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STRONG CONVERGENCE THEOREMS OF COMMON ELEMENTS FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS IN BANACH SPACES

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ABSTRACT. We introduce a new iterative algorithm for equilibrium and fixed point problems of three hemi-relatively nonexpansive mappings by the CQ hybrid method in Banach spaces, Our results improve and extend the corresponding results announced by Xiaolong Qin, Yeol Je Cho, Shin Min Kang [Xiaolong Qin, Yeol Je Cho, Shin Min Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, Journal of Computational and Applied Mathematics 225 (2009) 20-30], P. Kumam, K. Wattanawitoon [P. Kumam, K. Wattanawitoon, Convergence theorems of a hybrid algorithm for equilibrium problems, Nonlinear Analysis: Hybrid Systems (2009), doi:10.1016/j.nahs.2009.02.006], W. Takahashi, K. Zembayashi [W. Takahashi, K. Zembayashi, Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, Fixed Point Theory Appl. (2008) doi:10.1155/2008/528476] and others therein.

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1. Introduction

Let E be a real Banach space, E^* the dual space of E. let C be a nonempty closed convex subset of E. Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \geq 0 \textit{ for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(f)$. Given a mapping $T : C \to H$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(f)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality.

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Equilibrium problems which were introduced by Blum and Oettli [1] in 1994 have had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, physics, image reconstruction, ecology, transportation, network, elasticity and optimization. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [2-7] and the references therein.

Recall that the mapping T of C into H is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \ for\ all\ x,\ y \in C.$$

We denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

Recently, many authors studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces, respectively, see for instance, [4, 8-12, 16-18] and the references therein.

In 2004, Matsushita and Takahashi [10] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \tag{1.2}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in [0,1], T is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They prove that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Later, many authors studied the problem of finding a common element of the set of fixed points of a relatively nonexpansive mapping or two relatively nonexpansive mappings and the set of solutions of an equilibrium problem in the framework of Banach spaces. see, for instance, [11-13, 16, 18] and the references therein.

In 2009, Xiaolong Qin, Yeol Je Cho, Shin Min Kang [12] proposed the following modification of iteration (1.2) for two hemi-relatively nonexpansive mappings (Called quasi-ϕ-nonexpansive mapping in [12]):

$$\begin{cases} x_0 \in E \ chosen\ arbitrarily, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \ \ such\ that\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \ \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{cases} \tag{1.3}$$

where J is the duality mapping on E, and Π_C is the generalized projection from E onto a closed convex subset C of E and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $F = F(S) \cap F(T) \cap EP(f)$.

Recently, Poom Kumam, Kriengsak Wattanawitoon [16], introduced the modification for two hemi-relatively nonexpansive mappings as follows:

$$\begin{cases} x_0 \in C \ \textit{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S z_n), \\ z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J T x_n), \\ u_n \in C \ \ \textit{such that} \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.4}$$

they also proved that the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $F = F(S) \cap F(T) \cap EP(f)$.

Motivated and inspired by the research going on in this direction, we introduce a hybrid projection algorithm to find a common element of the set of solutions of an equilibrium problem and the set of common fixed points of three hemi-relatively nonexpansive mappings by the monotone CQ hybrid method in the framework of Banach spaces.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. Denote by $\langle \cdot, \cdot \rangle$ the duality product. The normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

for $x \in E$.

Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty closed convex subset of E. Throughout this paper, we denote by ϕ the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x),$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [11]). It is observe from the definition of the function ϕ that it has the properties as follows:

- $(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2$,
- $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,
- $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$,

for all $x, y \in E$, see[19] for more details. If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$.

Remark 2.1 If E is a strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From $(1')$, we have $\|x\| = \|y\|$, this implies $\langle y, Jx\rangle = \|y\|^2 = \|Jx\|^2$. From the definition of J, we have $Jx = Jy$. Since J is one to one, we have $x = y$. see [20,21] for more details.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T, if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denote by $\widehat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive [10, 11, 19] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively n onexpansive mapping was studied [10, 11, 19]. A point p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is called relatively weak nonexpansive if $\widetilde{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. A mapping T is called hemi-relatively nonexpansive if $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

It is obvious that a relatively nonexpansive mapping is a relatively weak nonexpansive mapping. and a relatively weak nonexpansive mapping is a hemi-relatively nonexpansive mapping.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|\frac{x_n - y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E, then the Banach space E is said to be smooth provided

$$\lim_{t\to 0}\frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$.

It is well known that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E, more properties of the duality mapping have been given in [20, 21].

We also need some definitions and lemmas which will be used in the proofs for the main results in the next section.

Lemma 2.1([19]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Lemma 2.2([22]). *Let C be a nonempty closed convex subset of a smooth real Banach space E and $x \in E$, then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0$$

Lemma 2.3(Alber[26]). *Let E be a reflexive ,strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$, then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$$

for all $y \in C$.

Lemma 2.4([12]). *Let E be a strictly convex and smooth real Banach space, let C be a closed convex subset of E, and let T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.5([23])). *Let X be uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.6 ([19])). *Let E be uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ and $g(\|x - y\|) \leq \phi(x, y)$ for all x, y $in B_r$.*

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A_1). $f(x, x) = 0$, *for all* $x \in C$;

(A_2). f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$, *for all* $x, y \in C$;

(A_3). For each $x, y, z \in C$, $\lim_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$;

(A_4). For each $x \in C$, the function $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 2.7([1]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $r > 0$ and $x \in E$, then, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad for\ all\ y \in C. \tag{2.4}$$

Lemma 2.8([24]). *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C\} \tag{2.5}$$

for all $z \in C$. Then, the following hold:

1. *T_r is single-valued;*

2. T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

3. $F(T_r) = EP(f)$;
4. $EP(f)$ is closed and convex.

Lemma 2.9([25]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$ and let $r > 0$. Then for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

3. Main results

In this section, we establish strong convergence theorem for equilibrium problems and fixed point problems of three hemi-relatively nonexpansive mappings which are more general than relatively non-expansive mappings in Banach spaces.

Theorem 3.1. *Let E a uniformly convex and uniformly smooth real Banach space, let C be a nonempty and closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $A_1 - A_4$ and let T, S, $W : C \to C$ are closed hemi-relatively nonexpansive mappings such that $F := F(S)T \cap F(T) \cap F(W) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ z_n = J^{-1}(\beta_n^1 Jx_n + \beta_n^2 JSx_n + \beta_n^3 JWx_n), \\ u_n \in C \;\text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.1}$$

where J is the duality mapping on E. Assume that α_n and β_n^i, where $i = 1, 2, 3$ are four sequences in [0, 1] satisfying the restrictions:
(a) $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$, $\lim_{n\to\infty} \beta_n^2 = \lim_{n\to\infty} \beta_n^3 = 0$;
(b) $\liminf_{n\to\infty}(1-\alpha_n)\beta_n^1\beta_n^2 > 0$, $\limsup_{n\to\infty} \alpha_n < 1$;
(c) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

If T is uniformly continuous, Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto $F := F(S)T \cap F(T) \cap F(W) \cap EP(f)$.

Proof. First, we show that C_n is closed and convex for all $n \geq 0$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For all $z \in C_k$, one obtains that

$$\phi(z, u_k) \leq \phi(z, x_k)$$

is equivalent to

$$2(\langle z, Jx_k\rangle - 2\langle z, Ju_k\rangle) \le \|x_k\|^2 - \|u_k\|^2.$$

It is easy to see that C_{k+1} is closed and convex. Then, for all $n \ge 1$, C_n is closed and convex. This shows that $\Pi_{C_{n+1}} x_0$ is well defined.

Next, we show that $F \subset C_n$ for all $n \ge 0$. $F \subset C_1 = C$ is obvious. Suppose $F \subset C_k$ for some $k \in \mathbb{N}$. Notice that $u_n = T_{r_n} y_n$ for all $n \ge 0$, On the other hand, from Lemma 2.8, one has T_{r_n} is a hemi-relatively nonexpansive mapping. Then, for $\forall p \in F \subset C_k$, one has

$$\begin{aligned}
\phi(p, u_k) &= \phi(p, T_{r_k} y_k)\\
&\le \phi(p, y_k)\\
&= \phi(p, J^{-1}(\alpha_k Jx_k + (1-\alpha_k)JTz_k))\|\\
&= \|p\|^2 - 2\langle p, \alpha_k Jx_k + (1-\alpha_k)JTz_k\rangle + \|\alpha_k Jx_k + (1-\alpha_k)JTz_k\|^2\\
&\le \|p\|^2 - 2\langle p, \alpha_k Jx_k + (1-\alpha_k)JTz_k\rangle + \alpha_k\|Jx_k\| + (1-\alpha_k)\|JTz_k\|^2\\
&= \alpha_k\phi(p, x_k) + (1-\alpha_k)\phi(p, Tz_k)\\
&\le \alpha_k\phi(p, x_k) + (1-\alpha_k)\phi(p, z_k)
\end{aligned} \tag{3.2}$$

and then

$$\begin{aligned}
\phi(p, z_k) &= \phi(p, J^{-1}(\beta_k^1 Jx_k + \beta_k^2 JSx_k + \beta_k^3 JWx_k))\\
&= \|p\|^2 - 2\beta_k^1\langle p, Jx_k\rangle - 2\beta_k^2\langle p, JSx_k\rangle - 2\beta_k^3\langle p, JWx_k\rangle\\
&\quad + \|\beta_k^1 Jx_k + \beta_k^2 JSx_k + \beta_k^3 JWx_k\|^2\\
&\le \|p\|^2 - 2\beta_k^1\langle p, Jx_k\rangle - 2\beta_k^2\langle p, JSx_k\rangle - 2\beta_k^3\langle p, JWx_k\rangle\\
&\quad + \beta_k^1\|Jx_k\|^2 + \beta_k^2\|JSx_k\|^2 + \beta_k^3\|JWx_k\|^2\\
&= \beta_k^1\phi(p, x_k) + \beta_k^2\phi(p, Sx_k) + \beta_k^3\|JWx_k\|^2\\
&\le \phi(p, x_k)
\end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), one has

$$\phi(p, u_k) \le \alpha_k\phi(p, x_k) + (1-\alpha_k)\phi(p, x_k) \le \phi(p, x_k), \tag{3.4}$$

that is $p \in C_{n+1}$. This implies that $F \subset C_n$, for all $n \ge 0$.

From $x_n = \Pi_{C_n} x_0$, one sees

$$\langle x_n - z, Jx_0 - Jx_n\rangle \ge 0 \quad \forall\, z \in C_n. \tag{3.5}$$

Since $F \subset C_n$ for all $n \ge 0$, one arrives at

$$\langle x_n - p, Jx_0 - Jx_n\rangle \ge 0 \quad \forall\, p \in F. \tag{3.6}$$

From Lemma 2.3, one has

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \le \phi(p, x_0) - \phi(p, x_n) \le \phi(p, x_0), \tag{3.7}$$

for each $p \in F \subset C_n$ and $n \geq 1$. Therefore, the sequence $\phi(x_n, x_0)$ is bounded. On the other hand, noticing that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, one has

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \tag{3.8}$$

for all $n \geq 0$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, one has $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \tag{3.9}$$

One has $\phi(x_m, x_n) \to 0$, as m, $n \to \infty$ in above inequality. It follows from Lemma 2.1 that

$$x_m - x_n \to 0, \quad as\ m,\ n \to \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \to q \in C$ as $n \to \infty$. Similar to (3.9), by analogy, one can obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.10}$$

From Lemma 2.2, we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.11}$$

Noticing that $x_{n+1} \in C_{n+1}$, we obtain

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n). \tag{3.12}$$

It follows from (3.10) that

$$\phi(x_{n+1}, u_n) \to 0, \quad as\ n \to \infty.$$

From Lemma 2.1, one has

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.13}$$

Combining (3.11) whit (3.13), one gets

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.14}$$

It follows from $x_n \to q$ as $n \to \infty$, that $u_n \to q$ as $n \to \infty$.

On the other hand, since J is uniformly norm-to-norm continuous on bounded sets and $\lim_n \|x_n - u_n\| = 0$, one has

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.15}$$

, Since E is a uniformly smooth Banach space, one knows that E^* is a uniformly convex Banach space. Let $r = \sup_{n\geq 0}\{\|x_n\|,\ \|Tx_n\|,\ \|Sx_n\|,\ \|Wx_n\|\}$. From

Lemma 2.5, one has

$$\begin{aligned}\phi(p,z_n) &= \phi(p, J^{-1}(\beta_n^1 Jx_n + \beta_n^2 JSx_n + \beta_n^3 JWx_n)) \\ &= \|p\|^2 - 2\beta_n^1\langle p, Jx_n\rangle - 2\beta_n^2\langle p, JSx_n\rangle - 2\beta_n^3\langle p, JWx_n\rangle \\ &\quad + \|\beta_n^1 Jx_n + \beta_n^2 JSx_n + \beta_n^3 JWx_n\|^2 \\ &\le \|p\|^2 - 2\beta_n^1\langle p, Jx_n\rangle - 2\beta_n^2\langle p, JSx_n\rangle - 2\beta_n^3\langle p, JWx_n\rangle \\ &\quad + \beta_n^1\|Jx_n\|^2 + \beta_n^2\|JSx_n\|^2 + \beta_n^3\|JWx_n\|^2 - \beta_n^1\beta_n^2 g(\|JSx_n - Jx_n\|) \\ &= \beta_n^1\phi(p,x_n) + \beta_n^2\phi(p,Sx_n) + \beta_k^3\|JWx_n\|^2 - \beta_n^1\beta_n^2 g(\|JSx_n - Jx_n\|) \\ &\le \phi(p,x_n) - \beta_n^1\beta_n^2 g(\|JSx_n - Jx_n\|)\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\phi(p,u_n) = \phi(p, T_{r_n}y_n) &\le \phi(p,y_n) \\ &\le \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p,z_n).\end{aligned} \tag{3.17}$$

Substituting (3.16) into (3.17), one has

$$\begin{aligned}\phi(p,u_n) &\le \alpha_n\phi(p,x_n) + (1-\alpha_n)(\phi(p,x_n) - \beta_n^1\beta_n^2 g(\|JSx_n - Jx_n\|)) \\ &\le \phi(p,x_n) - (1-\alpha_n)\beta_n^1\beta_n^2 g(\|JSx_n - Jx_n\|).\end{aligned} \tag{3.18}$$

It follows that

$$(1-\alpha_n)\beta_n^1\beta_n^2 g(\|JSx_n - Jx_n\|) \le \phi(p,x_n) - \phi(p,u_n). \tag{3.19}$$

On the other hand, one has

$$\begin{aligned}\phi(p,x_n) - \phi(p,u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n\rangle \\ &\le \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\|.\end{aligned}$$

It follows from $\|x_n - u_n\| \to 0$ and $\|Jx_n - Ju_n\| \to 0$ that

$$\phi(p,x_n) - \phi(p,u_n) \to 0 \quad as\ n \to \infty. \tag{2.20}$$

Observing the assumption $\liminf_{n\to\infty}(1-\alpha_n)\beta_n^1\beta_n^2 > 0$, (3.19) and (3.20), one has

$$g(\|JSx_n - Jx_n\|) \to 0, \quad as\ n \to \infty.$$

It follows from the property of g that

$$\|JSx_n - Jx_n\| \to 0, \quad as\ n \to \infty.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, one sees that

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0.$$

By the same analogy, one can obtain

$$\lim_{n\to\infty}\|x_n - Wx_n\| = 0.$$

From the closedness of S and W, one has $q \in F(S) \cap F(W)$. One obtain

$$\begin{aligned} &\phi(x_{n+1}, z_n) \\ &= \phi(x_{n+1}, J^{-1}(\beta_n^1 Jx_n + \beta_n^2 JSx_n + \beta_n^3 JWx_n)) \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n^1 Jx_n + \beta_n^2 JSx_n + \beta_n^3 JWx_n\rangle \\ &\quad + \|\beta_n^1 Jx_n + \beta_n^2 JSx_n + \beta_n^3 JWx_n\|^2 \\ &\le \|x_{n+1}\|^2 - 2\beta_n^1\langle x_{n+1}, Jx_n\rangle - 2\beta_n^2\langle x_{n+1}, JSx_n\rangle - 2\beta_n^3\langle x_{n+1}, JWx_n\rangle \\ &\quad + \beta_n^1\|Jx_n\|^2 + \beta_n^2\|JSx_n\|^2 + \beta_n^3\|JWx_n\|^2 \\ &= \beta_n^1\phi(x_{n+1}, x_n) + \beta_n^2\phi(x_{n+1}, Sx_n) + \beta_n^3\phi(x_{n+1}, Wx_n). \end{aligned} \tag{3.21}$$

Since $\lim_{n\to\infty}\beta_n^2 = \lim_{n\to\infty}\beta_n^3 = 0$, $\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0$ and $\{x_n\}$ is bounded, therefore, $\phi(x_{n+1}, z_n) \to 0$, as $n \to \infty$. Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, from (3.2) and (3.3), one has

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n),$$

for all $n \ge 0$. Hence $\phi(x_{n+1}, y_n) \to 0$, as $n \to \infty$. By using Lemma 2.1, one also has

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = \lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|x_{n+1} - z_n\| = 0 \tag{3.22}$$

Since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty}\|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty}\|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty}\|Jx_{n+1} - Jz_n\| = 0 \tag{3.23}$$

Observing that

$$\begin{aligned} \|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1-\alpha_n)JTz_n)\| \\ &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JTz_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JTz_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\ &\ge (1-\alpha_n)\|Jx_{n+1} - JTz_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|. \end{aligned}$$

It follows that

$$\|Jx_{n+1} - JTz_n\| \le \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|).$$

By (3.23) and $\limsup_{n\to\infty}\alpha_n < 1$, one sees $\lim_{n\to\infty}\|Jx_{n+1} - JTz_n\| = 0$. Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty}\|x_{n+1} - Tz_n\| = 0. \tag{3.24}$$

Since

$$\|z_n - x_n\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

in view of (3.22), one obtain

$$\lim\|z_n - x_n\| = 0. \tag{3.25}$$

By using the triangle inequality, we get

$$\|x_n - Tx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - Tz_n\| + \|Tz_n - Tx_n\|,$$

since T is uniformly continuous, it follows from (3.22), (3.24) and (3.25) that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Since T is also closed operator and $x_n \to q$, then q is also a fixed point of T. Hence $q \in F(T) \cap F(S) \cap F(W)$.

Next, we show $q \in EP(f) = F(T_r)$. From $u_n = T_{r_n} y_n$ and Lemma 2.9, one obtains

$$\begin{aligned}\phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \le \phi(q, y_n) - \phi(q, T_{r_n} y_n)\\ &\le \phi(q, x_n) - \phi(q, T_{r_n} y_n)\\ &= \phi(q, x_n) - \phi(q, u_n).\end{aligned}$$

It follows from (3.20) that $\phi(u_n, y_n) \to 0,\ as\ n \to \infty$. Noticing that Lemma 2.1, we get

$$\|u_n - y_n\| \to 0, \quad as\ n \to \infty.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0.$$

From the (A_2), we note that

$$\begin{aligned}\|y - u_n\|\frac{\|Ju_n - Jy_n\|}{r_n} &\ge \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle\\ &\ge -f(u_n, y)\\ &\ge f(y, u_n), \quad \forall\ y \in C.\end{aligned}$$

By taking the limit as $n \to \infty$ in above inequality and from (A_4) and $u_n \to q$, one has $f(y, q) \le 0$, $\forall y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)q$. Noticing $y,\ q \in C$, we obtain $y_t \in C$, which yields that $f(y_t, q) \le 0$. It follows from (A_1) that

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1-t)f(y_t, q) \le tf(y_t, y).$$

It follows that $f(y_t, y) \ge 0$. Let $t \downarrow 0$ from (A_3), we obtain $f(q, y) \ge 0$ for $\forall y \in C$. This implies that $q \in EP(f)$. This shows that $q \in F$.

Finally, we prove $q = \Pi_F x_0$.

By taking limit in (3.6), one has

$$\langle q - p, Jx_0 - Jq\rangle \ge 0, \quad \forall p \in F.$$

In view of Lemma 2.2, one sees that $q = \Pi_F x_0$. This completes the proof. □

According to Theorem 3.1, one can obtain the following corollaries directly.

Corollary 3.2. *Let E a uniformly convex and uniformly smooth real Banach space, let C be a nonempty and closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $A_1 - A_4$ and let $T,\ S : C \to C$ are closed hemi-relatively nonexpansive mappings such that $F := F(S)T \cap F(T) \cap EP(f) \ne \emptyset$.*

Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T z_n), \\ z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S x_n), \\ u_n \in C \ \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.1}$$

where J is the duality mapping on E. Assume that α_n and β_n are two sequences in [0, 1] satisfying the restrictions:

(a) $\lim_{n\to\infty} \beta_n = 1$, $\limsup_{n\to\infty} \alpha_n < 1$;

(b) $\liminf_{n\to\infty} (1-\alpha_n)\beta_n(1-\beta_n) > 0$,

(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

If T is uniformly continuous, Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto $F := F(S)T \cap F(T) \cap EP(f)$.

Remark 1. Corollary 3.2 is the same as Theorem 3.1 in Poom Kumam, Kriengsak Wattanawitoon [16], so it is a special case in our Theorem 3.1.

Corollary 3.3. *Let E a uniformly convex and uniformly smooth real Banach space, let C be a nonempty and closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $A_1 - A_4$ and let T, $S : C \to C$ are closed hemi-relatively nonexpansive mappings such that $F := F(S)T \cap F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S x_n), \\ u_n \in C \ \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 \end{cases}$$

where J is the duality mapping on E. Assume that α_n, β_n and γ_n are three sequences in [0, 1] satisfying the restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$

(b) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$;

(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto $F := F(S)T \cap F(T) \cap EP(f)$.

Proof. Put $\alpha_n \equiv 0$ and $T \equiv I$ in Theorem 3.1, we can get the result directly. □

Remark 2. Corollary 3.3 is the main result in Xiaolong Qin, Yeol Je Cho, Shin Min Kang [12], so it is also a special case in our Theorem 3.1.

Remark 3. Our Theorem 3.1 have improved and extended many prevenient results, whereat, we cannot list them all.

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