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GLOBAL ASYMPTOTIC STABILITY OF A SECOND ORDER RATIONAL DIFFERENCE EQUATION

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ABSTRACT. The aim of this paper is to investigate the global stability, periodic nature, oscillation and the boundedness of solutions of the difference equation

$$x_{n+1} = \frac{A + Bx_{n-1}}{C + Dx_n^2} \qquad ,n = 0, 1, 2, \ldots$$

where A, B are nonnegative real numbers and $C, D > 0$.

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1. Introduction and Preliminaries

Difference equations, although their forms look very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions. One can refer to [5, 4]. The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations. It is worthwhile to point out that although several approaches have been developed for finding the global character of difference equations [4, 6, 7], relatively a large number of difference equations have not been thoroughly understood yet [3, 8, 10].

C.H.Gibbons et all [2] investigated the global asymptotic behavior of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\gamma + x_n}, \quad n = 0, 1, 2, \ldots \tag{1}$$

where $\beta > 0$ and $\alpha, \gamma \geq 0$.

In [5] the global asymptotic behavior of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + Cx_n^2}, \; n = 0, 1, 2, \ldots$$

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was discussed, where A, B, C are nonnegative real numbers and the initial conditions x_{-1}, x_0 are nonnegative real numbers.

In this paper, we study the global asymptotic stability of all solutions of the difference equation

$$x_{n+1} = \frac{A + Bx_{n-1}}{C + Dx_n^2} \qquad , n = 0, 1, 2, \ldots \tag{2}$$

where A, B are nonnegative real numbers and $C, D > 0$.
we give some preliminaries which will be needed in this paper.

Let I be a real interval and let $f : I \times I \to I$ be a continuous function. For every pair of initial conditions $\langle x_{-1}, x_0 \rangle \in I \times I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, \ldots \tag{3}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$. An equilibrium point of equation (3) is a point $\bar{x} \in I$ with $\bar{x} = f(\bar{x}, \bar{x})$.

Definition 1.1. Let $\bar{x}$ be an equilibrium point of equation (3).

(1) $\bar{x}$ is stable if for every $\epsilon > 0, \exists \delta > 0$ such that for any initial conditions $\langle x_{-1}, x_0 \rangle \in I \times I$ with $\mid x_{-1} - \bar{x} \mid + \mid x_0 - \bar{x} \mid < \delta, \mid x_n - \bar{x} \mid < \varepsilon$ holds for $n = 1, 2, \ldots$
(2) $\bar{x}$ is a local attractor if there exists $\gamma > 0$ such that $x_n \to \bar{x}$ holds for any initial conditions $\langle x_{-1}, x_0 \rangle \in I \times I$ with $\mid x_{-1} - \bar{x} \mid + \mid x_0 - \bar{x} \mid < \gamma$.
(3) $\bar{x}$ is locally asymptotically stable if it is stable and is a local attractor.
(4) $\bar{x}$ is a global attractor if $x_n \to \bar{x}$ holds for any $\langle x_{-1}, x_0 \rangle \in I \times I$.
(5) $\bar{x}$ is globally asymptotically stable if it is stable and is a global attractor.
(6) $\bar{x}$ is a repeller if there exists $\gamma > 0$ such that for each $\langle x_{-1}, x_0 \rangle \in I \times I$ with $\mid x_{-1} - \bar{x} \mid + \mid x_0 - \bar{x} \mid < \gamma$, there exists N such that $\mid x_N - \bar{x} \mid \geq \gamma$.
(7) $\bar{x}$ is a saddle point if it is neither a local attractor nor a repeller.

Assume that $\bar{x}$ is an equilibrium point of equation (3). Let $r = \frac{\partial f(\bar{x}, \bar{x})}{\partial x_n}$ and $s = \frac{\partial f(\bar{x}, \bar{x})}{\partial x_{n-1}}$. Then the linearized equation associated with equation (3) about the equilibrium $\bar{x}$ is

$$z_{n+1} - r z_n - s z_{n-1} = 0. \tag{4}$$

The characteristic equation associated with equation (4) is

$$\lambda^2 - r\lambda - s = 0. \tag{5}$$

Theorem 1.2. (*The linearized stability theorem* [6]).

(1) *If* $\mid r \mid < 1 - s < 2$, *then* $\bar{x}$ *is locally asymptotically stable.*
(2) *If* $\mid r \mid < \mid 1 - s \mid$ *and* $\mid s \mid > 1$, *then* $\bar{x}$ *is a repeller.*
(3) *If* $\mid r \mid > \mid 1 - s \mid$ *and* $r^2 + 4s > 0$, *then* $\bar{x}$ *is a saddle point.*

Now we give the definitions for the positive and negative semicycle of a solution of equation (3) relative to an equilibrium point $\bar{x}$.

Definition 1.3. [8] A positive semicycle of a solution $\{x_n\}_{n=-1}^{\infty}$ of equation (3) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all greater than or equal to the equilibrium $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

either $l = -1$, or $l > -1$ and $x_{l-1} < \bar{x}$

and

either $m = \infty$, or $m < \infty$ and $x_{m+1} < \bar{x}$.

Definition 1.4.[8] A negative semicycle of a solution $\{x_n\}_{n=-1}^{\infty}$ of equation (3) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all less than or equal to the equilibrium $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

either $l = -1$, or $l > -1$ and $x_{l-1} \geq \bar{x}$

and

either $m = \infty$, or $m < \infty$ and $x_{m+1} \geq \bar{x}$.

Theorem 1.5. [8] *Assume that $f \in C[(0,\infty) \times (0,\infty), (0,\infty)]$ is such that: $f(x,y)$ is decreasing in x for each fixed y, and $f(x,y)$ is increasing in y for each fixed x. Let $\bar{x}$ be a positive equilibrium of equation (3). Then except possibly for the first semicycle, every solution of equation (3) has semicycles of length one.*

Theorem 1.5. [9] *Let $\bar{x}$ be an equilibrium point of the equation*

$$x_{n+1} = f(x_n).$$

Suppose that $f \in C^3(R)$ and $f'(\bar{x}) = -1$.

(1) *If $-3(f''(\bar{x}))^2 - 2f''(\bar{x}) < 0$, then $\bar{x}$ is locally asymptotically stable.*
(2) *If $-3(f''(\bar{x}))^2 - 2f''(\bar{x}) > 0$, then $\bar{x}$ is unstable.*

2. Linearized stability analysis

Consider the difference equation

$$x_{n+1} = \frac{A + Bx_{n-1}}{C + Dx_n^2} \qquad , n = 0, 1, 2, \dots$$

where A, B are nonnegative real numbers and $C, D > 0$. The change of variables $x_n = \sqrt{\frac{C}{D}}y_n$ reduces equation (2) to the difference equation

$$y_{n+1} = \frac{p + qy_{n-1}}{1 + y_n^2} \qquad , n = 0, 1, 2, \dots \tag{6}$$

where $p = \frac{A}{C}\sqrt{\frac{D}{C}}, q = \frac{B}{C}$. Now we determine the equilibrium points of equation (6) and discuss their local asymptotic behavior. It is clear that the values of the equilibrium points depends on p and q.

The equilibrium points of equation (6) are the zeros of the function $f(y) = \bar{y}^3 + (1-q)\bar{y} - p$. When $q < 1$, then equation (6) has a unique equilibrium point $\bar{y}$, such that $\bar{y} > \sqrt{1-q}$ if $p > 2(1-q)^{\frac{3}{2}}$ and $0 < \bar{y} < \sqrt{1-q}$ if $p < 2(1-q)^{\frac{3}{2}}$. When $q > 1$, then equation (6) has a unique positive equilibrium point $\bar{y} > \sqrt{q-1}$ if

$p > 2(\frac{q-1}{3})^{\frac{3}{2}}$ and three equilibrium points $\bar{y} > \sqrt{q-1}$, $-\sqrt{\frac{q-1}{3}} < \bar{y_1} < 0$ and $-\sqrt{q-1} < \bar{y_2} < -\sqrt{\frac{q-1}{3}}$ if $p < 2(\frac{q-1}{3})^{\frac{3}{2}}$.

The linearized equation associated with equation (6) about $\bar{y}$ is

$$z_{n+1} + \frac{2\bar{y}^2}{1+\bar{y}^2} z_n - \frac{q}{1+\bar{y}^2} z_{n-1} = 0 \qquad , n = 0, 1, 2, \ldots \tag{7}$$

The characteristic equation associated with this equation is

$$\lambda^2 + \frac{2\bar{y}^2}{1+\bar{y}^2}\lambda - \frac{q}{1+\bar{y}^2} = 0. \tag{8}$$

We summarize the results of this section in the following theorem.

Theorem 2.1.

(1) *Assume that $q < 1$ and let $\bar{y}$ be the unique positive equilibrium point of equation (6). Then*
   (a) *$\bar{y}$ is locally asymptotically stable if $p < 2(1-q)^{\frac{3}{2}}$.*
   (b) *$\bar{y}$ is a saddle point if $p > 2(1-q)^{\frac{3}{2}}$.*
   (c) *$\bar{y}$ is nonhyperbolic point if $p = 2(1-q)^{\frac{3}{2}}$.*

(2) *Assume that $q > 1$. Then*
   (a) *The unique positive equilibrium point of equation (6) is a saddle point.*
   (b) *If $p < 2(\frac{q-1}{3})^{\frac{3}{2}}$, then $\bar{y_1}$ is a repeller and $\bar{y_2}$ is a saddle point.*
   (c) *If $p = 2(\frac{q-1}{3})^{\frac{3}{2}}$, then $\bar{y_1} = \bar{y_2} = -\sqrt{\frac{q-1}{3}}$ are unstable equilibrium points.*

Proof. Let $r = -\frac{2\bar{y}^2}{1+\bar{y}^2}$ and $s = \frac{q}{1+\bar{y}^2}$.

(1) Assume that $q < 1$ and let $\bar{y}$ be the unique positive equilibrium point of equation (6). Then
   (a) $\mid r \mid -(1-s) = \frac{2\bar{y}^2}{1+\bar{y}^2} - (1 - \frac{q}{1+\bar{y}^2}) = \frac{2\bar{y}^2}{1+\bar{y}^2} - \frac{1+\bar{y}^2-q}{1+\bar{y}^2} = \frac{\bar{y}^2-1+q}{1+\bar{y}^2} < 0$ when $p < 2(1-q)^{\frac{3}{2}}$.
   (b) $\mid r \mid - \mid 1-s \mid = \frac{2\bar{y}^2}{1+\bar{y}^2} - \frac{1+\bar{y}^2-q}{1+\bar{y}^2} = \frac{\bar{y}^2-1+q}{1+\bar{y}^2} > 0$ when $p > 2(1-q)^{\frac{3}{2}}$.
   (c) It is easy to show that equation (7) has the root $\lambda = -1$ and other root with modulus less than one.

(2) Assume that $q > 1$. Then
   (a) $\mid r \mid - \mid 1-s \mid = \frac{2\bar{y}^2}{1+\bar{y}^2} - \frac{1+\bar{y}^2-q}{1+\bar{y}^2} = \frac{\bar{y}^2-1+q}{1+\bar{y}^2} > 0$, since $\bar{y} > \sqrt{q-1}$ and $r^2 + 4s > 0$.
   (b) It is clear that $\mid 1-s \mid = \mid \frac{1+\bar{y_1}^2-q}{1+\bar{y_1}^2} \mid = -\frac{1+\bar{y_1}^2-q}{1+\bar{y_1}^2}$, since $0 > \bar{y_1} > -\sqrt{\frac{q-1}{3}} > -\sqrt{q-1}$. Then $\mid r \mid - \mid 1-s \mid = \frac{\bar{y_1}^2}{1+\bar{y_1}^2} - (-\frac{1+\bar{y_1}^2-q}{1+\bar{y_1}^2})$ $= \frac{3\bar{y_1}^2+1-q}{1+\bar{y_1}^2} < 0$. Moreover, we have $\mid s \mid = \frac{q}{1+\bar{y_1}^2} > 1$.

Now for $\bar{y_2}$, we have $\mid r \mid - \mid 1-s \mid= \frac{3\bar{y_2}^2+1-q}{1+\bar{y_2}^2} > 0$, since $\bar{\bar{y_2}} < -\sqrt{\frac{q-1}{3}}$. Moreover, we have $r^2+4s>0$.

(c) If $p = 2(\frac{q-1}{3})^{\frac{3}{2}}$, then $\bar{y_1} = \bar{y_2} = -\sqrt{\frac{q-1}{3}}$. We can show that equation (8) with $\bar{y} = \bar{y_1} = \bar{y_2} = -\sqrt{\frac{q-1}{3}}$ has the root $\lambda = 1$ and other root with modulus greater than one.

The proof is complete in view of theorem (1). □

3. Periodic nature

Theorem 3.1. *Equation (6) has the periodic solution* $\ldots, \varphi, \psi, \varphi, \psi, \ldots$ *where* φ, ψ *are the roots of the equation*

$$t^2 + \frac{1}{q-1}t - q + 1 = 0$$

if either $q<1$ *and* $p > 2(1-q)^{\frac{3}{2}}$ *or* $q>1$.

Proof. Let $\{\ldots, \varphi, \psi, \varphi, \psi, \ldots\}$ be a periodic solution of equation (6) with $q<1$. This implies that $\varphi = \frac{p+q\varphi}{1+\psi^2}, \psi = \frac{p+q\psi}{1+\varphi^2}$. Hence we have, $\varphi\psi = 1-q$ and $\varphi + \psi = \frac{p}{1-q}$. Therefore, φ, ψ are the roots of the equation

$$t^2 + \frac{1}{q-1}t - q + 1 = 0.$$

Now consider the discriminate $L = (\frac{p}{1-q})^2 - 4(1-q)$. It is clear that $L>0$ if either $q>1$ or $q<1$ and $p > 2(1-q)^{\frac{3}{2}}$. □

4. Global behavior of equation (6)

Theorem 4.1. *Assume that* $q<1$ *and* $p < 2(1-q)^{\frac{3}{2}}$. *Then the positive equilibrium point* $0 < \bar{y} < \sqrt{1-q}$ *is globally asymptotically stable.*

Proof. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of equation (6). Then

$$y_{n+1} = \frac{p+qy_{n-1}}{1+y_n^2} < p + qy_{n-1}, \quad n = 0,1,2,\ldots.$$

Then there exists a real number $\beta > 0$ such that $y_n < \beta$, $n = -1, 0, 1, 2, \ldots$ This implies that

$$y_{n+1} = \frac{p+qy_{n-1}}{1+y_n^2} > \frac{p}{1+\beta^2}.$$

Let $\lambda = \liminf y_n$ and $\Lambda = \limsup y_n$. Hence we have

$$\frac{p+q\lambda}{1+\Lambda^2} \le \lambda \le \Lambda \le \frac{p+q\Lambda}{1+\lambda^2}.$$

This implies that $p + q\lambda \le \lambda + \lambda\Lambda^2$ and $\Lambda + \Lambda\lambda^2 \le p + q\Lambda$. Then

$$p\lambda + q\lambda^2 \le \lambda^2 + \lambda^2\Lambda^2$$

and

$$\Lambda^2 + \Lambda^2\lambda^2 \le p\Lambda + q\Lambda^2.$$

Then we get that

$$p\lambda + \lambda^2(q-1) \le p\Lambda + \Lambda^2(q-1)$$

or

$$\lambda^2(1-q) - p\lambda \ge \Lambda^2(1-q) - p\Lambda \qquad (9)$$

Consider the function $g(x) = (1-q)x^2 - px$. As $p < 2(1-q)^{\frac{3}{2}}$, we have $g(\frac{p}{2(1-q)}) < 0$. That is $\frac{p}{2(1-q)} < \bar{y} < \sqrt{1-q}$, and $g(x)$ is increasing on $(\frac{p}{2(1-q)}, \infty)$. In view of equation (9), we have a contradiction. Therefore $\lambda = \Lambda = \bar{x}$. This completes the proof. □

5. Semicycle analysis

Theorem 5.1. *Let $\{y_n\}_{n=-1}^{\infty}$ be a nontrivial solution of equation (6) and let $\bar{y}$ denote the unique positive equilibrium of equation (6) such that either,*

(C_1) $y_{-1} < \bar{y} < y_0$ *or*

(C_2) $y_0 < \bar{y} < y_{-1}$.

is satisfied. Then $\{y_n\}_{n=-1}^{\infty}$ oscillates about $\bar{y}$ with semicycles of length one.

Proof. The proof is a direct consequence using theorem (1). □

6. case $q = 0$

When $q = 0$, equation (6) becomes

$$y_{n+1} = \frac{p}{1+y_n^2} \qquad , n = 0, 1, 2, \ldots \qquad (10)$$

It is clear that equation (10) has the unique positive equilibrium point $0 < \bar{y} < p$.

Theorem 6.1.

(1) *If $p \le 2$, then the equilibrium point $\bar{y}$ is locally asymptotically stable.*
(2) *If $p > 2$, then the equilibrium point $\bar{y}$ is unstable (saddle point).*

Proof. It is sufficient to consider the linearized equation

$$z_{n+1} + \frac{2\bar{y}^2}{1+\bar{y}^2} z_n = 0 \qquad , n = 0, 1, 2, \ldots$$

and its associated characteristic equation

$$\lambda + \frac{2\bar{y}^2}{1+\bar{y}^2} = 0.$$

It is easy to establish the proof if $p < 2$ and $p > 2$. When $p = 2$ we can apply theorem (1) and the result follows. □

Theorem 6.2. *The following statements are true:*

(1) *The subsequences* $\{y_{2n}\}_{n=0}^{\infty}$ *and* $\{y_{2n+1}\}_{n=0}^{\infty}$ *are neither increase together nor decrease together. Moreover,*
    (a) *If* $y_0 < \bar{y}$, *Then* $\{y_{2n}\}_{n=0}^{\infty}$ *increases and* $\{y_{2n+1}\}_{n=0}^{\infty}$ *decreases.*
    (b) *If* $y_0 > \bar{y}$, *Then* $\{y_{2n}\}_{n=0}^{\infty}$ *decreases and* $\{y_{2n+1}\}_{n=0}^{\infty}$ *increases.*

(2) *Assume that* $p < 2$. *Then the equilibrium point* $\bar{y}$ *is globally asymptotically stable.*

(3) *Assume that* $p > 2$. *Then equation (10) has the unique periodic solution* $\{\dots, \varphi, \psi, \varphi, \psi, \dots\}$, *where* φ, ψ *are the roots of the equation*

$$t^2 - pt + 1 = 0.$$

(4) *Every solution of equation (10) oscillates about* $\bar{y}$ *with semicycles of length one.*

Proof. (1) The result follows from the equality

$$y_{2n+3} - y_{2n+1} = -\frac{p(y_{2n+2} + y_{2n})}{(1 + y_{2n}^2)(1 + y_{2n+2}^2)}(y_{2n+2} - y_{2n}).$$

The proof of (2), (3) and (4) are easy to establish and will be omitted. □

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