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A NOTE ON THE ZEROS OF THE q-BERNOULLI POLYNOMIALS

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ABSTRACT. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the q-Bernoulli polynomials $B_{n,q}(x)$ for $-1 < q < 0$ in complex plane. Observe that the structure of the zeros of the Genocchi polynomials $G_n(x)$ resembles the structure of the zeros of the q-Bernoulli polynomials $B_{n,q}(x)$ as $q \to -1$.

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1. Introduction

In [8], we introduced the q-Bernoulli numbers and q-Bernoulli polynomials and observed the behavior of complex roots of q-Bernoulli polynomials, using numerical investigation. Using computer experiments, S.C.Won [10] verified a remarkably regular structure of the complex roots of Bernoulli polynomials. Also, A.P. Veselov and J.P. Ward [9] proved the regular lattice behaviour of almost all of the real roots of the Bernoulli polynomials. C.S.Ryoo [8] demonstrated a remarkably regular structure of the complex roots of q-Bernoulli polynomials. Therefore, using computer, a realistic study for q-Bernoulli polynomials $B_{n,q}(x)$ is very interesting. It is the aim of this paper to an interesting phenomenon of 'scattering' of the zeros of the q-Bernoulli polynomials $B_{n,q}(x)$ for $-1 < q < 0$ in complex plane. Observe that the structure of the zeros of the Genocchi polynomials $G_n(x)$ resembles the structure of the zeros of the q-Bernoulli polynomials $B_{n,q}(x)$ as $q \to -1$. We introduce the q-Bernoulli polynomials $B_{n,q}(x)$ and Genocchi polynomials $G_n(x)$. q-Bernoulli polynomials $B_{n,q}(x)$ was introduced

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in [8]. First, we introduce the q-Bernoulli numbers and q-Bernoulli polynomials. Let q be a complex number with $|q|<1$. Let us define the q-Bernoulli numbers $B_{n.q}$ and polynomials $B_{n,q}(x)$ as follows:

$$F_q(t)=\frac{t}{qe^t-1}=\sum_{n=0}^{\infty}B_{n,q}\frac{t^n}{n!},\ \text{cf. } [1, 8] \tag{1.1}$$

$$F_q(x,t)=\frac{t}{qe^t-1}e^{xt}=\sum_{n=0}^{\infty}B_{n,q}(x)\frac{t^n}{n!}. \tag{1.2}$$

Now, we introduce the Genocchi numbers and Genocchi polynomials. The Genocchi numbers G_n are defined by the generating function:

$$F(t)=\frac{2t}{e^t+1}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!},(|t|<\pi) \tag{1.3}$$

where we use the technique method notation by replacing G^n by $G_n(n\geq 0)$ symbolically. For $x\in\mathbb{C}$, we consider the Genocchi polynomials as follows:

$$F(x,t)=F(t)e^{xt}=\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\ \text{see } [8]. \tag{1.5}$$

Since

$$\begin{aligned}\sum_{n=0}^{\infty}G_n(1-x)\frac{(-t)^n}{n!}&=F(1-x,-t)=\frac{-2t}{e^{-t}+1}e^{(1-x)(-t)}\\&=\frac{-2t}{e^t+1}e^{xt}=-F(x,t)=-\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\end{aligned}$$

we obtain

$$G_n(x)=(-1)^{n+1}G_n(1-x). \tag{1.6}$$

In [5], we observed the behavior of complex roots of the Genocchi polynomials $G_n(x)$, using numerical investigation. Prove that $G_n(x), x\in\mathbb{C}$, has $Re(x)=1/2$ reflection symmetry in addition to the usual $Im(x)=0$ reflection symmetry analytic complex functions. The obvious corollary is that the zeros of $G_n(x)$ will also inherit these symmetries.

$$\text{If } G_n(x_0)=0,\ \text{then } G_n(1-x_0)=0=G_n(x_0^*)=G_n(1-x_0^*) \tag{1.7}$$

$*$ denotes complex conjugation (see [5], Figure 2). Prove that $G_n(x)=0$ has $n-1$ distinct solutions. If $G_{2n}(x)$ has $Re(x)=1/2$ and $Im(x)=0$ reflection symmetries, and $2n-1$ non-degenerate zeros, then $2n-2$ of the distinct zeros will satisfy (1.7). If the remaining one zero is to satisfy (1.7) too, it must reflect into itself, and therefore it must lie at $1/2$ (see Figure 9), the center of the structure of the zeros, ie.,

$$G_n(1/2)=0\quad\forall\text{ even } n.$$

We calculated an approximate solution satisfying $G_n(x), x \in \mathbb{R}$. The results are given in Table 1.

Table 1. Approximate solutions of $G_n(x) = 0, x \in \mathbb{R}$

degree n	x
2	0.500000000
3	0.000000000, 1.000000000
4	−0.366025404, 0.500000000, 1.366025404
5	−0.61803399, 0.0000000, 1.00000000, 1.61803399
6	−0.6180, −0.6180, 0.50000000, 1.618, 1.618
7	0.0000000, 1.0000000
8	0.4977314, 0.5000000, 1.497731
9	−0.932328, 0.0000000, 1.000000, 1.9323
10	−1.21973, −0.500080, 0.500000, 1.5001, 2.220
11	−1.3652, −1.0150, 0.0000000, 1.0000, 2.015, 2.37

Since

$$\sum_{n=0}^{\infty}(G_n(x+1)+G_n(x))\frac{t^n}{n!} = \sum_{n=0}^{\infty} 2nx^{n-1}\frac{t^n}{n!},$$

we have the following difference equation.

$$G_n(x+1)+G_n(x) = 2nx^{n-1} \tag{1.8}$$

It is known that from (1.8),

$$G_n(1) = G_n(0) = G_n = 0 \quad \forall \text{ odd } n \geq 3.$$

This shows the distribution of real zeros of $G_n(x)$ for $1 \leq n \leq 30$(Figure 1). We investigate the beautiful zeros of the $G_n(x)$ by using a computer. We plot the zeros of $G_{30}(x)$(Figure 2).

2. Zeros of the q-Bernoulli polynomials $B_{n,q}(x)$

In this section, observer that the structure of the zeros of the Genocchi polynomials $G_n(x)$ resembles the structure of the zeros of the q-Bernoulli polynomials $B_{n,q}(x)$ as $q \to -1$. Since

$$\sum_{n=0}^{\infty}(qB_{n,q}(x+1)-B_{n,q}(x))\frac{t^n}{n!} = \sum_{n=0}^{\infty} nx^{n-1}\frac{t^n}{n!},$$

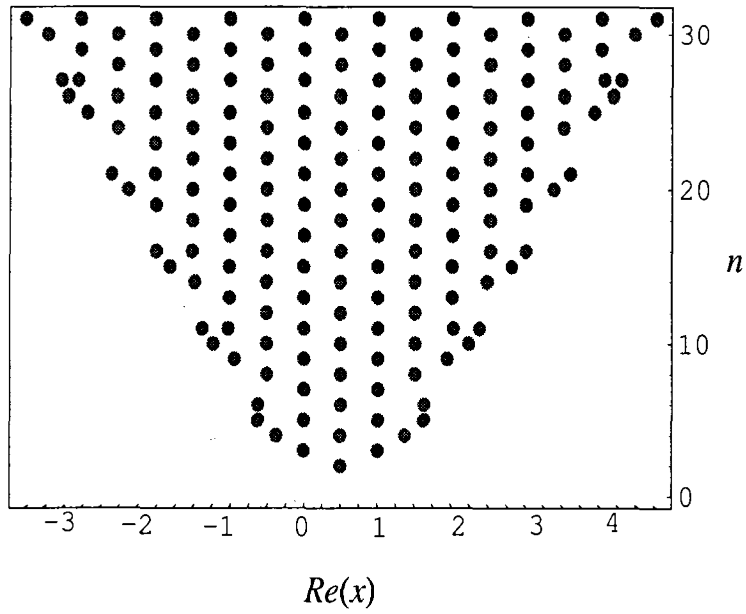

FIGURE 1. Plot of real zeros of $G_n(x), 1 \leq n \leq 30$

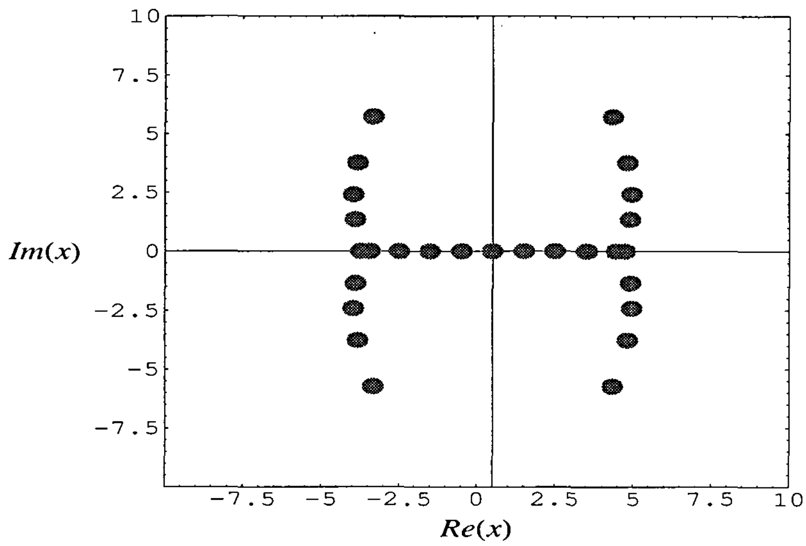

FIGURE 2. Zeros of $G_{30}(x)$

we have the following difference equation.

$$qB_{n,q}(x+1) - B_{n,q}(x) = nx^{n-1} \tag{2.1}$$

It is known that from (2.1),

$$qB_{n,q}(1) = B_{n,q}(0). \tag{2.2}$$

We plot the zeros of q-Bernoulli polynomials $B_{n,q}(x)$ for $n = 30, q = -0.2, -0.6, -0.999, -0.999999999$ and $x \in \mathbb{C}$ (Figure 3). Since

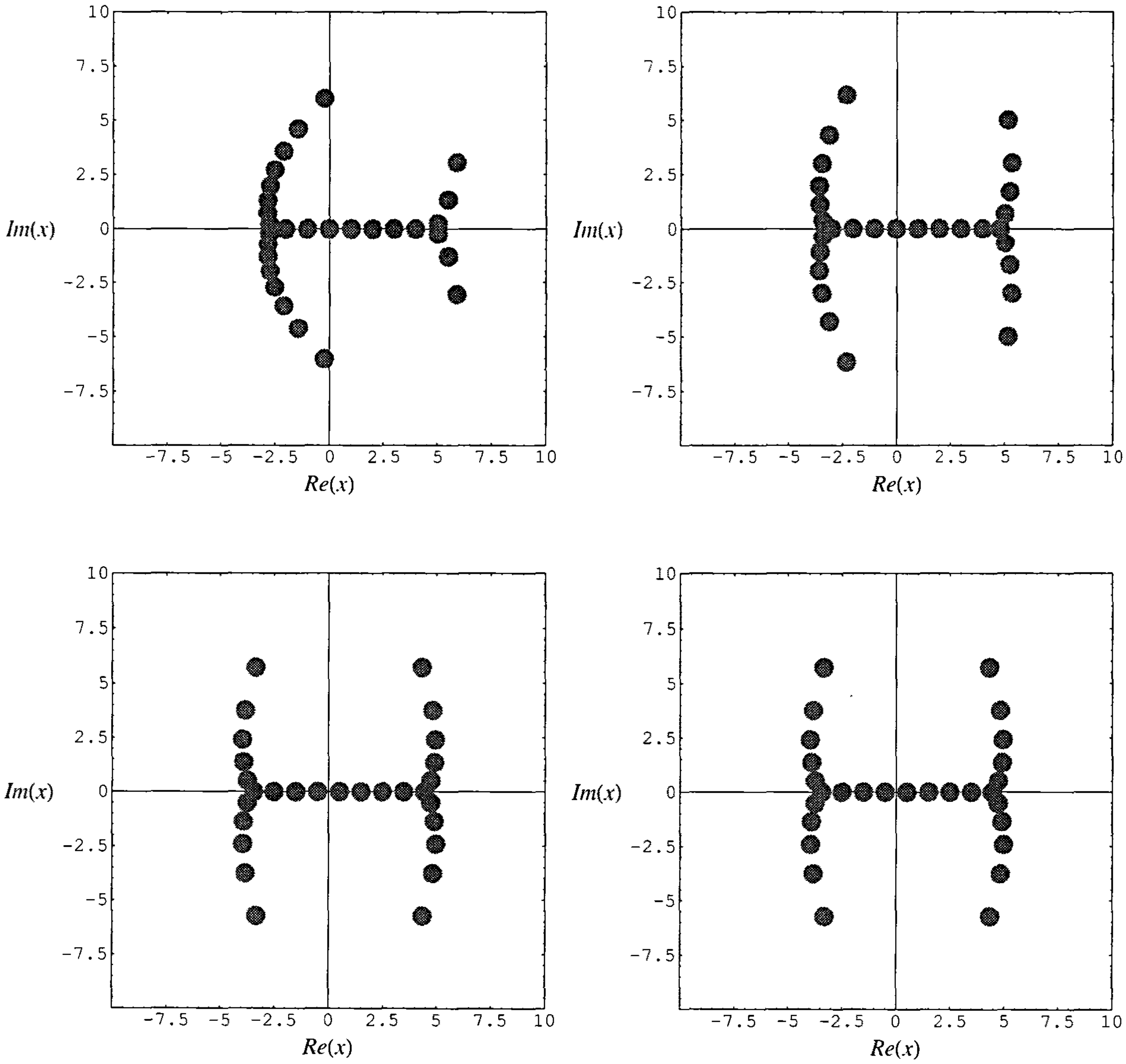

FIGURE 3. Zeros of $B_{30}(x)$

$$\sum_{n=0}^{\infty} B_{n,q^{-1}}(1-x)\frac{(-t)^n}{n!} = F_q(1-x,-t) = \frac{-t}{q^{-1}e^{-t}-1}e^{(1-x)(-t)}$$

$$= q\frac{t}{qe^t-1}e^{xt} = F_q(x,t) = q\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!},$$

we obtain

$$B_{n,q}(x) = (-1)^n q^{-1} B_{n,q^{-1}}(1-x). \tag{2.3}$$

Clearly,

$$\lim_{q\to -1} B_{n,q}(x) = -\frac{1}{2}G_n(x), \quad \lim_{q\to -1} B_{n,q} = -\frac{1}{2}G_n.$$

By (2.3), we obtain

$$\lim_{q\to -1} B_{n,q}(1) = \lim_{q\to -1} B_{n,q^{-1}}(0) = 0 \quad \forall \text{ odd } n \geq 3.$$

Figures 4 and 5 show the distribution of real zeros of $B_{n,q}(x)$ for $1 \leq n \leq 30$.

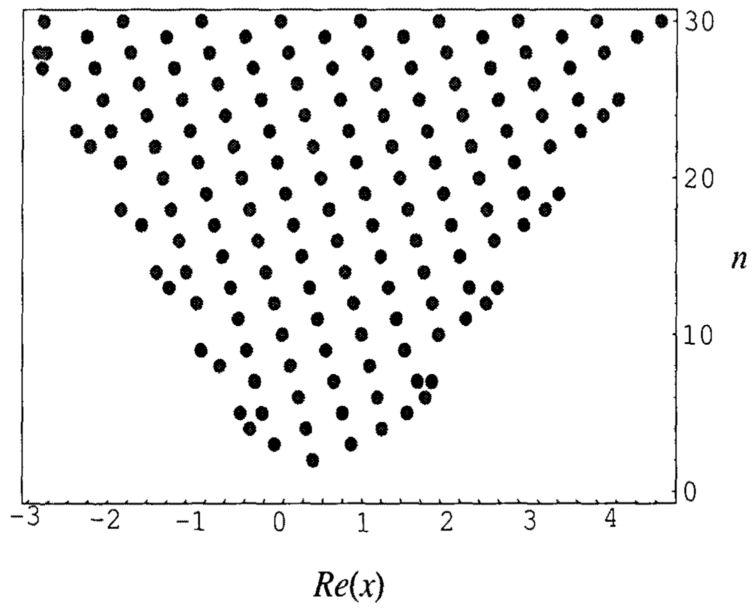

FIGURE 4. Plot of real zeros of $B_{n,-0.6}(x), 1 \leq n \leq 30$

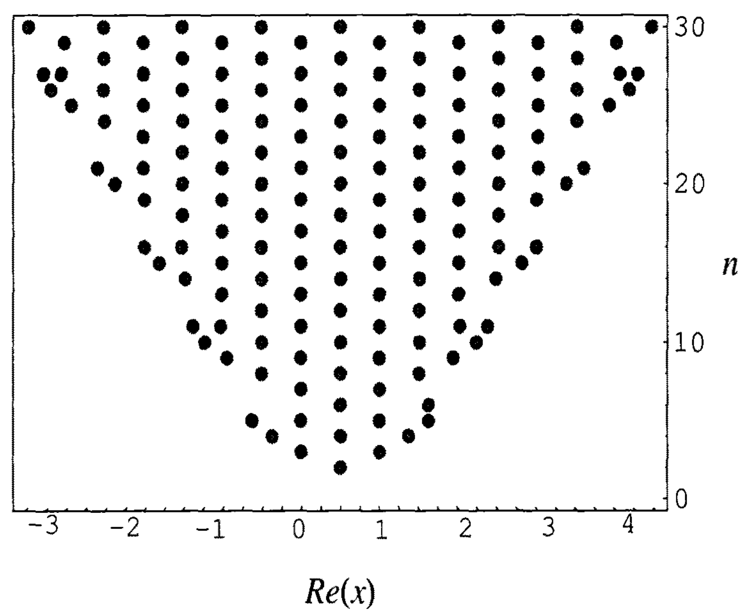

FIGURE 5. Plot of real zeros of $B_{n,-0.999999999}(x), 1 \leq n \leq 30$

Observe that the structure of the zeros of the Genocchi polynomials $G_n(x)$ resembles the structure of the zeros of the q-Bernoulli polynomials $B_{n,q}(x)$ as

$q \to -1$ (see Figures 1, 2, 3, 4, 5). The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the q-Bernoulli polynomials $B_{n,q}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [2, 3, 4, 5, 6, 7, 8, 9, 10].

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