)}}$ is a locally integrable increasing process and for all $t \in [0,T]$, $\int_{\mathbb{R}} |\theta_3(t,x)| m_1(dx) < \infty$ a.s., where $1_{(\cdot)}$ denotes the indicator function.

• $L^2(\widetilde{\mathcal{P}})$: the set of all $\widetilde{\mathcal{P}}$-measurable, $\mathbb{R}$-valued functions θ_3 defined on $\Omega \times [0,T] \times \mathbb{R}$ such that $E\left[\int_0^T \int_{\mathbb{R}} |\theta_3(t,x)|^2 m(dx) dt\right] < \infty$

• $\Theta := \{\theta = (\theta_1,\theta_2,\theta_3,\theta_4) \in L(\mathcal{P}) \times L(\mathcal{P}) \times L(\widetilde{\mathcal{P}}) \times L(\widetilde{\mathcal{P}})\}$

• $\Theta^2 := \{\theta = (\theta_1,\theta_2,\theta_3,\theta_4) \in L^2(\mathcal{P}) \times L^2(\mathcal{P}) \times L^2(\widetilde{\mathcal{P}}) \times L^2(\widetilde{\mathcal{P}})\}$

• $L^2_{\mathcal{F}}$: the set of all $(\mathcal{F})_t$-adapted processes (X_t) with cádlág paths such that $E\left[\sup_{0\le t\le T} |X_t|^2\right] < \infty$.

It is well known that every square-integrable martingale belongs to $L^2_{\mathcal{F}}$, and it is easy to see that if $\pi \in \Pi$ then $X^{x_0,\pi} \in L^2_{\mathcal{F}}$. The following result of Cont and Tankov [12] Proposition 9.4 is the critical tool to obtain our main result.

Lemma 1. *(Martingale Representation). For any local (resp. square-integrable) martingale (Z_t), there exists some $\theta = (\theta_1,\theta_2,\theta_3,\theta_4) \in \Theta$ (resp. $\theta = (\theta_1,\theta_2,\theta_3,\theta_4) \in \Theta^2$) such that*

$$Z_t = Z_0 + \sum_{i=1}^{2} \int_0^t \theta_i(s) dW_s^{(i)} + \sum_{i=3}^{4} \int_0^t \int_{\mathbb{R}} \theta_i(s,x)\left(\mu_{i-2}(ds,dx) - \nu_{i-2}(ds,dx)\right), \tag{9}$$

for all $t \in [0,T]$.

3. Mean-variance criterion

The problem of mean-variance portfolio choice is to maximize the expected terminal wealth $E\left[X_T^{x_0,\pi}\right]$ and, in the meantime, to minimize the variance of the terminal wealth $Var\left[X_T^{x_0,\pi}\right]$ over $\pi \in \Pi$. This is a multi-objective optimization problem with two conflicting criteria. The trading strategy $\pi^* \in \Pi$ is said to be mean-variance efficient if there does not exist a strategy $\pi \in \Pi$ such that

$$E\left[X_T^{x_0,\pi}\right] \geq E\left[X_T^{x_0,\pi^*}\right], \quad Var\left[X_T^{x_0,\pi}\right] \leq Var\left[X_T^{x_0,\pi^*}\right]. \tag{10}$$

with at least one inequality holds strictly. In this case, we call $\left(E\left[X_T^{x_0,\pi}\right], Var\left[X_T^{x_0,\pi}\right]\right) \in \mathbb{R}^2$ an efficient point. The set of all efficient points is called the efficient frontier. It is well-known that to find a mean-variance efficient strategy is equivalent to maximize the expected quadratic utility. Thus, in this section, we consider the problem (7) for the quadratic utility function $U(x) = x - \frac{\gamma}{2}x^2$, where $\gamma > 0$ is a parameter.

3.1. Efficient strategies

For the utility function $U(x) = x - \frac{\gamma}{2}x^2$, we have $U'(x) = 1 - \gamma x$ and condition (8) can be written as

$$E\left[\left(1 - \gamma X_T^{x_0,\pi^*}\right) X_T^{x_0,\pi}\right],$$

is constant over $\pi \in \Pi$. By (6), it is equivalent to that

$$E\left[\left(1 - \gamma X_T^{x_0,\pi^*}\right) \int_0^T \left((b-r)\pi_s e^{-rs} ds + \sigma \pi_s e^{-rs} dW_s^{(1)} + \sigma \pi_s e^{-rs} dL_s^{(1)}\right)\right], \tag{11}$$

is constant over $\pi \in \Pi$. Put $Z_t^* = E[1 - \gamma X_T^{x_0,\pi^*} | \mathcal{F}_t]$, $\quad t \in [0,T]$, then $Z_T^* = 1 - \gamma X_T^{x_0,\pi^*}$ and $Z_\tau^* = E\left[Z_T^* | \mathcal{F}_\tau\right]$ a.s. for any stopping time $\tau \leq T$ a.s. Furthermore, we have the following lemma.

Lemma 2. *Let $\pi^* \in \Pi$, then π^* satisfies condition* (11) *if and only if there exists a $(\theta_2^*, \theta_3^*, \theta_4^*) \in L^2(\mathcal{P}) \times L^2(\widetilde{\mathcal{P}}) \times L^2(\widetilde{\mathcal{P}})$ such that $\left(X^{x_0,\pi^*}, \pi^*, Z^*, \theta_2^*, \theta_3^*, \theta_4^*\right)$ solves the following forward-backward stochastic differential equation (FBSDE)*

$$\begin{cases} dX_t = rX_t dt + (b-r)\pi_t dt + \sigma \pi_t dW_t^{(1)} + c dt + +\beta \rho dW_t^{(1)} \\ \qquad\qquad +\beta\sqrt{1-\rho^2} dW_t^{(2)} + \sigma dL_t^{(1)} - dL_t^{(2)}, \\ X_0 = x_0, \\ dZ_t = -\frac{b-r}{\sigma} Z_{t-} dW_t^{(1)} + \theta_2(t) dW_t^{(2)} \\ \qquad\qquad +d\left(\sum_{i=3}^4 \int_0^t \int_R \theta_i(s,x) \left(\mu_{i-2}(ds,dx) - \nu_{i-2}(ds,dx)\right)\right), \\ Z_T = 1 - \gamma X_T. \end{cases} \tag{12}$$

for $(X, \pi, Z, \theta_2, \theta_3, \theta_4) \in L^2_{\mathcal{F}} \times \Pi \times L^2_{\mathcal{F}} \times L^2(\mathcal{P}) \times L^2(\widetilde{\mathcal{P}}) \times L^2(\widetilde{\mathcal{P}})$.

Proof. Suppose π^* satisfies condition (11). It is clear $X^{x_0,\pi^*} \in L^2_{\mathcal{F}}$ solves the forward SDE in (12) for X and (Z_t^*) is a square-integrable martingale. For any stopping time $\tau \leq T$, let $\pi_t^\tau = 1_{(t \leq \tau)}$, then $\pi^\tau \in \Pi$. Substituting π^τ into (11), we have

$$\begin{aligned} & E\left[Z_T^* \int_0^T \left((b-r)e^{-rs}ds + \sigma e^{-rs}dW_s^{(1)} + \sigma e^{-rs}dL_s^{(1)}\right)\right], \\ & = E\left[E\left[Z_T^*|\mathcal{F}_\tau\right] \int_0^\tau \left((b-r)e^{-rs}ds + \sigma e^{-rs}dW_s^{(1)} + \sigma e^{-rs}dL_s^{(1)}\right)\right], \\ & = E\left[Z_\tau^* \int_0^\tau \left((b-r)e^{-rs}ds + \sigma e^{-rs}dW_s^{(1)} + \sigma e^{-rs}dL_s^{(1)}\right)\right], \end{aligned} \tag{13}$$

is constant over all stopping time $\tau \leq T$ a.s., which implies that

$$Z_t^* \int_0^t \left((b-r)e^{-rs}ds + \sigma e^{-rs}dW_s^{(1)} + \sigma e^{-rs}dL_s^{(1)}\right), \tag{14}$$

is a martingale. Since Z_t^* is a square integrable martingale, then by Lemma 1, there exists $\theta = (\theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*) \in \Theta^2$ such that

$$\begin{aligned} dZ_t^* = & \sum_{i=1}^{2} \theta_i^*(t) dW_t^{(i)} \\ & + d\left(\sum_{i=3}^{4} \int_0^t \int_R \theta_i(s,x) \left(\mu_{i-2}(ds,dx) - \nu_{i-2}(ds,dx)\right)\right). \end{aligned} \tag{15}$$

By Itô's formula, we have

$$\begin{aligned} & d\left(Z_t^* \int_0^t \left((b-r)e^{-rs}ds + \sigma e^{-rs}dW_s^{(1)} + \sigma e^{-rs}dL_s^{(1)}\right)\right), \\ & = \left((b-r)e^{-rt}Z_{t-}^* + \sigma e^{-rt}\theta_1^*(t)\right)dt + local \quad martingale, \end{aligned} \tag{16}$$

which together with (14) implies, $(b-r)Z_{t-}^* + \sigma\theta_1^*(t) = 0$, i.e.

$$\theta_1^*(t) = -\frac{b-r}{\sigma} Z_{t-}^*. \tag{17}$$

By (15), we know $(Z^*, \theta_2^*, \theta_3^*, \theta_4^*)$ solves the backward SDE in (12), and hence $\left(X^{x_0,\pi^*}, \pi^*, Z^*, \theta_2^*, \theta_3^*, \theta_4^*\right)$ solves the following forward-backward stochastic differential equation (12).

Conversely, suppose there exists $(Z^*, \theta_2^*, \theta_3^*, \theta_4^*) \in L_{\mathcal{F}}^2 \times L^2(\mathcal{P}) \times L^2(\widetilde{\mathcal{P}}) \times L^2(\widetilde{\mathcal{P}})$ such that $\left(X^{x_0,\pi^*}, \pi^*, Z^*, \theta_2^*, \theta_3^*, \theta_4^*\right)$ solves the following forward-backward stochastic differential equation (12). It is easy to verify that for any $\pi \in \Pi$, by Itô's formula that $(Z_t^\pi M_t^\pi)$ is a local martingale, where

$$M_t^\pi := \int_0^t \left((b-r)\pi_s e^{-rs}ds + \sigma\pi_s e^{-rs}dW_s^{(1)} + \sigma\pi_s e^{-rs}dL_s^{(1)}\right).$$

Furthermore, for any $\pi \in \Pi$, we have $M^\pi \in L_{\mathcal{F}}^2$ and hence

$$E\left[\sup_{0\leq t\leq T} |Z_t^* M_t^\pi|\right] \leq \left(E\left[\sup_{0\leq t\leq T} |Z_t^*|^2\right]\right)^{\frac{1}{2}} \left(E\left[\sup_{0\leq t\leq T} |M_t^*|^2\right]\right)^{\frac{1}{2}} < \infty,$$

which implies that the family

$$\{Z_\tau^* M_\tau^\pi : \tau \quad is \quad a \quad stopping \quad time \quad and \quad \tau \leq T\},$$

is uniformly integrable and hence $Z^* M^\pi$ is a martingale. Thus we have $E[Z_T^* M_T^*] = 0$ for any $\pi \in \Pi$, implying that π^* satisfies condition (11). □

Now let us give the main result of this note.

Theorem 1. *π^* is the optimal strategy for the quadratic utility function $U(x) = x - \frac{\gamma}{2}x^2$ where*

$$\begin{cases} \pi_t^* = \frac{(b-r)Z_{t-}^*}{\gamma\sigma^2} \exp\left((T-t)\left(\frac{(b-r)^2}{\sigma^2} - r\right)\right) - \frac{\beta\rho}{\sigma}, \\ Z_0^* = \left(1 - \gamma e^{rT} x_0 - \gamma \frac{c(e^{rT}-1)}{r} + \gamma(b-r)\beta\rho\frac{e^{rT}-1}{r\sigma}\right) e^{-\frac{(b-r)^2}{\sigma^2}T}, \\ M_t = \exp\left(-\frac{b-r}{\sigma}W_t^{(1)} - \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 t\right), \\ Z_t^* = M_t(Z_0^* - \beta\gamma\sqrt{1-\rho^2}\int_0^t M_s^{-1} \exp\left((T-s)\left(r - \frac{(b-r)^2}{\sigma^2}\right) dW_s^{(2)}\right) \\ \qquad -\sigma\gamma \int_0^t M_s^{-1}\pi_s \exp\left((T-s)\left(r - \frac{(b-r)^2}{\sigma^2}\right)\right) dL_s^{(1)} \\ \qquad +\gamma \int_0^t M_s^{-1} \exp\left((T-s)\left(r - \frac{(b-r)^2}{\sigma^2}\right)\right) dL_s^{(2)}). \end{cases} \tag{18}$$

Proof. In what follows, we will solve FBSDE (12) by two steps, first, conjecture the form of solution; next, verify it.

Step 1: Put

$$A_t = \exp\left(\int_0^t a_s ds\right), \quad t \in [0, T],$$

where (a_t) is a non-random Lebesgue-integrable function to be determined. If $(X, \pi, Z, \theta_2, \theta_3, \theta_4)$ solves FBSDE (12), then by Itô's formula, we have

$$\begin{aligned} A_T Z_T &= Z_0 + \int_0^T A_s dZ_s + \int_0^T Z_{s-} dA_s, \\ &= Z_0 - \int_0^T \frac{b-r}{\sigma} A_s Z_{s-} dW_s^{(1)} + \int_0^T A_s \theta_2(s) dW_s^{(2)} + \int_0^T Z_{s-} A_s a_s ds \\ &\quad + \left(\sum_{i=3}^4 \int_0^T \int_{\mathbb{R}} A_s \theta_i(s,x)\left(\mu_{i-2}(ds,dx) - \nu_{i-2}(ds,dx)\right)\right), \end{aligned} \tag{19}$$

which implies

$$\begin{aligned} \frac{1-Z_T}{\gamma} &= \frac{1}{\gamma} - \frac{Z_0}{\gamma A_T} + \frac{1}{\gamma A_T}\int_0^T \frac{b-r}{\sigma} A_s Z_{s-} dW_s^{(1)} \\ &\quad - \frac{1}{\gamma A_T}\int_0^T A_s \theta_2(s) dW_s^{(2)} - \frac{1}{\gamma A_T}\int_0^T Z_{s-} A_s a_s ds \\ &\quad - \frac{1}{\gamma A_T}\left(\sum_{i=3}^4 \int_0^T \int_{\mathbb{R}} A_s \theta_i(s,x)\left(\mu_{i-2}(ds,dx) - \nu_{i-2}(ds,dx)\right)\right). \end{aligned} \tag{20}$$

Let $(X, \pi, Z, \theta_2, \theta_3, \theta_4)$ solves FBSDE (12), then there must be $X_T = \frac{1-Z_T}{\gamma}$. Comparing $dW_t^{(1)}$-term, $dW_t^{(2)}$-term and $d(\mu_i - \nu_i)$-term ($i = 1, 2$) respectively in (20) with those in (6). It is reasonable to conjecture that

$$\frac{b-r}{\gamma\sigma A_T} A_t Z_{t-} = e^{r(T-t)}(\sigma\pi_t + \beta\rho), \quad -\frac{A_t\theta_2(t)}{\gamma A_T} = \beta\sqrt{1-\rho^2} e^{r(T-t)},$$

$$-\frac{A_t\theta_3(t,x)}{\gamma A_T}=x\sigma\pi_t e^{r(T-t)},\quad \frac{A_t\theta_4(t,x)}{\gamma A_T}=xe^{r(T-t)}.$$

i.e.,

$$\begin{cases}\pi_t=\frac{(b-r)A_t}{\gamma\sigma^2 A_T}Z_{t-}e^{-r(T-t)}-\frac{\beta\rho}{\sigma},\\ \theta_2(t)=-\frac{\beta\gamma\sqrt{1-\rho^2}A_T}{A_t}e^{r(T-t)},\\ \theta_3(t,x)=-\frac{\gamma\sigma\pi_t A_T}{A_t}e^{r(T-t)}x,\\ \theta_4(t,x)=\frac{\gamma A_T}{A_t}e^{r(T-t)}x.\end{cases}\tag{21}$$

Substituting (21) into (20), then we have

$$\begin{aligned}\frac{1-Z_T}{\gamma}&=\frac{1}{\gamma}-\frac{Z_0}{\gamma A_T}+\int_0^T(\sigma\pi_s+\beta\rho)e^{r(T-s)}dW_s^{(1)}+\int_0^T\beta\sqrt{1-\rho^2}e^{r(T-s)}dW_s^{(2)}\\&\quad+\int_0^T\sigma\pi_s e^{r(T-s)}dL_s^{(1)}-\int_0^T e^{r(T-s)}dL_s^{(2)}-\frac{1}{\gamma A_T}\int_0^T Z_{s-}A_s a_s ds,\\&=\frac{1}{\gamma}-\frac{Z_0}{\gamma A_T}+X_T-e^{rT}x_0-\frac{c(e^{rT}-1)}{r}-(b-r)\int_0^T\pi_s e^{r(T-s)}ds\\&\quad-\frac{1}{\gamma A_T}\int_0^T Z_{s-}A_s a_s ds,\\&=X_T+\frac{1}{\gamma}-\frac{Z_0}{\gamma A_T}-e^{rT}x_0-\frac{c(e^{rT}-1)}{r}-\int_0^T\frac{(b-r)^2A_sZ_{s-}}{\gamma\sigma^2A_T}ds\\&\quad+\int_0^T\frac{(b-r)\beta\rho e^{r(T-s)}}{\sigma}ds-\frac{1}{\gamma A_T}\int_0^T Z_{s-}A_s a_s ds,\end{aligned}\tag{22}$$

where the second equality follows from (6). If we take $a_t=-\frac{(b-r)^2}{\sigma^2}$ and

$$\begin{aligned}Z_0&=A_T-\gamma A_T e^{rT}x_0-\gamma A_T\frac{c(e^{rT}-1)}{r}+\gamma A_T\int_0^T\frac{(b-r)\beta\rho e^{r(T-s)}}{r}ds,\\&=\left(1-\gamma e^{rT}x_0-\gamma\frac{c(e^{rT}-1)}{r}+\gamma(b-r)\beta\rho\frac{e^{rT}-1}{\sigma r}\right)e^{-\frac{(b-r)^2}{\sigma^2}T}.\end{aligned}$$

then (22) is reduced as $\frac{1-Z_T}{\gamma}=X_T$, and FBSDE (12) is solved.

Step 2: Now let us verify the conjectures in **Step 1**. Let

$$\begin{aligned}\theta_2^*(t)&=-\beta\gamma\sqrt{1-\rho^2}\exp\left((T-t)\left(r-\frac{(b-r)^2}{\sigma^2}\right)\right),\\ \theta_3^*(t,x)&=-\sigma\gamma x\pi_t\exp\left((T-t)\left(r-\frac{(b-r)^2}{\sigma^2}\right)\right),\\ \theta_4^*(t,x)&=\gamma x\exp\left((T-t)\left(r-\frac{(b-r)^2}{\sigma^2}\right)\right).\end{aligned}$$

and

$$Z_0^*=\left(1-\gamma e^{rT}x_0-\gamma\frac{c(e^{rT}-1)}{r}+\gamma(b-r)\beta\rho\frac{e^{rT}-1}{\sigma r}\right)e^{-\frac{(b-r)^2}{\sigma^2}T},\tag{23}$$

then the following SDE

$$dZ_t = -\frac{b-r}{\sigma} Z_{t-} dW_t^{(1)} + \theta_2^*(t) dW_t^{(2)} + d\left(\sum_{i=3}^{4} \int_0^t \int_{\mathbb{R}} \theta_i^*(s,x) \left(\mu_{i-2}(ds,dx) - \nu_{i-2}(ds,dx)\right)\right),$$

admits a solution

$$\begin{aligned} Z_t^* &= M_t \left(Z_0^* + \int_0^t M_s^{-1} \theta_2^*(s) dW_s^{(2)} + \left(\sum_{i=3}^{4} \int_0^t \int_{\mathbb{R}} M_s^{-1} \theta_i^*(s,x) \left(\mu_{i-2}(ds,dx) - \nu_{i-2}(ds,dx)\right) \right) \right) \\ &= M_t \left(Z_0^* - \beta\gamma\sqrt{1-\rho^2} \int_0^t M_s^{-1} \exp\left((T-s)\left(r - \frac{(b-r)^2}{\sigma^2} \right) \right) dW_s^{(2)} \right) \\ &\quad - \sigma\gamma M_t \int_0^t M_s^{-1} \pi_s \exp\left((T-s)\left(r - \frac{(b-r)^2}{\sigma^2} \right) \right) dL_s^{(1)} \\ &\quad + \gamma M_t \int_0^t M_s^{-1} \exp\left((T-s)\left(r - \frac{(b-r)^2}{\sigma^2} \right) \right) dL_s^{(2)}, \end{aligned} \tag{24}$$

where

$$M_t = \exp\left(-\frac{b-r}{\sigma} W_t^{(1)} - \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 t \right). \tag{25}$$

Furthermore, let

$$\pi_t^* = \frac{(b-r)Z_{t-}^*}{\gamma\sigma^2} \exp\left((T-t)\left(\frac{(b-r)^2}{\sigma^2} - r \right) \right) - \frac{\beta\rho}{\sigma}, \tag{26}$$

then by the same procedure as in (22), it is easy to verify that

$$\frac{1-Z_T}{\gamma} = X_T, \quad i.e. \quad 1 - \gamma X_T^{x_0,\pi^*} = Z_T^*.$$

Thus $\left(X^{x_0,\pi^*}, \pi^*, Z^*, \theta_2^*, \theta_3^*, \theta_4^*\right)$ solves the forward-backward stochastic differential equation (12).

Finally, by Proposition 1, Lemma 1 and the arguments in **Step 1-2**, we complete the proof of this theorem. □

3.2. Efficient frontier

In this subsection, we apply the results established in the previous subsection to the mean-variance problem.

Since Z^* is a square-integrable martingale and $X_T^{x_0,\pi^*} = \frac{1-Z_T^*}{\gamma}$, we have

$$\begin{cases} E\left[X_T^{x_0,\pi^*}\right] = \frac{1-Z_0^*}{\gamma}, \\ Var\left[X_T^{x_0,\pi^*}\right] = \frac{1}{\gamma^2} Var\left[Z_T^*\right] = \frac{1}{\gamma^2}\left(E[(Z_T^*)^2] - (Z_0^*)^2\right). \end{cases} \tag{27}$$

In order to avoid some tedious technical arguments in the following, we assume that Z^* is non-negative. Let $f(t) = E\left[(Z_t^*)^2\right]$. Since Z^* is a non-negative square-integrable martingale, we have

$$\begin{aligned} f(t) &= E\left[(Z_t^*)^2\right] \\ &= \int_0^t \left(1+\int_{\mathbb{R}} x^2 m_1(dx)\right)\left(\frac{b-r}{\sigma}\right)^2 f(s)ds + h(t), \end{aligned} \tag{28}$$

where

$$\begin{aligned} h(t) = &\frac{\sigma^2\gamma^2\left[\beta^2\rho^2\int_{\mathbb{R}} x^2 m_1(dx) + \int_{\mathbb{R}} x^2 m_2(dx) + \beta^2(1-\rho^2)\right]}{2(b-r)^2 - 2r\sigma^2} \\ &\times \left[\exp\left(2\left(r - \frac{(b-r)^2}{\sigma^2}\right)(T-t)\right) - \exp\left(2\left(r-\frac{(b-r)^2}{\sigma^2}\right)T\right)\right] \\ &- \frac{2\sigma\gamma\beta\rho(b-r)Z_0^*}{(b-r)^2 - r\sigma^2}\left[\exp\left(\left(r-\frac{(b-r)^2}{\sigma^2}\right)(T-t)\right) - \exp\left(\left(r-\frac{(b-r)^2}{\sigma^2}\right)T\right)\right]. \end{aligned} \tag{29}$$

Solving Eq. (28), we obtain

$$\begin{aligned} &E\left[(Z_T^*)^2\right] = f(T) \\ &= \exp\left(\left(1+\int_{\mathbb{R}} x^2 m_1(dx)\right)\left(\frac{b-r}{\sigma}\right)^2 T\right) \\ &\quad \times \left((Z_0^*)^2 + \int_0^T \exp\left(-\left(1+\int_{\mathbb{R}} x^2 m_1(dx)\right)\left(\frac{b-r}{\sigma}\right)^2 t\right) dh(t)\right). \end{aligned} \tag{30}$$

Substituting Eq. (29) into Eq. (30), we have

$$\begin{aligned} &E\left[(Z_T^*)^2\right] \\ &= \exp\left(\left(1+\int_{\mathbb{R}} x^2 m_1(dx)\right)\left(\frac{b-r}{\sigma}\right)^2 T\right)(Z_0^*)^2 \\ &+ \frac{\gamma^2\left[\beta^2\rho^2\int_{\mathbb{R}} x^2 m_1(dx) + \int_{\mathbb{R}} x^2 m_2(dx) + \beta^2(1-\rho^2)\right]}{\left(1-\int_{\mathbb{R}} x^2 m_1(dx)\right)\left(\frac{b-r}{\sigma}\right)^2 - 2r} \\ &\qquad \times \left[1 - \exp\left(T\left(2r + \int_{\mathbb{R}} x^2 m_1(dx)\left(\frac{b-r}{\sigma}\right)^2 - \left(\frac{b-r}{\sigma}\right)^2\right)\right)\right] \\ &+ \frac{2\gamma\beta\rho(b-r)(Z_0^*)}{\sigma\left(\int_{\mathbb{R}} x^2 m_1(dx)\left(\frac{b-r}{\sigma}\right)^2 + r\right)}\left[1 - \exp\left(\left(\int_{\mathbb{R}} x^2 m_1(dx)\left(\frac{b-r}{\sigma}\right)^2 + r\right)T\right)\right]. \end{aligned} \tag{31}$$

Substituting Eq. (31) and Eq. (23) into Eq. (27), and eliminating γ, we obtain the efficient frontier parameterized by z as follows

$$
\begin{cases}
E\left[X_T^{x_0,\pi^*}\right] = z, \\
Var\left[X_T^{x_0,\pi^*}\right] = \frac{\left[\beta^2\rho^2\int_{\mathbb{E}} x^2 m_1(dx) + \int_{\mathbb{E}} x^2 m_2(dx) + \beta^2(1-\rho^2)\right]}{\left(1-\int_{\mathbb{E}} x^2 m_1(dx)\right)\left(\frac{b-r}{\sigma}\right)^2 - 2r} \\
\quad \times \left[1 - \exp\left(T\left(2r + \int_{\mathbb{E}} x^2 m_1(dx)\left(\frac{b-r}{\sigma}\right)^2 - \left(\frac{b-r}{\sigma}\right)^2\right)\right)\right] \\
\quad + \frac{\exp\left(\left(1+\int_{\mathbb{E}} x^2 m_1(dx)\right)\left(\frac{b-r}{\sigma}\right)^2 T\right) - 1}{\left[\exp\left(\left(\frac{b-r}{\sigma}\right)^2 T\right) - 1\right]^2}\left(z - x_0 e^{rT} + \frac{e^{rT}-1}{r\sigma}[\beta\rho(b-r) - c\sigma]\right)^2 \\
\quad + \frac{2\beta\rho(b-r)}{\sigma\left[\int_{\mathbb{E}} x^2 m_1(dx)\left(\frac{b-r}{\sigma}\right)^2 + r\right]\left[\exp\left(\left(\frac{b-r}{\sigma}\right)^2 T\right) - 1\right]} \\
\quad \times \left[1 - \exp\left(\left(\int_{\mathbb{E}} x^2 m_1(dx)\left(\frac{b-r}{\sigma}\right)^2 + r\right)T\right)\right] \\
\quad \times \left(z - x_0 e^{rT} + \frac{e^{rT}-1}{r\sigma}[\beta\rho(b-r) - c\sigma]\right).
\end{cases}
\tag{32}
$$

Remark 3. From Equqtion(32), we can see that if we do not consider the term $\sigma S(t)dL_t^{(1)}$ in Equation(1), the efficient frontier in this paper is the same as that in Wang et al.[7].

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