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THE VARIATIONAL HOMOTOPY PERTURBATION METHOD FOR ANALYTIC TREATMENT FOR LINEAR AND NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. In a recent paper, M.A. Noor et al. (Hindawi publishing corporation, Mathematical Problems in Engineering, Volume 2008, Article ID 696734, 11 pages, doi:10.1155/2008/696734) proposed the variational homotopy perturbation method (VHPM) for solving higher dimentional initial boundary value problems. In this paper, we consider the proposed method for analytic treatment of the linear and nonlinear ordinary differential equations, homogeneous or inhomogeneous. The results reveal that the proposed method is very effective and simple and can be applied for other linear and nonlinear problems in mathematical.

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1. Introduction

The VHPM provides the solution in a rapid convergent series which may lead the solution in a closed form. A vast amount of research work has been invested in the study of the linear and nonlinear ordinary differential equations. There are standard methods that are used for ODEs where each method works for specific equations depending on its order. It was the aim for many to develop a powerful unified method to handle most of these equations. The VHPM can effectively help us to achieve this goal.
Numerical results reveal that the VHPM is easy to implement and reduces the computational work to a tangible level while still maintaining a very higher level of accuracy [5].

2. Variational homotopy perturbation method

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To convey the basic idea of the variational homotopy perturbation method, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is an inhomogeneous term. According to variational iteration method [1–3], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\left\{Lu_n + N\tilde{u}_n - g(\tau)\right\} d\tau, \tag{2}$$

where $\lambda(\tau)$ is a Lagrange multiplier [1–3] which can be identified optimally via the variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta\tilde{u}_n = 0$ and (2) is called a correct functional. Now, we apply the homotopy perturbation method;

$$\sum_{i=0}^{\infty} p^i u_i = u_0 + p\int_0^x \lambda(\tau)\left\{N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right\} d\tau - p\int_0^x \lambda(\tau) g(\tau) d\tau, \tag{3}$$

which is the variational homotopy perturbation method and is formulated by the coupling of variational iteration method and Adomian's polynomials. A comparison of like powers of p gives solutions of various orders.

3. First order ODEs

We first start our analysis by studying the first order linear ODE of a standard form

$$u' + p(x)u = q(x), u(0) = \alpha, \tag{4}$$

for solving equation (4) by using the VHPM, we consider

$$L(u) = u', \tag{5}$$

$$N(u) = p(x)u - q(x), \tag{6}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [1–3], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\left\{u_{n_\tau}(\tau) + p(\tau)\tilde{u}_n(\tau) - q(\tau)\right\} d\tau, \tag{7}$$

where $\tilde{u}_n$ is considered as a restricted variation and $u_{n_\tau} = \frac{\partial u_n}{\partial \tau}$. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -1$ (see [4]), which yields the following iteration formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left\{u_{n_\tau}(\tau) + p(\tau)u_n(\tau) - q(\tau)\right\} d\tau. \tag{8}$$

Applying the variational homotopy perturbation method, we have:

$$u_0 + p\,u_1 + p^2u_2 + \cdots = \alpha - p\int_0^x p(\tau)\left(u_0 + p\,u_1 + p^2u_2 + \cdots\right)\,d\tau + p\int_0^x q(\tau)d\tau. \qquad (9)$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned}
p^0 : u_0(x) &= \alpha,\\
p^1 : u_1(x) &= -\int_0^x p(\tau)u_0 d\tau + \int_0^x q(\tau)d\tau,\\
p^2 : u_2(x) &= -\int_0^x p(\tau)u_1 d\tau,\\
p^3 : u_3(x) &= -\int_0^x p(\tau)u_2 d\tau,\\
&\vdots
\end{aligned}$$

Thus we will have: $u(x) = u_0 + u_1 + u_2 + \cdots$, that may give the exact solution if a closed form solution exists or we can use the $(\sum_{i=0}^{n} u_i)$ as nth approximation for numerical purposes. In what follows, we will apply the VHPM method to two physical models to illustrate the strength of the method and to establish exact solutions for this models.

Example 1. Solve the following first order homogeneous ODE

$$u' - 2xu = 0, u(0) = 1, \qquad (10)$$

for solving equation (10) by using the VHPM, we consider

$$L(u) = u'(x), \qquad (11)$$

$$N(u) = -2xu(x), \qquad (12)$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [1–3], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\left\{u_{n_\tau}(\tau) - 2\tau\tilde{u}_n(\tau)\right\}d\tau, \qquad (13)$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -1$, which yields the following iteration formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left\{u_{n_\tau}(\tau) - 2\tau u_n(\tau)\right\}d\tau. \qquad (14)$$

Applying the variational homotopy perturbation method, we have:

$$u_0 + p\,u_1 + p^2u_2 + \cdots = 1 + p\int_0^x 2\tau\left(u_0 + p\,u_1 + p^2u_2 + \cdots\right)\,d\tau. \qquad (15)$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned}
p^0 : u_0(x) &= 1,\\
p^1 : u_1(x) &= \int_0^x 2\tau u_0 d\tau = x^2,\\
p^2 : u_2(x) &= \int_0^x 2\tau u_1 d\tau = \frac{1}{2!}x^4,\\
p^3 : u_3(x) &= \int_0^x 2\tau u_2 d\tau = \frac{1}{3!}x^6,\\
p^4 : u_4(x) &= \int_0^x 2\tau u_3 d\tau = \frac{1}{4!}x^8,\\
&\vdots
\end{aligned}$$

Thus we will have:

$$u(x) = u_0 + u_1 + u_2 + \cdots = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \cdots,$$

so the exact solution is obtained as follows:

$$u(x) = e^{x^2}. \tag{16}$$

Example 2. Solve the following first order nonhomogeneous ODE

$$u' - u = e^x, u(0) = 0, \tag{17}$$

for solving equation (17) by using the VHPM, we consider

$$L(u) = u'(x), \tag{18}$$

$$N(u) = -u(x) - e^x, \tag{19}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [1–3], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\left\{u_{n_\tau}(\tau) - \tilde{u}_n(\tau) - e^\tau\right\} d\tau, \tag{20}$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -1$, which yields the following iteration formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left\{u_{n_\tau}(\tau) - u_n(\tau) - e^\tau\right\} d\tau. \tag{21}$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned}
u_0 + p\, u_1 + p^2 u_2 + \cdots &= 0 + p\int_0^x \left(u_0 + p\, u_1 + p^2 u_2 + \cdots\right)\, d\tau\\
&\quad + p\int_0^x e^\tau d\tau.
\end{aligned} \tag{22}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned}
p^0 : u_0(x) &= 0, \qquad p^1 : u_1(x) = \int_0^x u_0 d\tau + \int_0^x e^\tau d\tau = e^x - 1,\\
p^2 : u_2(x) &= \int_0^x u_1 d\tau = e^x - x - 1,\\
p^3 : u_3(x) &= \int_0^x u_2 d\tau = e^x - \frac{1}{2}x^2 - x - 1,\\
p^4 : u_4(x) &= \int_0^x u_3 d\tau = e^x - \frac{1}{6}x^3 - \frac{1}{2}x^2 - x - 1,\\
&\vdots
\end{aligned}$$

Thus we will have:

$$\begin{aligned}
u(x) = u_0 + u_1 + u_2 + \cdots = 0 &+ (e^x - 1)\\
&+ (e^x - x - 1)\\
&+ \left(e^x - \frac{1}{2}x^2 - x - 1\right)\\
&+ \left(e^x - \frac{1}{6}x^3 - \frac{1}{2}x^2 - x - 1\right)\\
&+ \cdots,
\end{aligned}$$

summarizing the above, we have: $u(x) = x\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots\right)$, by using the Taylor series the exact solution is obtained as follows:

$$u(x) = xe^x. \tag{23}$$

4. Second order ODEs

We now extend our analysis to the second order linear ODE with constant coefficients given by

$$u''(x) + au'(x) + bu(x) = g(x), u(0) = \alpha, u'(0) = \beta, \tag{24}$$

for solving equation (24) by using the VHPM, we consider

$$L(u) = u''(x), \tag{25}$$

$$N(u) = au'(x) + bu(x) - g(x), \tag{26}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [1–3], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\left\{u_{n_{\tau\tau}}(\tau) + a\tilde{u}_{n_\tau}(\tau) + b\tilde{u}_n(\tau) - g(\tau)\right\} d\tau, \tag{27}$$

where $\tilde{u}_n$ is considered as a restricted variation and $u_{n_\tau} = \frac{\partial u_n}{\partial \tau}, u_{n_{\tau\tau}} = \frac{\partial^2 u_n}{\partial \tau^2}$. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \tau - x$ (see [4]), which yields the following iteration formula:

$$u_{n+1}(x) = u_n(x) + \int_0^x (\tau - x)\left\{u_{n_{\tau\tau}}(\tau) + a u_{n_\tau}(\tau) + b u_n(\tau) - g(\tau)\right\} d\tau. \tag{28}$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned} u_0 + p\, u_1 + p^2 u_2 + \cdots &= \alpha + \beta x \\ &+ ap \int_0^x (\tau - x)\left(u_{0_\tau} + p\, u_{1_\tau} + p^2 u_{2_\tau} + \cdots\right) d\tau \\ &+ bp \int_0^x (\tau - x)\left(u_0 + p\, u_1 + p^2 u_2 + \cdots\right) d\tau \\ &- p \int_0^x (\tau - x) g(\tau) d\tau. \end{aligned} \tag{29}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} p^0 : u_0(x) &= \alpha + \beta x, \\ p^1 : u_1(x) &= a \int_0^x (\tau - x) u_{0_\tau} d\tau + b \int_0^x (\tau - x) u_0 d\tau - \int_0^x (\tau - x) g(\tau) d\tau, \\ p^2 : u_2(x) &= a \int_0^x (\tau - x) u_{1_\tau} d\tau + b \int_0^x (\tau - x) u_1 d\tau, \\ p^3 : u_3(x) &= a \int_0^x (\tau - x) u_{2_\tau} d\tau + b \int_0^x (\tau - x) u_2 d\tau, \\ &\vdots \end{aligned}$$

Thus we will have: $u(x) = u_0 + u_1 + u_2 + \cdots$, that may give the exact solution if a closed form solution exists or we can use the $(\sum_{i=0}^{n} u_i)$ as nth approximation for numerical purposes. In what follows, we will apply the VHPM method as presented before to two models of the second order ODEs.

Example 1. Solve the following second order homogeneous ODE

$$u''(x) + u(x) = 0, u(0) = 1, u'(0) = 1, \tag{30}$$

for solving equation (30) by using the VHPM, we consider

$$L(u) = u''(x), \tag{31}$$

$$N(u) = u(x), \tag{32}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [1–3], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\left\{u_{n_{\tau\tau}}(\tau) + \tilde{u}_n(\tau)\right\} d\tau, \tag{33}$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \tau - x$, which yields the following iteration formula:

$$u_{n+1}(x) = u_n(x) + \int_0^x (\tau - x)\left\{u_{n_{\tau\tau}}(\tau) + u_n(\tau)\right\} d\tau. \tag{34}$$

Applying the variational homotopy perturbation method, we have:

$$u_0 + p\, u_1 + p^2 u_2 + \cdots = 1 \; + \; x \\ + \; p \int_0^x (\tau - x)\left(u_0 + p\, u_1 + p^2 u_2 + \cdots\right) d\tau. \tag{35}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned}
p^0 : u_0(x) &= 1 + x,\\
p^1 : u_1(x) &= \int_0^x (\tau - x) u_0 d\tau = -\frac{1}{2!}x^2 - \frac{1}{3!}x^3,\\
p^2 : u_2(x) &= \int_0^x (\tau - x) u_1 d\tau = \frac{1}{4!}x^4 + \frac{1}{5!}x^5,\\
p^3 : u_3(x) &= \int_0^x (\tau - x) u_2 d\tau = -\frac{1}{6!}x^6 - \frac{1}{7!}x^7,\\
&\vdots
\end{aligned}$$

Thus we will have:

$$u(x) = u_0 + u_1 + u_2 + \cdots = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right),$$

so the exact solution is obtained as follows:

$$u(x) = \cos(x) + \sin(x). \tag{36}$$

Example 2. Solve the following second order nonhomogeneous ODE

$$u''(x) - 3u'(x) + 2u(x) = 2x - 3, u(0) = 1, u'(0) = 2, \tag{37}$$

for solving equation (37) by using the VHPM, we consider

$$L(u) = u''(x), \tag{38}$$

$$N(u) = -3u'(x) + 2u(x) - 2x + 3, \tag{39}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [1–3], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\left\{u_{n_{\tau\tau}}(\tau) - 3\tilde{u}_{n_\tau}(\tau) + 2\tilde{u}_n(\tau) - 2\tau + 3\right\} d\tau, \tag{40}$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \tau - x$, which yields the following iteration formula:

$$u_{n+1}(x) = u_n(x) + \int_0^x (\tau - x)\left\{u_{n_{\tau\tau}}(\tau) - 3u_{n_\tau}(\tau) + 2u_n(\tau) - 2\tau + 3\right\} d\tau. \tag{41}$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned} u_0 + p\,u_1 + p^2 u_2 + \cdots = 1 &+ 2x \\ &- 3p\int_0^x (\tau - x)\left(u_{0_\tau} + p\,u_{1_\tau} + p^2 u_{2_\tau} + \cdots\right) d\tau \\ &+ 2p\int_0^x (\tau - x)\left(u_0 + p\,u_1 + p^2 u_2 + \cdots\right) d\tau \\ &+ p\int_0^x (\tau - x)(-2\tau + 3)\ d\tau. \end{aligned} \tag{42}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} p^0 : u_0(x) &= 1 + 2x, \\ p^1 : u_1(x) &= -3\int_0^x (\tau - x) u_{0_\tau} d\tau + 2\int_0^x (\tau - x) u_0 d\tau + \int_0^x (\tau - x)(-2\tau + 3)\ d\tau \\ &= \frac{1}{2!}x^2 - \frac{1}{3}x^3, \\ p^2 : u_2(x) &= -3\int_0^x (\tau - x) u_{1_\tau} d\tau + 2\int_0^x (\tau - x) u_1 d\tau \\ &= \frac{1}{2}x^3 - \frac{1}{3}x^4 + \frac{1}{30}x^5, \\ p^3 : u_3(x) &= -3\int_0^x (\tau - x) u_{2_\tau} d\tau + 2\int_0^x (\tau - x) u_2 d\tau \\ &= \frac{3}{8}x^4 - \frac{1}{4}x^5 + \frac{7}{180}x^6 - \frac{1}{630}x^7, \\ p^4 : u_4(x) &= -3\int_0^x (\tau - x) u_{3_\tau} d\tau + 2\int_0^x (\tau - x) u_3 d\tau \\ &= \frac{5103}{22680}x^5 - \frac{3402}{22680}x^6 + \frac{648}{22680}x^7 - \frac{45}{22680}x^8 + \frac{1}{22680}x^9, \\ &\ \ \vdots \end{aligned}$$

Thus we will have:

$$u(x) = u_0 + u_1 + u_2 + \cdots = x + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right),$$

so the exact solution is obtained as follows:

$$u(x) = x + e^x. \tag{43}$$

In next section we consider the third order linear ODEs with constant coefficients given by

$$u'''(x) + au''(x) + bu'(x) + cu(x) = g(x), u(0) = \alpha, u'(0) = \beta, u''(0) = \gamma.$$

5. Third order ODEs

We now consider the third order linear ODE with constant coefficients given by

$$u'''(x) + au''(x) + bu'(x) + cu(x) = g(x), u(0) = \alpha, u'(0) = \beta, u''(0) = \gamma, \tag{44}$$

for solving equation (44) by using the VHPM, we consider

$$L(u) = u'''(x), \tag{45}$$

$$N(u) = au''(x) + bu'(x) + cu(x) - g(x), \tag{46}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [1–3], we can construct a correct functional as follows:

$$\begin{aligned} u_{n+1}(x) &= u_n(x) \qquad (47) \\ &+ \int_0^x \lambda(\tau)\left\{u_{n_{\tau\tau\tau}}(\tau) + a\tilde{u}_{n_{\tau\tau}}(\tau) + b\tilde{u}_{n_\tau}(\tau) + c\tilde{u}_n(\tau) - g(\tau)\right\} d\tau, \end{aligned}$$

where $\tilde{u}_n$ is considered as a restricted variation and
$u_{n_\tau} = \frac{\partial u_n}{\partial \tau}, u_{n_{\tau\tau}} = \frac{\partial^2 u_n}{\partial \tau^2}, u_{n_{\tau\tau\tau}} = \frac{\partial^3 u_n}{\partial \tau^3}$.
Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -\frac{1}{2!}(\tau - x)^2$ (see [4]), which yields the following iteration formula:

$$\begin{aligned} &u_{n+1}(x) = u_n(x) \qquad (48) \\ &- \int_0^x \frac{1}{2!}(\tau - x)^2 \left\{u_{n_{\tau\tau\tau}}(\tau) + au_{n_{\tau\tau}}(\tau) + bu_{n_\tau}(\tau) + cu_n(\tau) - g(\tau)\right\} d\tau. \end{aligned}$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned} u_0 + p\, u_1 + p^2 u_2 + \cdots = \alpha + \beta x + \frac{1}{2}\gamma x^2 & \\ -ap \int_0^x \frac{1}{2!}(\tau - x)^2 \left(u_{0_{\tau\tau}} + p\, u_{1_{\tau\tau}} + p^2 u_{2_{\tau\tau}} + \cdots\right) d\tau & \\ -bp \int_0^x \frac{1}{2!}(\tau - x)^2 \left(u_{0_\tau} + p\, u_{1_\tau} + p^2 u_{2_\tau} + \cdots\right) d\tau & \\ -cp \int_0^x \frac{1}{2!}(\tau - x)^2 \left(u_0 + p\, u_1 + p^2 u_2 + \cdots\right) d\tau & \\ +p \int_0^x \frac{1}{2!}(\tau - x)^2 g(\tau) d\tau. & \qquad (49) \end{aligned}$$

Comparing the coefficient of like powers of p, we have:

$$
\begin{aligned}
p^0 : u_0(x) &= \alpha + \beta x + \frac{1}{2}\gamma x^2,\\
p^1 : u_1(x) &= -a\int_0^x \frac{1}{2!}(\tau - x)^2 u_{0_{\tau\tau}} d\tau - b\int_0^x \frac{1}{2!}(\tau - x)^2 u_{0_\tau} d\tau\\
&- c\int_0^x \frac{1}{2!}(\tau - x)^2 u_0 d\tau + \int_0^x \frac{1}{2!}(\tau - x)^2 g(\tau) d\tau,\\
p^2 : u_2(x) &= -a\int_0^x \frac{1}{2!}(\tau - x)^2 u_{1_{\tau\tau}} d\tau - b\int_0^x \frac{1}{2!}(\tau - x)^2 u_{1_\tau} d\tau\\
&- c\int_0^x \frac{1}{2!}(\tau - x)^2 u_1 d\tau,\\
p^3 : u_3(x) &= -a\int_0^x \frac{1}{2!}(\tau - x)^2 u_{2_{\tau\tau}} d\tau - b\int_0^x \frac{1}{2!}(\tau - x)^2 u_{2_\tau} d\tau\\
&- c\int_0^x \frac{1}{2!}(\tau - x)^2 u_2 d\tau,\\
&\vdots
\end{aligned}
$$

Thus we will have: $u(x) = u_0 + u_1 + u_2 + \cdots$, that may give the exact solution if it exists, otherwise we can use the $(\sum_{i=0}^{n} u_i)$ as nth approximation for numerical purposes. In what follows, we will apply the VHPM method as presented before to two models of the third order ODEs.

Example 1. Solve the following third order homogeneous ODE

$$u'''(x) - u'(x) = 0, u(0) = 1, u'(0) = 0, u''(0) = 1, \tag{50}$$

for solving equation (50) by using the VHPM, we consider

$$L(u) = u'''(x), \tag{51}$$

$$N(u) = -u'(x), \tag{52}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [1–3], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\{u_{n_{\tau\tau\tau}}(\tau) - \tilde{u}_{n_\tau}(\tau)\}\, d\tau, \tag{53}$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -\frac{1}{2!}(\tau - x)^2$, which yields the following iteration formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x \frac{1}{2!}(\tau - x)^2 \{u_{n_{\tau\tau\tau}}(\tau) - u_{n_\tau}(\tau)\}\, d\tau. \tag{54}$$

Applying the variational homotopy perturbation method, we have:

$$u_0 + p\,u_1 + p^2 u_2 + \cdots = 1 + \frac{1}{2}x^2 + p\int_0^x \frac{1}{2!}(\tau - x)^2 (u_{0_\tau} + p\,u_{1_\tau} + p^2 u_{2_\tau} + \cdots)d\tau. \quad (55)$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned}
p^0 : u_0(x) &= 1 + \frac{1}{2}x^2,\\
p^1 : u_1(x) &= \int_0^x \frac{1}{2!}(\tau - x)^2 u_{0_\tau} d\tau = \frac{1}{4!}x^4,\\
p^2 : u_2(x) &= \int_0^x \frac{1}{2!}(\tau - x)^2 u_{1_\tau} d\tau = \frac{1}{6!}x^6,\\
p^3 : u_3(x) &= \int_0^x \frac{1}{2!}(\tau - x)^2 u_{2_\tau} d\tau = \frac{1}{8!}x^8,\\
&\vdots
\end{aligned}$$

Thus we will have:

$$u(x) = u_0 + u_1 + u_2 + \cdots = \left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots\right),$$

so the exact solution is obtained as follows:

$$u(x) = \cosh(x). \quad (56)$$

Example 2. Solve the following third order nonhomogeneous ODE

$$u'''(x) - 2u''(x) + u'(x) = 1, u(0) = 0, u'(0) = 2, u''(0) = 2, \quad (57)$$

for solving equation (57) by using the VHPM, we consider

$$L(u) = u'''(x), \quad (58)$$

$$N(u) = -2u''(x) + u'(x) - 1, \quad (59)$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [1–3], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\left\{u_{n_{\tau\tau\tau}}(\tau) - 2\tilde{u}_{n_{\tau\tau}}(\tau) + \tilde{u}_{n_\tau}(\tau) - 1\right\} d\tau, \quad (60)$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -\frac{1}{2!}(\tau - x)^2$, which yields the following iteration formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x \frac{1}{2!}(\tau - x)^2 \left\{u_{n_{\tau\tau\tau}}(\tau) - 2u_{n_{\tau\tau}}(\tau) + u_{n_\tau}(\tau) - 1\right\} d\tau. \quad (61)$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned} u_0 + p\, u_1 + p^2 u_2 + \cdots = 2x + x^2 & \\ +2p \int_0^x \frac{1}{2!}(\tau - x)^2 \left(u_{0_{\tau\tau}} + p\, u_{1_{\tau\tau}} + p^2 u_{2_{\tau\tau}} + \cdots\right)\, d\tau & \\ -p \int_0^x \frac{1}{2!}(\tau - x)^2 \left(u_{0_{\tau}} + p\, u_{1_{\tau}} + p^2 u_{2_{\tau}} + \cdots\right)\, d\tau & \\ +p \int_0^x \frac{1}{2!}(\tau - x)^2\, d\tau. & \qquad (62) \end{aligned}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} p^0 : u_0(x) &= 2x + x^2, \\ p^1 : u_1(x) &= 2\int_0^x \frac{1}{2!}(\tau - x)^2 u_{0_{\tau\tau}} d\tau - \int_0^x \frac{1}{2!}(\tau - x)^2 u_{0_{\tau}} d\tau + \int_0^x \frac{1}{2!}(\tau - x)^2 d\tau \\ &= \frac{1}{2}x^3 - \frac{1}{12}x^4, \\ p^2 : u_2(x) &= 2\int_0^x \frac{1}{2!}(\tau - x)^2 u_{1_{\tau\tau}} d\tau - \int_0^x \frac{1}{2!}(\tau - x)^2 u_{1_{\tau}} d\tau \\ &= \frac{1}{4}x^4 - \frac{7}{120}x^5 + \frac{1}{360}x^6, \\ p^3 : u_3(x) &= 2\int_0^x \frac{1}{2!}(\tau - x)^2 u_{2_{\tau\tau}} d\tau - \int_0^x \frac{1}{2!}(\tau - x)^2 u_{2_{\tau}} d\tau \\ &= \frac{1}{10}x^5 - \frac{56}{2016}x^6 + \frac{44}{20160}x^7 - \frac{1}{20160}x^8, \\ &\vdots \end{aligned}$$

Thus we will have:

$$\begin{aligned} u(x) &= u_0 + u_1 + u_2 + \cdots \\ &= \left(2x + x^2 + \frac{1}{2}x^3 - \frac{1}{12}x^4 + \frac{1}{4}x^4 - \frac{7}{120}x^5 + \frac{1}{360}x^6 + \frac{1}{10}x^5 + \cdots\right), \end{aligned}$$

summarizing the above, we have:

$$u(x) = x + x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right),$$

so the exact solution is obtained as follows:

$$u(x) = x\left(1 + e^x\right). \qquad (63)$$

In next section we will apply the VHPM to two Euler ODEs, the first is of second-order and the second is of third-order.

6. The Euler equations

In this section, we consider two Euler ODEs and apply the VHPM to them.

Example 1. Solve the second-order Euler equation

$$x^2y''(x) - 2xy'(x) + 2y(x) = 0, y(1) = 2, y'(1) = 3, x > 0. \tag{64}$$

We first should overcome the difficulties encountered by the singularity at $x = 0$. To achieve this goal, we use the transformation

$$z = \ln(x), x = e^z, \tag{65}$$

so that

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x}\frac{dy}{dz}, \quad \frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}, \\ \frac{d^3y}{dx^3} &= \frac{1}{x^3}\frac{d^3y}{dz^3} - \frac{3}{x^3}\frac{d^2y}{dz^2} + \frac{2}{x^3}\frac{dy}{dz}. \end{aligned} \tag{66}$$

Using (65) and (66) into (64) gives

$$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = 0, \tag{67}$$

with the conditions

$$y(z = 0) = 2, y'(z = 0) = 3. \tag{68}$$

The VHPM can be employed in the same manner used before for second order equations with constant coefficients. So we will have:

$$\lambda = \tau - z. \tag{69}$$

Using this value of the Lagrange multiplier gives the iteration formula:

$$y_{n+1}(z) = y_n(z) + \int_0^z (\tau - z)\left\{y_{n_{\tau\tau}}(\tau) - 3y_{n_\tau}(\tau) + 2y_n(\tau)\right\} d\tau. \tag{70}$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned} y_0 + p\, y_1 + p^2y_2 + \cdots = 2 + 3z & \\ -3p\int_0^z (\tau - z)\left(y_{0_\tau} + p\, y_{1_\tau} + p^2y_{2_\tau} + \cdots\right)\, d\tau & \\ +2p\int_0^z (\tau - z)\left(y_0 + p\, y_1 + p^2y_2 + \cdots\right)\, d\tau. & \end{aligned} \tag{71}$$

Comparing the coefficient of like powers of p, we have:

$$
\begin{aligned}
p^0 &: y_0(z) = 2+3z,\\
p^1 &: y_1(z) = -3\int_0^z (\tau - z)y_{0_\tau} + 2\int_0^z (\tau - z)y_0 d\tau\\
&= \frac{5}{2}z^2 - z^3,\\
p^2 &: y_2(z) = -3\int_0^z (\tau - z)y_{1_\tau} d\tau + 2\int_0^z (\tau - z)y_1 d\tau\\
&= \frac{5}{2}z^3 - \frac{7}{6}z^4 + \frac{1}{10}z^5,\\
p^3 &: y_3(z) = -3\int_0^z (\tau - z)y_{2_\tau} d\tau + 2\int_0^z (\tau - z)y_2 d\tau\\
&= \frac{15}{8}z^4 - \frac{19}{20}z^5 + \frac{23}{180}z^6 - \frac{1}{210}z^7,\\
p^4 &: y_4(z) = -3\int_0^z (\tau - z)y_{3_\tau} d\tau + 2\int_0^z (\tau - z)y_3 d\tau\\
&= \frac{8505}{7560}z^5 - \frac{4536}{7560}z^6 + \frac{756}{7560}z^7 - \frac{48}{7560}z^8 + \frac{1}{7560}z^9, \qquad (72)\\
&\vdots
\end{aligned}
$$

Thus we will have: $y(z) = y_0+y_1+y_2+\cdots = 2+3z+\frac{5}{2}z^2+\frac{3}{2}z^3+\frac{17}{24}z^4+\frac{11}{40}z^5+\cdots$, so we have:

$$y(z) = e^z + e^{2z}. \qquad (73)$$

Recall from (65) that $z = \ln(x)$, so the exact solution

$$y(x) = x + x^2, \qquad (74)$$

follows immediately.

Example 2. Solve the third-order Euler equation

$$x^3y'''(x) - 3x^2y'' + 6xy' - 6y = 0, y(1) = 3, y'(1) = 6, y''(1) = 8, x > 0. \qquad (75)$$

To overcome the difficulties encountered by the singularity at $x = 0$, we set the transformation

$$z = \ln(x), x = e^z, \qquad (76)$$

so that

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{x}\frac{dy}{dz}, \qquad \frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz},\\
\frac{d^3y}{dx^3} &= \frac{1}{x^3}\frac{d^3y}{dz^3} - \frac{3}{x^3}\frac{d^2y}{dz^2} + \frac{2}{x^3}\frac{dy}{dz}. \qquad (77)
\end{aligned}
$$

Using (76) and (77) into (75) gives

$$\frac{d^3y}{dz^3} - 6\frac{d^2y}{dz^2} + 11\frac{dy}{dz} - 6y = 0, \tag{78}$$

with the conditions

$$y(z=0) = 3, y'(z=0) = 6, y''(z=0) = 14. \tag{79}$$

The VHPM can be employed in the same manner used before for third-order equations with constant coefficients. So we will have:

$$\lambda = -\frac{1}{2}(\tau - z)^2. \tag{80}$$

Using this value of the Lagrange multiplier gives the iteration formula:

$$y_{n+1}(z) = y_n(z) - \int_0^z \frac{1}{2}(\tau - z)^2 \left\{ y_{n_{\tau\tau\tau}}(\tau) - 6y_{n_{\tau\tau}}(\tau) + 11y_{n_\tau}(\tau) - 6y_n(\tau) \right\} d\tau. \tag{81}$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned} y_0 + p\, y_1 + p^2 y_2 + \cdots = 3 + 6z + 7z^2 & \\ +3p \int_0^z (\tau - z)^2 \left(y_{0_{\tau\tau}} + p\, y_{1_{\tau\tau}} + p^2 y_{2_{\tau\tau}} + \cdots \right) d\tau & \\ -\frac{11}{2} p \int_0^z (\tau - z)^2 \left(y_{0_\tau} + p\, y_{1_\tau} + p^2 y_{2_\tau} + \cdots \right) d\tau & \\ +3p \int_0^z (\tau - z)^2 \left(y_0 + p\, y_1 + p^2 y_2 + \cdots \right) d\tau. & \end{aligned} \tag{82}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} p^0 : y_0(z) &= 3 + 6z + 7z^2, \\ p^1 : y_1(z) &= 3\int_0^z (\tau - z)^2 y_{0_{\tau\tau}} d\tau - \frac{11}{2}\int_0^z (\tau - z)^2 y_{0_\tau} d\tau + 3\int_0^z (\tau - z)^2 y_0 d\tau \\ &= 6z^3 - \frac{59}{12}z^4 + \frac{7}{10}z^5, \\ p^2 : y_2(z) &= 3\int_0^z (\tau - z)^2 y_{1_{\tau\tau}} d\tau - \frac{11}{2}\int_0^z (\tau - z)^2 y_{1_\tau} d\tau + 3\int_0^z (\tau - z)^2 y_1 d\tau \\ &= 9z^4 - \frac{46}{5}z^5 + \frac{1009}{360}z^6 - \frac{34}{105}z^7 + \frac{1}{80}z^8, \\ p^3 : y_3(z) &= 3\int_0^z (\tau - z)^2 y_{2_{\tau\tau}} d\tau - \frac{11}{2}\int_0^z (\tau - z)^2 y_{2_\tau} d\tau + 3\int_0^z (\tau - z)^2 y_2 d\tau \\ &= \frac{1197504}{1110880}z^5 - \frac{138600}{111088}z^6 + \frac{562056}{1110880}z^7 \\ &- \frac{1061885}{11108800}z^8 + \frac{10109}{1110880}z^9 - \frac{4686}{11108800}z^{10} + \frac{84}{11108800}z^{11}, \end{aligned} \tag{83}$$

$$\vdots$$

Thus we will have: $y(z) = y_0 + y_1 + y_2 + \cdots = 3 + 6z + 7z^2 + 6z^3 + \frac{49}{12}z^4 + \frac{23}{10}z^5 + \cdots,$

so we have:

$$y(z) = e^{z} + e^{2z} + e^{3z}. \tag{84}$$

Recall from (76) that $z = \ln(x)$, so the exact solution

$$y(x) = x + x^2 + x^3, \tag{85}$$

is readily obtained.

7. First order nonlinear ODEs

In this section we will focus our work on two well known first-order nonlinear equations, namely the logistic differential equation and the Riccati equation. Because the two models are first-order ODEs, then $\lambda = -1$.

7.1. The logistic differential equation

We first study the logistic nonlinear differential equation

$$u' = \mu u(1-u), u(0) = \frac{1}{2}, \tag{86}$$

where $\mu > 0$ is a positive constant. For solving the equation above, we use the following iteration formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x \{u_{n_\tau}(\tau) - \mu u_n(\tau)(1-u_n(\tau))\}\, d\tau. \tag{87}$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned} u_0 + p\, u_1 + p^2 u_2 + \cdots = \frac{1}{2} & \\ & + p\int_0^x \mu\left(u_0 + p\, u_1 + p^2 u_2 + \cdots\right)\, d\tau \\ & - p\int_0^x \mu\left(u_0 + p\, u_1 + p^2 u_2 + \cdots\right)^2\, d\tau. \end{aligned} \tag{88}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} p^0 : u_0(x) &= \frac{1}{2}, \\ p^1 : u_1(x) &= \int_0^x \mu u_0 d\tau - \int_0^x \mu u_0^2 d\tau = \frac{\mu}{4}x, \\ p^2 : u_2(x) &= \int_0^x \mu u_1 d\tau - 2\int_0^x \mu u_0 u_1 d\tau = -\frac{\mu^3}{48}x^3, \\ p^3 : u_3(x) &= \int_0^x \mu u_2 d\tau - \int_0^x \mu\left(u_1^2 + 2u_0u_2\right) d\tau = \frac{\mu^5}{480}x^5, \\ &\vdots \end{aligned}$$

Thus will have:

$$u(x) = u_0 + u_1 + u_2 + \cdots = \frac{1}{2} + \frac{\mu}{4}x - \frac{\mu^3}{48}x^3 + \frac{\mu^5}{480}x^5 + \cdots, \tag{89}$$

so the exact solution is obtained as follows:

$$u(x) = \frac{e^{\mu x}}{1+e^{\mu x}}. \tag{90}$$

7.2. The Riccati equation

We close our study by applying the VHPM to the Riccati equation

$$u' = u^2 - 2xu + x^2 + 1, u(0) = 1, \tag{91}$$

for solving the equation above, we use the following iteration formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left\{u_{n_\tau}(\tau) - u_n^2(\tau) + 2\tau u_n(\tau) - \tau^2 - 1\right\} d\tau. \tag{92}$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned} u_0 + p\, u_1 + p^2 u_2 + \cdots &= 1 \\ &+ p\int_0^x \left(u_0 + p\, u_1 + p^2 u_2 + \cdots\right)^2 d\tau \\ &- 2p\int_0^x \tau\left(u_0 + p\, u_1 + p^2 u_2 + \cdots\right) d\tau \\ &+ p\int_0^x \left(\tau^2+1\right) d\tau. \end{aligned} \tag{93}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} p^0 : u_0(x) &= 1, \\ p^1 : u_1(x) &= \int_0^x u_0^2 d\tau - 2\int_0^x \tau u_0 d\tau + \int_0^x (\tau^2+1)\, d\tau = 2x - x^2 + \frac{1}{3}x^3, \\ p^2 : u_2(x) &= \int_0^x 2u_0u_1 d\tau - 2\int_0^x \tau u_1 d\tau = 2x^2 - 2x^3 + \frac{2}{3}x^4 - \frac{2}{15}x^5, \\ p^3 : u_3(x) &= \int_0^x \left(u_1^2 + 2u_0u_2\right) d\tau - 2\int_0^x \tau u_2 d\tau \\ &= \frac{8}{3}x^3 - 3x^4 + \frac{23}{15}x^5 - \frac{17}{45}x^6 + \frac{17}{315}x^7, \\ &\vdots \end{aligned}$$

Thus we will have:

$$u(x) = u_0 + u_1 + u_2 + \cdots = x + \left(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots\right), \tag{94}$$

so the exact solution is obtained as follows:

$$u(x) = x + \frac{1}{1-x}, |x| < 1. \tag{95}$$

8. Conclusion

There are two main goals that we aimed for this work. The first is to employ the powerful variational homotopy perturbation method to investigate linear and nonlinear ordinary differential equations. The second is to show the power of this method and its significant features. The two goals are achieved. It is also notable that the computations in this paper are done by MATLAB software.

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