J. Appl. Math. & Informatics Vol. **28**(2010), No. 3 - 4, pp. 773 - 781
Website: http://www.kcam.biz

A GENERAL FORM OF MULTI-STEP ITERATIVE METHODS FOR NONLINEAR EQUATIONS†

SEYOUNG OH AND JAE HEON YUN*

ABSTRACT. Recently, Yun [8] proposed a new three-step iterative method with the fourth-order convergence for solving nonlinear equations. By using his ideas, we develop a general form of multi-step iterative methods with higher order convergence for solving nonlinear equations, and then we study convergence analysis of the multi-step iterative methods. Lastly, some numerical experiments are given to illustrate the performance of the multi-step iterative methods.

AMS Mathematics Subject Classification : 65H05.
Key word and phrases : Nonlinear equation, three-step iterative method, multi-step iterative method.

1. Introduction

In recent years, much attention has been given to develop several iterative methods for solving nonlinear equations [1, 3, 4, 5]. Abbasbandy [1] and Chun [3] have proposed and studied several one-step and two-step iterative methods with higher order convergence for nonlinear equations by using the decomposition technique of Adomain [2]. Noor et al [6] proposed a three-step iterative method with third-order convergence for solving nonlinear equations using a different type of decomposition, which is very simple as compared with Adomain decomposition method and does not involve the higher order derivatives. In 2008, Yun [8] proposed a new three-step iterative method with the fourth-order convergence for solving nonlinear equations, which improves the three-step iterative method of Noor et al [6]. In this paper, we first develop a general form of multi-step iterative methods with higher order convergence for solving nonlinear equations by using the ideas of Yun [8], and then we study convergence analysis of the multi-step iterative methods. Lastly, some numerical experiments are given to illustrate the performance of the multi-step iterative methods.

Received October 8, 2009. Accepted January 16, 2010. *Corresponding author.
†This work was supported by the research grant of the Chungbuk National University in 2009

2. Multi-step iterative methods

In this section, we first describe the derivation of multi-step iterative methods by using the ideas of Yun [8], and then we study convergence analysis of the multi-step iterative methods. Consider the nonlinear equation

$$f(x) = 0. \tag{1}$$

We assume that $f(x)$ has a simple root at α and γ is an initial guess sufficiently close to α. We can rewrite the nonlinear equation (1) into the following coupled system using the Taylor series:

$$f(\gamma) + f'(\gamma)(x - \gamma) + g(x) = 0, \tag{2}$$

$$g(x) = f(x) - f(\gamma) - f'(\gamma)(x - \gamma). \tag{3}$$

The equation (2) can be rewritten in the following form

$$x = c + N(x), \tag{4}$$

where

$$c = \gamma - \frac{f(\gamma)}{f'(\gamma)} \tag{5}$$

and

$$N(x) = -\frac{g(x)}{f'(\gamma)}. \tag{6}$$

Here $N(x)$ is a nonlinear function.

We now construct a sequence of multi-step iterative methods with higher order convergence by using the following decomposition method, which is mainly due to Daftardar-Gejji and Jafari [4]. This decomposition of the nonlinear function $N(x)$ is quite different from that of Adomain decomposition. The main idea of this technique is to look for a solution having the series form

$$x = \sum_{n=0}^{\infty} x_n, \tag{7}$$

and the nonlinear function $N(x)$ can be decomposed as

$$N(x) = N(x_0) + \sum_{n=1}^{\infty} \left\{ N(\sum_{j=0}^{n} x_j) - N(\sum_{j=0}^{n-1} x_j) \right\} \tag{8}$$

Substituting (7) and (8) into (4), one obtains

$$\sum_{n=0}^{\infty} x_n = c + N(x_0) + \sum_{n=1}^{\infty} \left\{ N(\sum_{j=0}^{n} x_j) - N(\sum_{j=0}^{n-1} x_j) \right\}. \tag{9}$$

It follows from (9) that

$$x_0 = c,$$
$$x_1 = N(x_0),$$

$$x_{m+1} = N\left(\sum_{j=0}^{m} x_j\right) - N\left(\sum_{j=0}^{m-1} x_j\right), \; m = 1, 2, \ldots.$$

Then

$$x_1 + x_2 + \cdots + x_{m+1} = N(x_0 + x_1 + \cdots + x_m), \; m = 1, 2, \ldots. \tag{10}$$

Since $x_0 = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}$, from (3) and (6) one obtains

$$g(x_0) = f(x_0) \tag{11}$$

and

$$x_1 = N(x_0) = -\frac{g(x_0)}{f'(\gamma)} = -\frac{f(x_0)}{f'(\gamma)}. \tag{12}$$

Since $x_0 + x_1 = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)}$, from (3) and (6)

$$\begin{aligned} N(x_0 + x_1) &= -\frac{g(x_0 + x_1)}{f'(\gamma)} \\ &= -\frac{f(x_0 + x_1) - f(\gamma) - f'(\gamma)(x_0 + x_1 - \gamma)}{f'(\gamma)} \\ &= -\frac{f(x_0 + x_1) + f(x_0)}{f'(\gamma)}. \end{aligned} \tag{13}$$

Since $x_2 = N(x_0 + x_1) - N(x_0)$, from (12) and (13)

$$x_2 = -\frac{f(x_0 + x_1)}{f'(\gamma)}. \tag{14}$$

From (12) and (14),

$$x_0 + x_1 + x_2 = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0 + x_1) + f(x_0)}{f'(\gamma)}. \tag{15}$$

From (3), (6) and (15),

$$\begin{aligned} N(x_0 + x_1 + x_2) &= -\frac{g(x_0 + x_1 + x_2)}{f'(\gamma)} \\ &= -\frac{f(x_0 + x_1 + x_2) - f(\gamma) - f'(\gamma)(x_0 + x_1 + x_2 - \gamma)}{f'(\gamma)} \\ &= -\frac{f(x_0 + x_1 + x_2) + f(x_0 + x_1) + f(x_0)}{f'(\gamma)}. \end{aligned} \tag{16}$$

Assume that $x \approx x_0 + x_1 + x_2 + x_3$. Then (10) and (16) imply that

$$\begin{aligned} x &\approx x_0 + x_1 + x_2 + x_3 = x_0 + N(x_0 + x_1 + x_2) \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0 + x_1 + x_2) + f(x_0 + x_1) + f(x_0)}{f'(\gamma)}, \end{aligned} \tag{17}$$

which suggests the following four-step iterative method

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= -\frac{f(y_n)}{f'(x_n)} \\ v_n &= -\frac{f(y_n+z_n)}{f'(x_n)} \\ x_{n+1} &= y_n + z_n + v_n - \frac{f(y_n+z_n+v_n)}{f'(x_n)}. \end{aligned} \tag{18}$$

Since $x_3 = N(x_0+x_1+x_2) - N(x_0+x_1)$, from (13) and (16)

$$x_3 = -\frac{f(x_0+x_1+x_2)}{f'(\gamma)}. \tag{19}$$

From (12), (14) and (19),

$$x_0+x_1+x_2+x_3 = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0+x_1+x_2)+f(x_0+x_1)+f(x_0)}{f'(\gamma)}. \tag{20}$$

From (3), (6) and (20),

$$\begin{aligned} &N(x_0+x_1+x_2+x_3) \\ &= -\frac{g(x_0+x_1+x_2+x_3)}{f'(\gamma)} \\ &= -\frac{f(x_0+x_1+x_2+x_3) - f(\gamma) - f'(\gamma)(x_0+x_1+x_2+x_3-\gamma)}{f'(\gamma)} \\ &= -\frac{f(x_0+x_1+x_2+x_3)+f(x_0+x_1+x_2)+f(x_0+x_1)+f(x_0)}{f'(\gamma)}. \end{aligned} \tag{21}$$

Assume that $x \approx x_0+x_1+x_2+x_3+x_4$. Then (10) and (21) imply that

$$\begin{aligned} x &\approx x_0+x_1+x_2+x_3 = x_0 + N(x_0+x_1+x_2+x_3) \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0+x_1+x_2+x_3)+f(x_0+x_1+x_2)+f(x_0+x_1)+f(x_0)}{f'(\gamma)}, \end{aligned} \tag{22}$$

which suggests the following five-step iterative method

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= -\frac{f(y_n)}{f'(x_n)} \\ v_n &= -\frac{f(y_n+z_n)}{f'(x_n)} \\ w_n &= -\frac{f(y_n+z_n+v_n)}{f'(x_n)} \\ x_{n+1} &= y_n + z_n + v_n + w_n - \frac{f(y_n+z_n+v_n+w_n)}{f'(x_n)}. \end{aligned} \tag{23}$$

By continuing the above arguments, we can easily obtain a general form of multi-step iterative methods. Specifically, the k-step iterative method for $k \geq 3$ is of the form

$$
\begin{aligned}
y_n^1 &= x_n - \frac{f(x_n)}{f'(x_n)} \\
y_n^2 &= -\frac{f(y_n^1)}{f'(x_n)} \\
&\vdots \\
y_n^{k-1} &= -\frac{f\left(\sum_{j=1}^{k-2} y_n^j\right)}{f'(x_n)} \\
x_{n+1} &= \sum_{j=1}^{k-1} y_n^j - \frac{f\left(\sum_{j=1}^{k-1} y_n^j\right)}{f'(x_n)}.
\end{aligned} \tag{24}
$$

Notice that the multi-step iterative method (24) requires the computation of only the first derivative of the function $f(x)$. The following theorem shows that the four-step iterative method (18) has fifth-order convergence.

Theorem 2.1. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to R$ for an open interval I. If x_0 is sufficiently close to α, then the four-step iterative method (18) has fifth-order convergence and satisfies the following error equation*

$$e_{n+1} = 8c_2^4 e_n^5 + O(e_n^6), \tag{25}$$

where $e_n = x_n - \alpha$ for $n = 0, 1, 2, \dots$ and $c_2 = \frac{f^{(2)}(\alpha)}{2f'(\alpha)}$.

Proof. Since f is sufficiently differentiable, by expanding $f(x_n)$ and $f'(x_n)$ about α, one obtains

$$f(x_n) = f'(\alpha)\left(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O(e_n^6)\right), \tag{26}$$

$$f'(x_n) = f'(\alpha)\left(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O(e_n^5)\right), \tag{27}$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$ for $k = 2, 3, \dots$. From (26) and (27), one obtains

$$
\begin{aligned}
\frac{f(x_n)}{f'(x_n)} &= e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 3c_4 - 4c_2^3)e_n^4 \\
&\quad + (10c_2c_4 - 20c_2^2c_3 + 8c_2^4 - 4c_5 + 6c_3^2)e_n^5 + O(e_n^6).
\end{aligned} \tag{28}
$$

From (18) and (28), one obtains

$$
\begin{aligned}
y_n &= \alpha + c_2 e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 3c_4 - 4c_2^3)e_n^4 \\
&\quad - (10c_2c_4 - 20c_2^2c_3 + 8c_2^4 - 4c_5 + 6c_3^2)e_n^5 + O(e_n^6).
\end{aligned} \tag{29}
$$

Expanding $f(y_n)$ about α and using (29),

$$f(y_n) = f'(\alpha)[c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 3c_4 - 5c_2^3)e_n^4 - (10c_2c_4 - 24c_2^2c_3 + 12c_2^4 - 4c_5 + 6c_3^2)e_n^5 + O(e_n^6)]. \tag{30}$$

From (27) and (30), one obtains

$$z_n = -c_2e_n^2 + 2(2c_2^2 - c_3)e_n^3 + (14c_2c_3 - 3c_4 - 13c_2^3)e_n^4 + (20c_2c_4 - 64c_2^2c_3 + 38c_2^4 - 4c_5 + 12c_3^2)e_n^5 + O(e_n^6). \tag{31}$$

Using (29) and (31),

$$y_n + z_n = \alpha + 2c_2^2e_n^3 + (7c_2c_3 - 9c_2^3)e_n^4 + (10c_2c_4 - 44c_2^2c_3 + 30c_2^4 + 6c_3^2)e_n^5 + O(e_n^6). \tag{32}$$

Expanding $f(y_n + z_n)$ about α and using (32),

$$f(y_n + z_n) = f'(\alpha)[2c_2^2e_n^3 + (7c_2c_3 - 9c_2^3)e_n^4 + (10c_2c_4 - 44c_2^2c_3 + 30c_2^4 + 6c_3^2)e_n^5 + O(e_n^6)]. \tag{33}$$

Using (27) and (33),

$$v_n = -2c_2^2e_n^3 - (7c_2c_3 - 13c_2^3)e_n^4 - (10c_2c_4 - 64c_2^2c_3 + 56c_2^4 + 6c_3^2)e_n^5 + O(e_n^6). \tag{34}$$

From (32) and (34),

$$y_n + z_n + v_n = \alpha + 4c_2^3e_n^4 + (20c_2^2c_3 - 26c_2^4)e_n^5 + O(e_n^6). \tag{35}$$

Expanding $f(y_n + z_n + v_n)$ about α and using (35),

$$f(y_n + z_n + v_n) = f'(\alpha)[4c_2^3e_n^4 + (20c_2^2c_3 - 26c_2^4)e_n^5 + O(e_n^6)]. \tag{36}$$

From (27) and (36),

$$\frac{f(y_n + z_n + v_n)}{f'(x_n)} = 4c_2^3e_n^4 + (20c_2^2c_3 - 34c_2^4)e_n^5 + O(e_n^6). \tag{37}$$

From (18), (35) and (37), one obtains

$$\begin{aligned} x_{n+1} &= y_n + z_n + v_n - \frac{f(y_n + z_n + v_n)}{f'(x_n)} \\ &= \alpha + 8c_2^4e_n^5 + O(e_n^6), \end{aligned}$$

which implies that $e_{n+1} = 8c_2^4e_n^5 + O(e_n^6)$. □

It was shown in [8] that the three-step iterative method (24) with $k = 3$ has fourth-order convergence, i.e., $e_{n+1} = 4c_2^3e_n^4 + O(e_n^5)$. Using MAPLE which is one of computer algebra systems, we can obtain that the k-step iterative method (24) with $k \geq 3$ has $(k+1)$th-order convergence (see [3] for computation of error equations by MAPLE). More specifically, the k-step iterative method (24) with $k \geq 3$ satisfies the following error equation

$$e_{n+1} = 2^{k-1}c_2^ke_n^{k+1} + O(e_n^{k+2}). \tag{38}$$

3. Numerical experiments

We present some numerical results to illustrate the efficiency of multi-step iterative method proposed in this paper. We compare the Newton method (NM), Variant of Newton method (VNM) in [7], Chun's method with fourth-order convergence (CM4) in [3], three-step iterative method (24) with $k = 3$ (TM), and four-step iterative method (18) (FM). The iteration has been stopped if one of the following stopping criteria is satisfied:

$$\text{(i) } |x_{n+1} - x_n| < 10^{-15},$$
$$\text{(ii) } |f(x_{n+1})| < 10^{-15}.$$

Let x_0 denote an initial approximation for the zero. The test functions to be used are defined as follows:

$$f_1(x) = \sin^2 x - x^2 + 1, \ x_0 = 1.2$$
$$f_2(x) = (x-1)^3 - 1, \ x_0 = 3.5$$
$$f_3(x) = x^3 - 10, \ x_0 = 1.2$$
$$f_4(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \ x_0 = -3$$
$$f_5(x) = e^{x^2+7x-30} - 1, \ x_0 = 4.0$$
$$f_6(x) = x^3 + 4x^2 - 10, \ x_0 = 3.5\,.$$

All numerical tests have been written in standard Fortran and carried out using 64-bit arithmetic. In Table 1, IT denotes the number of iterations, x_n the approximate zero, $f(x_n)$ the function value at x_n, and δ the distance of two consecutive approximations for the zero.

4. Conclusion

In this paper, we developed a general form of multi-step iterative methods with higher order convergence for solving nonlinear equations by using the ideas of Yun [8]. The derivation of multi-step iterative methods introduced in this paper is very simple as compared with the Adomain decomposition methods. Notice that the multi-step iterative methods requires the computation of *only the first order derivative* of the function $f(x)$ unlike other methods of the same order. With the help of MAPLE, we can obtain that the k-step iterative method (24) with $k \geq 3$ has the convergence of order $(k+1)$, i.e., $e_{n+1} = 2^{k-1}c_2^k e_n^{k+1} + O(e_n^{k+2})$. Numerical results for several nonlinear equations are provided in Table 1 to show the convergence of the multi-step iterative methods.

REFERENCES

1. S. Abbasbandy, *Improving Newton-Raphson method for nonlinear equations by modified Adomain decomposition method,* Appl. Math. Comput. 145 (2003) 887-893.

TABLE 1. Comparison of iterative methods

Function	Method	IT	x_n	$f(x_n)$	δ
$f_1(x)$	NM	5	1.40449164821534	-4.44×10^{-16}	2.04×10^{-12}
	VNM	3	1.40449164821534	3.33×10^{-16}	4.46×10^{-7}
	CM4	3	1.40449164821534	-4.44×10^{-16}	8.28×10^{-8}
	TM	3	1.40449164821534	-4.44×10^{-16}	1.19×10^{-8}
	FM	3	1.40449164821534	-4.44×10^{-16}	3.28×10^{-12}
$f_2(x)$	NM	7	2.0	0.0	2.88×10^{-11}
	VNM	5	2.0	0.0	6.55×10^{-13}
	CM4	4	2.0	0.0	8.61×10^{-7}
	TM	4	2.0	0.0	3.48×10^{-7}
	FM	4	2.0	0.0	3.04×10^{-12}
$f_3(x)$	NM	8	2.15443469003188	1.78×10^{-15}	0.0
	VNM	5	2.15443469003188	1.78×10^{-15}	8.88×10^{-16}
	CM4	9	2.15443469003188	1.78×10^{-15}	0.0
	TM	5	2.15443469003188	1.78×10^{-15}	0.0
	FM	5	2.15443469003188	1.78×10^{-15}	0.0
$f_4(x)$	NM	15	-1.20764782713092	3.55×10^{-15}	2.22×10^{-16}
	VNM	10	-1.20764782713092	3.55×10^{-15}	2.22×10^{-16}
	CM4	9	-1.20764782713092	-2.66×10^{-15}	0.0
	TM	9	-1.20764782713092	-2.66×10^{-15}	2.22×10^{-16}
	FM	8	-1.20764782713092	-2.66×10^{-15}	0.0
$f_5(x)$	NM	19	3.0	0.0	3.27×10^{-11}
	VNM	13	3.0	0.0	1.48×10^{-9}
	CM4	12	3.0	0.0	8.55×10^{-12}
	TM	12	3.0	0.0	4.66×10^{-14}
	FM	10	3.0	0.0	2.17×10^{-5}
$f_6(x)$	NM	7	1.36523001341410	0.0	2.22×10^{-16}
	VNM	4	1.36523001341410	0.0	1.07×10^{-7}
	CM4	4	1.36523001341410	0.0	3.10×10^{-10}
	TM	4	1.36523001341410	0.0	8.08×10^{-11}
	FM	3	1.36523001341410	0.0	2.24×10^{-4}

2. G. Adomain, *Nonlinear stochastic systems and applications to Physics,* Kluwer Academic Publishers, Dordrecht, 1989.
3. C. Chun, *Iterative methods improving Newton's method by the decomposition method,* Comput. Math. Appl. 50 (2005) 1559-1568.
4. V. Daftardar-Gejji and H. Jafari, *An iterative method for solving nonlinear functional equations,* J. Math. Anal. Appl. 316 (2006) 753-763.
5. M. A. Noor, *New family of iterative methods for nonlinear equations,* Appl. Math. Comput. 190 (2007) 553-558.
6. M. A. Noor and K. I. Noor, *Three-step iterative methods for nonlinear equations,* Appl. Math. Comput. 183 (2006) 322-327.
7. S. Weerakoon and T. G. I. Fernando, *A variant of Newton's method with accelerated third-order convergence,* Appl. Math. Lett. 13 (2000) 87-93.
8. J.H. Yun, *A note on three-step iterative method for nonlinear equations,* Appl. Math. Comput. 202 (2008) 401-405.

SeYoung Oh received M.Sc. from Seoul National University and Ph.D at University of Minnesota. Since 1992 he has been at Chungnam National University. His research interests include numerical optimization and biological computation.

Department of Mathematics, Chungnam National University, Daejeon 305-764, Korea
e-mail: soh@cnu.ac.kr

Jae Heon Yun received M.Sc. from Kyungpook National University, and Ph.D. from Iowa State University. He is currently a professor at Chungbuk National University since 1991. His research interests are computational mathematics, iterative method and parallel computation.

Department of Mathematics, College of Natural Sciences, Chungbuk National University, Cheongju 361-763, Korea
e-mail: gmjae@chungbuk.ac.kr