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HYPERBOLIC EQUATION FOR FOURTH ORDER WITH MULTIPLE CHARACTERISTICS

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ABSTRACT. In this paper, a class of initial value problem of hyperbolic equations for fourth order with multiple characteristics is considered and can be solved analytically by variable transforms. Also, similar to Goursat's problem we present a direct integration technique for finding a new solutions of an inhomogeneous hyperbolic equation of fourth order such that the attached conditions are given on its multiple characteristics.

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1. Introduction

Partial differential equations of higher order are encountered when studying mathematical models for certain natural and physical processes. The Cauchy problem for general linear hyperbolic differential equations has been studied much more thoroughly in the case of operators which are strictly hyperbolic, i.e., have simple real characteristic than in the case of operators with multiple characteristics. A. Lax [1] has studied hyperbolic equations with multiple characteristics involving one space variable.

In [6] a class of equations with fourth order partial differential equations with multiple characteristics and dominated low terms is considered. The existence and uniqueness of a Riemann function for this equation is proved. Also, in [5] Adomian decomposition method is applied to the solvability of nonlinear hyperbolic equations of higher order with initial conditions. In particular, the hyperbolic equation of fourth order is chosen as example to illustrate this method.

The reader is referred to[1], [2], [3], [6], [7], [8], [9], [10] for further studies. As an example of such type of equations, is the linear hyperbolic equation of

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fourth order with multiple characteristics $x-t=0$ and $x+t=0$,

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)^2 u=0.$$

This paper is concerned with a direct integration technique for solving this type of equations with initial conditions . Also, similar to Goursat's problem an inhomogeneous hyperbolic equation of fourth order with multiple real characteristics is considered and can be solved analytically.

2. The Cauchy problem for hyperbolic equation of fourth order in $\Re$

Consider the following hyperbolic equation of fourth order with multiple real characteristics

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)^2 u=0, \tag{1}$$

for $x\in\Re$ and $t\geq 0$ with initial conditions

$$\frac{\partial^i u}{\partial t^i}(0,x)=\varphi_i(x),\ \ i=0,...,3. \tag{2}$$

Our purpose is to look for the general solution of problem (1)-(2). The basic key of this method is to transform PDEs into integrable PDEs by introducing new variables $w=x+t$ and $z=x-t$. We shall prove

Theorem 1. *The solution of problem (1)-(2) can be expressed in the form*

$$u(t,x)=(x+t)\Phi_1(x-t)+(x-t)\Psi_1(x+t)+\Phi_2(x-t)+\Psi_2(x+t),$$

where the function Φ_i and Ψ_i, $i=1,2$ are given by

$$\begin{cases}\Phi_1(x) &=& \frac{1}{8}\left(\varphi_0'(x)-\varphi_1(x)-\int_0^x\varphi_2(x)dx+\int_0^x\int_0^x\varphi_3(x)dxdx\right),\\ \Psi_1(x) &=& \frac{1}{8}\left(\varphi_0'(x)+\varphi_1(x)-\int_0^x\varphi_2(x)dx-\int_0^x\int_0^x\varphi_3(x)dxdx\right),\\ \Phi_2(x) &=& \frac{1}{2}\varphi_0(x)-\frac{1}{2}\int_0^x\varphi_1(x)dx-\int_0^x\left(\Psi_1(x)+x\Phi_1'(x)\right)dx+\gamma_1,\\ \Psi_2(x) &=& \frac{1}{2}\varphi_0(x)+\frac{1}{2}\int_0^x\varphi_1(x)dx-\int_0^x\left(\Phi_1(x)+x\Psi_1'(x)\right)dx+\gamma_2,\end{cases}$$

where γ_i, $i=1,2$ are constants and $\gamma_1+\gamma_2=0$.

Proof. To find the general solution of problem (1)-(2), make the substitutions $w=x+t$ and $z=x-t$ into equation (1), and applying the chain rule to obtain

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial w}+\frac{\partial u}{\partial z},\qquad \frac{\partial u}{\partial t}=\frac{\partial u}{\partial w}-\frac{\partial u}{\partial z},$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial w^2}+2\frac{\partial^2 u}{\partial w\partial z}+\frac{\partial^2 u}{\partial z^2},\qquad \frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial w^2}-2\frac{\partial^2 u}{\partial w\partial z}+\frac{\partial^2 u}{\partial z^2},$$

$$\frac{\partial^3 u}{\partial x^3}=\frac{\partial^3 u}{\partial w^3}+3\frac{\partial^3 u}{\partial w^2\partial z}+3\frac{\partial^3 u}{\partial w\partial z^2}+\frac{\partial^3 u}{\partial z^3},$$

$$\frac{\partial^4 u}{\partial x^4}=\frac{\partial^4 u}{\partial w^4}+4\frac{\partial^4 u}{\partial w^3\partial z}+6\frac{\partial^4 u}{\partial w^2\partial z^2}+4\frac{\partial^4 u}{\partial w\partial z^3}+\frac{\partial^4 u}{\partial z^4},$$

$$\frac{\partial^4 u}{\partial t^4}=\frac{\partial^4 u}{\partial w^4}-4\frac{\partial^4 u}{\partial w^3\partial z}+6\frac{\partial^4 u}{\partial w^2\partial z^2}-4\frac{\partial^4 u}{\partial w\partial z^3}+\frac{\partial^4 u}{\partial z^4}$$

and

$$\frac{\partial^4 u}{\partial t^2\partial x^2}=\frac{\partial^4 u}{\partial w^4}-2\frac{\partial^4 u}{\partial w^2\partial z^2}+\frac{\partial^4 u}{\partial z^4}.$$

Then equation (1) becomes

$$\frac{\partial^4 u}{\partial w^2\partial z^2}=0. \tag{3}$$

The successive integrations of equation (3) give us

$$u=w\Phi_1(z)+z\Psi_1(w)+\Phi_2(z)+\Psi_2(w), \tag{4}$$

where Φ_i and $\Psi_i,\ i=1,2$ are arbitrary functions. Now, we return to the original variables t and x, we obtain

$$u(t,x)=(x+t)\Phi_1(x-t)+(x-t)\Psi_1(x+t)+\Phi_2(x-t)+\Psi_2(x+t). \tag{5}$$

Taking into account the initial conditions (2), and from equation (5) we obtain

$$x\Phi_1(x)+x\Psi_1(x)+\Phi_2(x)+\Psi_2(x)=\varphi_0(x), \tag{6}$$

$$\Phi_1(x)-x\Phi_1'(x)-\Psi_1(x)+x\Psi_1'(x)-\Phi_2'(x)+\Psi_2'(x)=\varphi_1(x), \tag{7}$$

$$-2\Phi_1'(x)-2\Psi_1'(x)+x\Phi_1''(x)+x\Psi_1''(x)+\Phi_2''(x)+\Psi_2''(x)=\varphi_2(x), \tag{8}$$

$$3\Phi_1''(x)-3\Psi_1''(x)-x\Phi_1'''(x)+x\Psi_1'''(x)-\Phi_2'''(x)+\Psi_2'''(x)=\varphi_3(x), \tag{9}$$

Taking the third, second and first derivatives of equations (6), (7) and (8) with respect to x respectively, we obtain

$$3\Phi_1''(x)+3\Psi_1''(x)+x\Phi_1'''(x)+x\Psi_1'''(x)+\Phi_2'''(x)+\Psi_2'''(x)=\varphi_0'''(x), \tag{10}$$

$$-\Phi_1''(x)-x\Phi_1'''(x)+\Psi_1''(x)+x\Phi_1'''(x)-\Phi_2'''(x)+\Psi_2'''(x)=\varphi_1''(x) \tag{11}$$

$$-\Phi_1''(x)-\Psi_1''(x)+\Phi_2'''(x)+\Psi_2'''(x)+x\Phi_1'''(x)+x\Psi_1'''(x)=\varphi_2'(x). \tag{12}$$

Now, solving equations (10) and (12) simultaneously for Φ_1'' and Ψ_1'' immediately gives

$$\Phi_1''(x)+\Psi_1''(x)=\frac{1}{4}(\varphi_0'''(x)-\varphi_2'(x)), \tag{13}$$

and equation (9) with equation (11) for Φ_1'' and Ψ_1'', we obtain

$$\Phi_1''(x)-\Psi_1''(x)=\frac{1}{4}(\varphi_3(x)-\varphi_1''(x)). \tag{14}$$

It follows that $\Phi_1''(x) = \frac{1}{8}\left(\varphi_0'''(x) - \varphi_2'(x) + \varphi_3(x) - \varphi_1''(x)\right)$ and

$$\Psi_1''(x) = \frac{1}{8}\left(\varphi_0'''(x) - \varphi_2'(x) - \varphi_3(x) + \varphi_1''(x)\right).$$

Thus

$$\Phi_1(x) = \frac{1}{8}\left(\varphi_0'(x) - \varphi_1(x) - \int_0^x \varphi_2(x)dx + \int_0^x \int_0^x \varphi_3(x)dxdx\right) \tag{15}$$

and

$$\Psi_1(x) = \frac{1}{8}\left(\varphi_0'(x) + \varphi_1(x) - \int_0^x \varphi_2(x)dx - \int_0^x \int_0^x \varphi_3(x)dxdx\right). \tag{16}$$

Also, from (6), we have

$$\Phi_2(x) + \Psi_2(x) = \varphi_0(x) - x\Phi_1(x) - x\Psi_1(x). \tag{17}$$

and from (7), we have

$$-\Phi_2'(x) + \Psi_2'(x) = \varphi_1(x) - \Phi_1(x) + x\Phi_1'(x) + \Psi_1(x) - x\Psi_1'(x). \tag{18}$$

Taking the first derivative of equation (17) with respect to x, then gives

$$\Phi_2'(x) + \Psi_2'(x) = \varphi_0'(x) - \Phi_1(x) - \Psi_1(x) - x\Phi_1'(x) - x\Psi_1'(x), \tag{19}$$

and solving equations (18) and (19) simultaneously for Φ_2' and Ψ_2', we obtain

$$\Phi_2' = \frac{1}{2}\left(\varphi_0'(x) - \varphi_1(x)\right) - \Psi_1(x) - x\Phi_1'(x),$$

and $\Psi_2' = \frac{1}{2}\left(\varphi_0'(x) + \varphi_1(x)\right) - \Phi_1(x) - x\Psi_1'(x)$. Thus

$$\Phi_2(x) = \frac{1}{2}\varphi_0(x) - \frac{1}{2}\int_0^x \varphi_1(x)dx - \int_0^x \left(\Psi_1(x) + x\Phi_1'(x)\right)dx + \gamma_1 \tag{20}$$

and

$$\Psi_2(x) = \frac{1}{2}\varphi_0(x) + \frac{1}{2}\int_0^x \varphi_1(x)dx - \int_0^x \left(\Phi_1(x) + x\Psi_1'(x)\right)dx + \gamma_2, \tag{21}$$

where $\gamma_i, i = 1, 2$ are constants. Further, we use equations (6), (20) and (21), we obtain $\varphi_0(0) = \Phi_2(0) + \Psi_2(0)$ and consequently $\gamma_1 + \gamma_2 = 0$.
So plugging equations (15), (16), (20) and (21) into equation (5) then gives the solution of (1)-(2) as (5). This completes the proof of Theorem 1. □

In order to illustrate a possible practical use of Theorem 1, we shall give in the following a simple example.

Example 1. Consider equation (1) with initial conditions

$$\varphi_i(x) = e^x - e^{-x}, \ i = 0, ..., 3.$$

Using Theorem 1, and straightforward computation yields

$$\Phi_1(x) = \frac{1}{8}\left(\varphi_0'(x) - \varphi_1(x) - \int_0^x \varphi_2(x)dx + \int_0^x \int_0^x \varphi_3(x)dxdx\right) = \frac{1}{4}(1-x),$$

$$\Psi_1(x) = \frac{1}{8}\left(\varphi_0'(x) + \varphi_1(x) - \int_0^x \varphi_2(x)dx - \int_0^x \int_0^x \varphi_3(x)dxdx\right) = \frac{1}{4}(1+x),$$

$$\begin{aligned}\Phi_2(x) &= \frac{1}{2}\varphi_0(x) - \frac{1}{2}\int_0^x \varphi_1(x)dx - \int_0^x \left(\Psi_1(x) + x\Phi_1'(x)\right)dx + \gamma_1 \\ &= -e^{-x} + 1 - \frac{1}{4}x + \gamma_1,\end{aligned}$$

$$\begin{aligned}\Psi_2(x) &= \frac{1}{2}\varphi_0(x) + \frac{1}{2}\int_0^x \varphi_1(x)dx - \int_0^x \left(\Phi_1(x) + x\Psi_1'(x)\right)dx + \gamma_2 \\ &= e^{x} - 1 - \frac{1}{4}x + \gamma_2.\end{aligned}$$

Then, the solution of this problem is expressed as

$$u(t,x) = (x+t)\Phi_1(x-t) + (x-t)\Psi_1(x+t) + \Phi_2(x-t) + \Psi_2(x+t).$$

Thus $u(t,x) = e^{x+t} - e^{t-x}$, which is the exact solution of this initial value problem.

3. The Goursat's problem for hyperbolic equation of fourth order

The Goursat's problem concerns a class of hyperbolic partial differential in two independent variables with given values on two characteristics curves. For example, for the hyperbolic equation

$$u_{tt} - \frac{1}{c^2}u_{xx} = h(t,x).$$

Goursat's problem is posed as follows:

$$\begin{cases} u_{tt} - \frac{1}{c^2}u_{xx} &= h(t,x) \\ u(t,x) &= f(x) \text{ for } x = ct, \\ u(t,x) &= g(x) \text{ for } x = -ct, \end{cases}$$

In a manner parallel to the Goursat's problem, we study a linear hyperbolic equation of fourth-order with multiple characteristics curves. The Goursat's problem for this equation is formulated as follows:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)^2 u = f(t,x). \tag{22}$$

To equation (22), we attach the following conditions

$$u(t,x) = \varphi_0(x) \text{ for } x - t = 0, \tag{23}$$

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = \varphi_1(x) \text{ for } x - t = 0, \tag{24}$$

$$u(t,x) = \varphi_2(x) \text{ for } x + t = 0, \tag{25}$$

and

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=\varphi_3(x) \text{ for } x+t=0. \tag{26}$$

We mention that the attached conditions are given on the multiple characteristics curves $x-t=0$ and $x+t=0$ of equation (22). We have

Theorem 2. *The solution of problem (22)-(26) can be expressed in the form*

$$\begin{aligned}u(x,t)&=\varphi_0(\frac{x+t}{2})+\varphi_2(\frac{x-t}{2})-u(0,0)\\&-\frac{x-t}{2}\varphi_1(\frac{x+t}{2})+\frac{x+t}{2}\varphi_3(\frac{x-t}{2})+\frac{x-t}{2}\varphi_1(0)\\&-\frac{x+t}{2}\varphi_3(0)+\frac{(x+t)(x-t)}{4}\varphi_1'(0)\\&+\frac{1}{16}\int_0^{x+t}dr\int_0^{x-t}ds\int_0^{x+t}dr\int_0^{x-t}f(r,s)ds.\end{aligned}$$

Proof. Let u be a solution of equation (22), make the transformations $z=x-t,\ w=x+t$, and applying the chain rule to obtain

$$\frac{\partial^4 u}{\partial w^2\partial z^2}=\frac{1}{16}f(\frac{w-z}{2},\frac{w+z}{2}), \tag{27}$$

and the given conditions (23)-(26) can be converted into

$$u(w,0)=\varphi_0(\frac{w}{2}), \tag{28}$$

$$\frac{\partial u}{\partial z}(w,0)=-\frac{1}{2}\varphi_1(\frac{w}{2}), \tag{29}$$

$$u(0,z)=\varphi_2(\frac{z}{2}), \tag{30}$$

$$\frac{\partial u}{\partial w}(0,z)=\frac{1}{2}\varphi_3(\frac{z}{2}). \tag{31}$$

Then, the boundary value problem (22)-(26) is now (27)-(31). The successive integrations of equation (27) give

$$\begin{aligned}u(w,z)&=u(w,0)+u(0,z)-u(0,0)\\&+z\frac{\partial u}{\partial z}(w,0)+w\frac{\partial u}{\partial w}(0,z)-z\frac{\partial u}{\partial z}(0,0)-w\frac{\partial u}{\partial w}(0,0)\\&-wz\frac{\partial^2 u}{\partial w\partial z}(0,0)+\frac{1}{16}\int_0^w\int_0^z\int_0^w\int_0^z f(\frac{w-z}{2},\frac{w+z}{2})dwdzdwdz,\end{aligned}$$

where $u(0,0)=\varphi_0(0)=\varphi_2(0)$, $\frac{\partial^2 u}{\partial w\partial z}(0,0)=-\frac{1}{4}\varphi_1'(0)$, $\frac{\partial u}{\partial z}(0,0)=-\frac{1}{2}\varphi_1(0)$ and $\frac{\partial u}{\partial w}(0,0)=\frac{1}{2}\varphi_3(0)$. We return to the original variables t and x, we obtain

$$
\begin{aligned}
u(x,t) = {}& \varphi_0(\frac{x+t}{2}) + \varphi_2(\frac{x-t}{2}) - u(0,0) \\
& -\frac{x-t}{2}\varphi_1(\frac{x+t}{2}) + \frac{x+t}{2}\varphi_3(\frac{x-t}{2}) + \frac{x-t}{2}\varphi_1(0) \\
& -\frac{x+t}{2}\varphi_3(0) + \frac{(x+t)(x-t)}{4}\varphi_1'(0) \\
& +\frac{1}{16}\int_0^{x+t} dr \int_0^{x-t} ds \int_0^{x+t} dr \int_0^{x-t} f(r,s)ds.
\end{aligned}
$$

This completes the proof. □

Example 2. Consider Goursat's problem (22)-(26) with

$$u(t,x) = \varphi_0(x) = x^4 \text{for } x - t = 0,$$

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = \varphi_1(x) = x^2 \text{ for } x - t = 0,$$

$$u(t,x) = \varphi_2(x) = \frac{1}{4}x^2 \text{ for } x + t = 0,$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \varphi_3(x) = 0 \text{ for } x + t = 0$$

and $f(t,x) = 0$. Then, from Theorem 2, and direct calculation produces the exact solution

$$u(t,x) = \frac{1}{16}\left[(x+t)^2 - (x-t)\right]^2.$$

Example 3. Consider Goursat's problem (22)-(26) with

$$\varphi_i(x) = 0 \text{ for } x - t = 0,\ i = 0,1,$$

$$\varphi_i(x) = 0 \text{ for } x + t = 0,\ i = 2,3$$

and $f(t,x) = 64$. Then, from Theorem 2, and direct calculation produces the exact solution $u(t,x) = (x+t)^2(x-t)^2$.

Example 4. Finally, Consider Goursat's problem (22)-(26) with

$$\varphi_0(x) = 2\cos 2x \text{ for } x - t = 0, \qquad \varphi_1(x) = 0 \text{ for } x - t = 0,$$

$$\varphi_2(x) = 2\cos 2x \text{ for } x + t = 0, \qquad \varphi_3(x) = 0 \text{ for } x + t = 0$$

and $f(t,x) = 16(\cos 2t + \cos 2x)$. Then, by Theorem 2, we obtain $u(t,x) = 2\cos(x+t) + 2\cos(x-t) - 2 + [\cos 2t + \cos 2x - 2\cos(x+t) - 2\cos(x-t) + 2]$ so that

$$u(t,x) = \cos 2t + \cos 2x,$$

which is the exact solution to this problem.

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