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REPRODUCING KERNEL METHOD FOR SOLVING TENTH-ORDER BOUNDARY VALUE PROBLEMS

FAZHAN GENG* AND MINGGEN CUI

ABSTRACT. In this paper, the tenth-order linear boundary value problems are solved using reproducing kernel method. The algorithm developed approximates the solutions, and their higher-order derivatives, of differential equations and it avoids the complexity provided by other numerical approaches. First a new reproducing kernel space is constructed to solve this class of tenth-order linear boundary value problems; then the approximate solutions of such problems are given in the form of series using the present method. Three examples compared with those considered by Siddiqi, Twizell and Akram [S.S. Siddiqi, E.H. Twizell, Spline solutions of linear tenth order boundary value problems, Int. J. Comput. Math. 68 (1998) 345-362; S.S.Siddiqi, G.Akram, Solutions of tenth-order boundary value problems using eleventh degree spline , Applied Mathematics and Computation 185 (1)(2007) 115-127] show that the method developed in this paper is more efficient.

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1. Introduction

Reproducing kernel theory has important application in numerical analysis, differential equation, probability and statistics and so on [1-3]. Recently, using reproducing kernel theory, we discussed singular linear two-point boundary value problem, singular nonlinear two-point periodic boundary value problem, nonlinear system of boundary value problems, nonlinear Burgers equation and ill-posed operator equations of the first kind [4-8]. In this paper, we consider the following class of tenth-order linear boundary value problems in a reproducing

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*Corresponding author.
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kernel space

$$\begin{cases} Ly(x) \equiv y^{(10)}(x) + b(x)y(x) = g(x), a \leq x \leq b, \\ y(a) = \alpha_0, y(b) = \alpha_1, \\ y'(a) = \beta_0, y'(b) = \beta_1, \\ y^{(2)}(a) = \gamma_0, y^{(2)}(b) = \gamma_1, \\ y^{(3)}(a) = \xi_0, y^{(3)}(b) = \xi_1, \\ y^{(4)}(a) = \varsigma_0, y^{(4)}(b) = \varsigma_1, \end{cases} \tag{1}$$

where $\alpha_i, \beta_i, \gamma_i, \xi_i, \varsigma_i, i = 0, 1$ are finite real constants and $b(x)$, $g(x)$ are continuous on [a,b].

Higher order differential equations arise in many fields e.g. when an infinite horizontal layer of fluid is heated from below and a uniform magnetic field is also applied across the fluid in the same direction as gravity under the action of rotation, instability sets in. When instability sets in as ordinary convection, it is modelled by a tenth-order boundary value problems. However, there are few literature on the numerical solutions of tenth-order boundary value problems and associated eigenvalue problems. Higher order boundary value problems were researched in [9-16]. Wazwaz [10] presented a modified Adomian Decomposition method for tenth-order and twelfth-order boundary value problems. Twizell et al. [13,15] developed numerical methods for eighth, tenth and twelfth-order eigenvalue problems arising in thermal instability and boundary value problems with order 2m. Siddiqi and Twizell [9,16] gave the solution of sixth-order boundary value problems and tenth-order linear boundary value problems using spline technique. Siddiqi and Akram [11,12] gave the solution of tenth-order linear boundary value problems using non-polynomial spline technique and eleventh degree spline.

In the present paper, we shall give the solution of $Eq.(1)$ in the form of series using reproducing kernel method. Through a simple transformation of function, $Eq.(1)$ can be converted into the form as follows:

$$\begin{cases} Lu(x) \equiv u^{(10)}(x) + b(x)u(x) = f(x), a \leq x \leq b, \\ u(a) = 0, u(b) = 0, \\ u'(a) = 0, u'(b) = 0, \\ u^{(2)}(a) = 0, u^{(2)}(b) = 0, \\ u^{(3)}(a) = 0, u^{(3)}(b) = 0, \\ u^{(4)}(a) = 0, u^{(4)}(b) = 0, \end{cases} \tag{2}$$

where $u(x) \in W^{11}[a,b]$, $b(x)$, $f(x)$ are continuous on [a,b]. Therefore, to solve $Eq.(1)$, it suffices to solve $Eq.(2)$.

This paper is arranged as follows. Some reproducing kernel spaces and corresponding reproducing kernel are constructed in Section 2. In Section 3, the tenth-order linear boundary value problems are solved using reproducing kernel method. In Section 4, numerical experiments are studied to demonstrated the efficiency of the proposed method, and the results obtained using present method are compared with other methods. Results of experiments are discussed and conclusion are included in Section 5.

2. Several reproducing kernel spaces

1. *The reproducing kernel space* $W^{11}[a,b]$. The inner product space $W^{11}[a,b]$ is defined as $W^{11}[a,b] = \{u^{(i)}, i = 0, 1, \cdots, 10$ are absolutely continuous real value functions, $u^{(11)} \in L^2[a,b], u^{(i)}(a) = 0, u^{(i)}(b) = 0, i = 0, 1, 2, 3, 4\}$. The inner product in $W^{11}[a,b]$ is given by

$$(u(y), v(y))_{W^{11}} = \sum_{i=0}^{5} u^{(i)}(a)v^{(i)}(a) + \sum_{i=0}^{4} u^{(i)}(b)v^{(i)}(b) + \int_a^b u^{(11)}v^{(11)}dy, \quad (3)$$

and the norm $\| u \|_{W^{11}}$ is denoted by $\| u \|_{W^{11}} = \sqrt{(u,u)_{W^{11}}}$, where $u, v \in W^{11}[a,b]$.

Theorem 1. *The space $W^{11}[a,b]$ is a reproducing kernel space. That is, there exists $R_x(y) \in W^{11}[a,b]$, $y \in [a,b]$, for any $u(y) \in W^{11}[a,b]$ and each fixed $x \in [a,b]$, such that $(u(y), R_x(y))_{W^{11}} = u(x)$. The reproducing kernel $R_x(y)$ can be denoted by*

$$R_x(y) = \begin{cases} \sum\limits_{i=0}^{21} c_i y^i, & y \le x, \\ \sum\limits_{i=0}^{21} d_i y^i, & y > x. \end{cases} \quad (4)$$

The proof of Theorem 1 and the method of obtaining coefficients of the reproducing kernel $R_x(y)$ are given in Appendix.

2. *The reproducing kernel space* $W^1[a,b]$

The inner product space $W^1[a,b]$ is defined by $W^1[a,b] = \{u(x) \mid u$ is an absolutely continuous real value function, $u' \in L^2[a,b]\}$. The inner product and norm in $W^1[a,b]$ are given respectively by

$$(u(x), v(x))_{W^1} = \int_a^b (uv + u'v')dx, \quad \| u \|_{W^1} = \sqrt{(u,u)_{W^1}},$$

where $u(x), v(x) \in W^1[a,b]$. In [17], the authors proved that $W^1[a,b]$ is a complete reproducing kernel space and its reproducing kernel is

$$\overline{R}_x(y) = \frac{1}{2\sinh(b-a)}[\cosh(x+y-a-b) + \cosh(|x-y|-b+a)].$$

3. The solution of *Eq.*(2)

In this section, the solution of *Eq.*(2) is given using reproducing kernel method in the reproducing kernel space $W^{11}[a,b]$.

In *Eq.*(2), it is clear that $L : W^{11}[a,b] \to W^1[a,b]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{R}_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$ where L^* is the adjoint operator of L . The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W^{11}[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), (\beta_{ii} > 0, i = 1, 2, ...). \tag{5}$$

Theorem 2. *For Eq.(2), if $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W^{11}[a,b]$ and $\psi_i(x) = L_y R_x(y)|_{y=x_i}$.*

Proof. Note here that

$$\begin{aligned} \psi_i(x) &= (L^*\varphi_i)(x) = ((L^*\varphi_i)(y), R_x(y)) \\ &= (\varphi_i(y), L_y R_x(y)) = L_y R_x(y)|_{y=x_i}. \end{aligned}$$

The subscript y by the operator L indicates that the operator L applies to the function of y. Clearly, $\psi_i(x) \in W^{11}[a,b]$.

For each fixed $u(x) \in W^{11}[a,b]$, let $(u(x), \psi_i(x)) = 0, (i = 1, 2, ...)$, which means that

$$(u(x), (L^*\varphi_i)(x)) = (Lu(\cdot), \varphi_i(\cdot)) = (Lu)(x_i) = 0. \tag{6}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of the Theorem 2 is complete. □

Theorem 3. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ and the solution of Eq.(2) is unique, then the solution of Eq.(2) is*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x). \tag{7}$$

Proof. Applying Theorem 2, it is easy to know that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W^{11}[a,b]$.
Note that $(v(x), \varphi_i(x)) = v(x_i)$ for each $v(x) \in W^1[a,b]$, hence we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} (u(x), \overline{\psi}_i(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(u(x), L^*\varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(Lu(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(f(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x) \end{aligned} \tag{8}$$

and the proof of the theorem is complete. □

Now, the approximate solution $u_n(x)$ can be obtained by the n-term intercept of the exact solution $u(x)$ and

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x). \tag{9}$$

Remark 1. Put $Q = \overline{Span\{\{\overline{\psi}_i\}_{i=1}^{n}\}}$. Clearly, $Q \subset W^{11}[a,b]$. In fact, $u_n(x)$ is the projection of exact solution $u(x)$ onto space Q.

Theorem 4. *Assume $u(x)$ is the solution of Eq.(2) and $r_n(x)$ is the error between the approximate $u_n(x)$ and the exact solution $u(x)$. Then the error $r_n(x)$ is monotone decreasing in the sense of $\|\cdot\|_{W^{11}}$.*

Proof. From (7), (9), it follows that

$$\begin{aligned} \| r_n \|_{W^{11}} &= \| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x) \|_{W^{11}} \\ &= \sum_{i=n+1}^{\infty} (\sum_{k=1}^{i} \beta_{ik} f(x_k))^2. \end{aligned} \tag{10}$$

(10) shows that the error r_n is monotone decreasing in the sense of $\|\cdot\|_{W^{11}}$ and the proof is complete. □

Lemma 1. *If $u(x) \in W^{11}[a,b]$, then $|u(x)| \le M \| u(x) \|_{W^{11}}$, $|u^{(i)}(x)| \le M \| u(x) \|_{W^{11}}$, $i = 1,2,3,4,5,6,7,8,9,10$.*

From Lemma 1, it is easy to obtain the following Theorem.

Theorem 5. $u_n^{(i)}(x) \to u^{(i)}(x)$, $i = 1,2,3,4,5,6,7,8,9,10$ *as* $n \to \infty$.

Remark 2. In this paper, although only special linear tenth-order boundary value problems are considered, the method presented in this paper can be also applied to general linear tenth-order boundary value problems.

4. Numerical examples

In this section, some numerical examples are studied to demonstrate the efficiency of the present method. The examples are computed using Mathematica 5.0. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

Example 1. Consider the following singularly perturbed boundary value problem:

$$\begin{cases} y^{(10)}(x) + (x^2 - 2x)y(x) = 10\cos(x) - (x-1)^3\sin(x), -1 \le x \le 1, \\ y(-1) = 2\sin(1), y(1) = 0, \\ y'(-1) = 2\cos(1) - \sin(1), y'(1) = \sin(1), \\ y^{(2)}(-1) = 2\cos(1) - 2\sin(1), y^{(2)}(1) = 2\cos(1), \\ y^{(3)}(-1) = 2\cos(1) + 3\sin(1), y^{(3)}(1) = -3\sin(1), \\ y^{(4)}(-1) = -4\cos(1) + 2\sin(1), y^{(4)}(1) = -4\cos(1), \end{cases}$$

whose exact solution is $y(x) = (x-1)\sin(x)$. The maximum error in absolute value $| y^{(\mu)}(x) - y_n^{(\mu)}(x) |$ associated with $y_n^{(\mu)}, \mu = 0,2,4,6,8$ are summarized in Table 1, 2. Table 1 is the results obtained using method in [12] and Table 2 is the results obtained using present method.

Example 2. Consider the following singularly perturbed boundary value problem:

$$\begin{cases} y^{(10)}(x) - xy(x) = -(89+21x+x^2-x^3)e^x, -1 \leq x \leq 1, \\ y(-1) = 0, y(1) = 0, \\ y'(-1) = \frac{2}{e}, y'(1) = -2e, \\ y^{(2)}(-1) = \frac{2}{e}, y^{(2)}(1) = -6e, \\ y^{(3)}(-1) = 0, y^{(3)}(1) = -12e, \\ y^{(4)}(-1) = -\frac{4}{e}, y^{(4)}(1) = -20e, \end{cases}$$

whose exact solution is $y(x) = (1-x^2)e^x$. The maximum error in absolute value $| y^{(\mu)}(x) - y_n^{(\mu)}(x) |$ associated with $y_n^{(\mu)}, \mu = 0,2,4,6,8$ compared with those considered by Siddiqi and Twizell [9], Siddiqi and Akram [12] corresponding the method developed for $h = 1/9 (n = 19)$ are shown in Table 3. It is evident from Table 3 that the maximum absolute errors are less than those presented by Siddiqi and Twizell [9], Siddiqi and Akram [12].

Example 3. Consider the following singularly perturbed boundary value problem:

$$\begin{cases} y^{(10)}(x) + y(x) = -10(2x\sin(x) - 9\cos(x)), -1 \leq x \leq 1, \\ y(-1) = 2\sin(1), y(1) = 0, \\ y'(-1) = -2\cos(1), y'(1) = 2\cos(1), \\ y^{(2)}(-1) = 2\cos(1) - 4\sin(1), y^{(2)}(1) = 2\cos(1) - 4\cos(1), \\ y^{(3)}(-1) = 6\cos(1) + 6\sin(1), y^{(3)}(1) = -6\cos(1) - 6\sin(1), \\ y^{(4)}(-1) = -12\cos(1) + 8\sin(1), y^{(4)}(1) = -12\cos(1) + 8\sin(1), \end{cases}$$

whose exact solution is $y(x) = (x^2-1)\cos(x)$. The maximum error in absolute value $| y^{(\mu)}(x) - y_n^{(\mu)}(x) |$ associated with $y_n^{(\mu)}, \mu = 0,2,4,6,8$ compared with those considered by Siddiqi and Twizell [9], Siddiqi and Akram [12] corresponding the method developed for $h = 1/16 (n = 33)$ are shown in Table 4. It is evident from Table 4 that the maximum absolute errors are less than those presented by Siddiqi and Twizell [9], Siddiqi and Akram [12].

4. Conclusion

In this paper, we construct a new reproducing kernel space and solve a class of linear tenth-order boundary value problems using reproducing kernel method. The proposed method is implemented on three test examples. Comparing with other methods, we can see that the results obtained using this method are more accurate than the stated existing methods with same numbers of nodal points. And this method avoids the complexity provided by other numerical approaches.

TABLE 1. Maximum absolute errors for Example 1 in $y_n^{(\mu)}, \mu = 0, 2, 4, 6, 8$ using method in[12]

$y_n^{(\mu)}$	n=14	n=28	n=42	n=56
$\mu = 0$	5.96×10^{-6}	7.99×10^{-7}	1.72×10^{-7}	3.73×10^{-8}
$\mu = 2$	1.79×10^{-6}	7.10×10^{-6}	1.45×10^{-6}	4.83×10^{-7}
$\mu = 4$	$4.31\times10^{+1}$	2.35×10^{-0}	4.28×10^{-1}	1.93×10^{-1}
$\mu = 6$	4.42×10^{-1}	2.84×10^{-2}	6.70×10^{-3}	7.30×10^{-3}
$\mu = 8$	$5.55\times10^{+2}$	$3.52\times10^{+2}$	$4.46\times10^{+2}$	$6.36\times10^{+2}$

TABLE 2. Maximum absolute errors for Example 1 in $y_n^{(\mu)}, \mu = 0, 2, 4, 6, 8$ using present method

$y_n^{(\mu)}$	n=14	n=28	n=42	n=56
$\mu = 0$	5.37×10^{-9}	1.34×10^{-9}	5.98×10^{-10}	3.36×10^{-10}
$\mu = 2$	3.19×10^{-8}	7.98×10^{-9}	3.55×10^{-9}	1.99×10^{-9}
$\mu = 4$	8.38×10^{-7}	2.10×10^{-7}	9.32×10^{-8}	5.25×10^{-8}
$\mu = 6$	3.02×10^{-4}	7.54×10^{-5}	3.35×10^{-5}	1.88×10^{-5}
$\mu = 8$	7.94×10^{-3}	1.99×10^{-3}	8.83×10^{-4}	4.97×10^{-4}

TABLE 3. Comparison of maximum absolute errors of the present method with other methods for Example 2 in $y_n^{(\mu)}$

$y_n^{(\mu)}$	[9]$x \in [x_5, x_{k-5}]$	[9]$x \notin [x_5, x_{k-5}]$	[12]	Present method
$\mu = 0$	2.65×10^{-4}	$4.16\times10^{+13}$	3.28×10^{-6}	3.92×10^{-8}
$\mu = 2$	6.55×10^{-4}	$2.41\times10^{+16}$	1.40×10^{-3}	2.37×10^{-7}
$\mu = 4$	1.02×10^{-3}	$7.30\times10^{+17}$	$7.76\times10^{+2}$	5.76×10^{-6}
$\mu = 6$	4.04×10^{-3}	$3.83\times10^{+14}$	1.97×10^{-1}	1.96×10^{-3}
$\mu = 8$	1.10×10^{-2}	$3.17\times10^{+17}$	$2.73\times10^{+4}$	5.47×10^{-2}

TABLE 4. Comparison of maximum absolute errors of the present method with other methods for Example 3 in $y_n^{(\mu)}$

$y_n^{(\mu)}$	[9]$x \in [x_5, x_{k-5}]$	[9]$x \notin [x_5, x_{k-5}]$	[12]	Present method
$\mu = 0$	2.65×10^{-4}	$4.16\times10^{+13}$	8.85×10^{-8}	1.13×10^{-8}
$\mu = 2$	6.55×10^{-4}	$2.48\times10^{+16}$	3.65×10^{-6}	6.71×10^{-8}
$\mu = 4$	1.62×10^{-3}	$5.75\times10^{+17}$	5.92×10^{-0}	1.77×10^{-6}
$\mu = 6$	4.04×10^{-3}	$1.65\times10^{+16}$	1.78×10^{-2}	6.40×10^{-4}
$\mu = 8$	1.10×10^{-2}	$3.20\times10^{+19}$	$2.08\times10^{+3}$	1.70×10^{-2}

Moreover, the higher-order derivatives of approximate solutions can also approximate the higher-order derivatives of exact solutions well. Therefore, our conclusion is that the present reproducing kernel method is a satisfactory method for solving linear tenth-order boundary value problems.

Appendix

The proof of Theorem 1:

Integrations by parts for (3) gives

$$\begin{aligned}
&(u(y), R_x(y))_{W^{11}} \\
&= \sum_{i=0}^{5} u^{(i)}(a)R_x^{(i)}(a) + \sum_{i=0}^{4} u^{(i)}(b)R_x^{(i)}(b) + \int_a^b u(y)(-R_x^{(22)}(y))dy \\
&+u(y)R_x^{(21)}(y)|_a^b - u'(y)R_x^{(20)}(y)|_a^b + u^{(2)}(y)R_x^{(19)}(y)|_a^b \\
&-u^{(3)}(y)R_x^{(18)}(y)|_a^b + u^{(4)}(y)R_x^{(17)}(y)|_a^b - u^{(5)}(y)R_x^{(16)}(y)|_a^b \\
&+u^{(6)}(y)R_x^{(15)}(y)|_a^b - u^{(7)}(y)R_x^{(14)}(y)|_a^b + u^{(8)}(y)R_x^{(13)}(y)|_a^b \\
&-u^{(9)}(y)R_x^{(12)}(y)|_a^b + u^{(10)}(y)R_x^{(11)}(y)|_a^b.
\end{aligned} \tag{11}$$

Since $R_x(y) \in W^{11}[a, b]$, it follows that

$$R_x^{(i)}(a) = 0, R_x^{(i)}(b) = 0, i = 0, 1, 2, 3, 4. \tag{12}$$

Since $u \in W^{11}[a, b]$, one obtains $u^{(i)}(a) = 0, u^{(i)}(b) = 0, i = 0, 1, 2, 3, 4$. If

$$R_x^{(16)}(b) - R_x^{(5)}(b) = 0, R_x^{(16)}(a) + R_x^{(5)}(a) = 0, \tag{13}$$

$$R_x^{(i)}(a) = R_x^{(i)}(b) = 0,\ i = 11, 12, \cdots, 15, \tag{14}$$

then (11) implies that

$$(u(y), R_x(y))_{W^{11}} = \int_a^b u(y)(-R_x^{(22)}(y))dy.$$

For $\forall x \in [a, b]$, if $R_x(y)$ also satisfies

$$-R_x^{(22)}(y) = \delta(y - x), \tag{15}$$

then

$$(u(y), R_x(y))_{W^{11}} = u(x).$$

Characteristic equation of (15) is given by $\lambda^{22} = 0$, then we can obtain characteristic values $\lambda = 0$(a repeated root of multiplicity 22). So, let

$$R_x(y) = \begin{cases} \sum\limits_{i=0}^{21} c_i y^i, & y \leq x, \\ \sum\limits_{i=0}^{21} d_i y^i, & y > x. \end{cases}$$

On the other hand, for (15), let $R_x(y)$ satisfies

$$R_x^{(k)}(x + 0) = R_x^{(k)}(x - 0), k = 0, 1, 2, \cdots, 20. \tag{16}$$

Integrating (15) from $x - \varepsilon$ to $x + \varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $R_x^{(21)}(y)$ at $y = x$

$$R_x^{(21)}(x - 0) - R_x^{(21)}(x + 0) = 1. \tag{17}$$

From (12),(13), (14), (16), (17), the unknown coefficients of (4) can be obtained.

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Fazhan Geng received his Ph.D at Harbin Insitute of Technology under the direction of Minggen Cui. His research interests focus on the approximation theory, reproducing kerenl theory and application.

Department of Mathematics, Changshu Insitute of Technology, Changshu, Jiangsu 215500, China

e-mail: `gengfazhan@sina.com`

Minggen Cui received his Ph.D at Harbin Insitute of Technology under the direction of Congxin Wu. His research interests focus on the wavelet analysis, approximation theory, reproducing kerenl theory and application.

Department of Mathematics, Harbin Insitute of Technology, Weihai, Shandong 264209, China

e-mail: `cmgyfs@263.net`