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TRIPLE POSITIVE SOLUTIONS OF SECOND ORDER SINGULAR NONLINEAR THREE-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. This paper deals with the existence of triple positive solutions for the nonlinear second-order three-point boundary value problem

$$z''(t) + a(t)f(t, z(t), z'(t)) = 0, \quad t \in (0,1),$$
$$z(0) = \nu z(1) \geq 0, \ z'(\eta) = 0,$$

where $0 < \nu < 1$, $0 < \eta < 1$ are constants. $f : [0,1] \times [0,+\infty) \times R \longrightarrow [0,+\infty)$ and $a : (0,1) \longrightarrow [0,+\infty)$ are continuous. First, Green's function for the associated linear boundary value problem is constructed, and then, by means of a fixed point theorem due to Avery and Peterson, sufficient conditions are obtained that guarantee the existence of triple positive solutions to the boundary value problem. The interesting point is that the nonlinear term f is involved with the first-order derivative explicitly.

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1. Introduction

The purpose of this paper is to establish the existence of triple positive solutions for the following singular three-point boundary value problem (BVP)

$$\begin{cases} z''(t) + a(t)f(t, z(t), z'(t)) = 0, & t \in (0,1), \\ z(0) = \nu z(1) \geq 0, \ z'(\eta) = 0, \end{cases} \tag{1.1}$$

where $0 < \nu < 1$, $0 < \eta < 1$ are constants. $f : [0,1] \times [0,+\infty) \times R \longrightarrow [0,+\infty)$, $a : (0,1) \longrightarrow [0,+\infty)$ are continuous.

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Three-point boundary value problems (BVPs, for short) of differential equations or difference equations arise in a variety of different areas of applied mathematics and physics. The study of three-point boundary value problems for nonlinear ordinary differential equations was initiated by Il'in and Moiseer [11]. Since then, nonlinear multi-point boundary value problems have been widely studied (see [1 – 8, 12 – 18] and the references therein). In [12], by applying the fixed point theorem, Liu proved the existence of at least one or two positive solutions for the following three-point boundary value problem

$$\begin{cases} z'' + h(t)f(z(t)) = 0, \quad t \in (0,1), \\ z(0) = 0, \ z(1) = \xi z(\eta) = 0, \end{cases}$$

where $0 < \eta < 1$, $0 < \xi < \frac{1}{\eta}$, $f \in C([0,+\infty), [0,+\infty))$, $h \in C([0,1], [0,+\infty))$ and there exists $t_0 \in [\eta, 1]$ such that $h(t_0) > 0$. In [17], Webb considered the existence of multiple positive solutions for the following nonlinear heat flow problem

$$-z'' = g(t)f(t,z), \ t \in (0, 1), \tag{1.2}$$

subject to the following nonlocal boundary conditions

$$z'(0) = 0, \quad \beta z'(1) + z(\eta) = 0, \tag{1.3}$$

where $0 < \eta < 1$, $g \in L^1[0,1]$, $f : [0,1] \times R \longrightarrow R$ satisfies carathéodory conditions.

The problem (1.2) and (1.3) is a model for stationary solutions of a heated bar, with a controller at 1 removing or adding heat dependent on the temperature detected by a sensor at η, the boundary condition at 0 corresponds to that end being insulated.

In [16], by using Leray-Schauder nonlinear alternative, Sun and Liu established the existence of nontrivial solution for the three-point boundary value problem

$$\begin{cases} z'' + f(t, z) = 0, \ 0 < t < 1, \\ z'(0) = 0, \ z(1) = \xi z(\eta), \end{cases}$$

where $\eta \in (0, 1)$, $\xi \in R$, $\xi \neq 1$, $f \in C([0,1] \times R, R)$.

Motivated greatly by the works mentioned above, the aim of the present paper is to improve and generalize the results in the above mentioned references. Obviously, what we discuss is different from those mentioned above and our positive solutions are nontrivial ones. The main new features presented in this paper are as follows: Firstly, the problem (1.1) has more general form in which $a(t)$ possesses singularity, that is $a(t)$ may be singular at $t = 0$ and/or $t = 1$. Secondly, the conditions imposed on nonlinear term are growth conditions. Thirdly, the main technique used in the analysis will depend on an application of a fixed-point theorem due to Avery and Peterson [6] which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. The emphasis is put on the nonlinear term involved with all lower-order derivatives explicitly. This paper is organized as follows. Some preliminaries and a number of lemmas are given in Section 2. The main results on the existence of triple positive solutions are stated and proved in Section 3.

2. Preliminaries and Lemmas

In this section, we present some notations and lemmas that will be used in the proof of our main results.

Definition 2.1. By a nonzero solution, also called a $C[0,1]$ solution of the problem (1.1), we mean a function $z \in C[0,1] \cap C^2(0,1)$ satisfying the problem (1.1) with $z(t)$ not identically zero on $(0,1)$. $z(t)$ is called a $C^1[0,1]$ solution, we mean that $z'(0+)$ and $z'(1-0)$ exist. $z(t)$ is called a positive solution of the problem (1.1) if $z(t)$ is a solution of the problem (1.1) and $z(t) > 0$ for $t \in (0,1)$.

Definition 2.2. Let E be a real Banach space and $P \subset E$ be a nonempty closed convex set. P is called a cone if the following two conditions are satisfied:
(i) $z \in P,\ \lambda > 0$ implies $\lambda z \in P$;
(ii) $z \in P,\ -z \in P$ implies $z = 0$.

Definition 2.3. *Let E be a Banach space and $P \subset E$ be a cone. The map $\alpha : P \longrightarrow [0,+\infty)$ is called a nonnegative continuous concave function on P if α is continuous and*

$$\alpha(\lambda x + (1-\lambda)y) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y)$$

for all $x,\ y \in P$ and $\lambda \in [0,1]$. Similarly, we say the map $\beta : P \longrightarrow [0,+\infty)$ is a nonnegative continuous convex function on P if β is continuous and

$$\beta(\lambda x + (1-\lambda)y) \leq \lambda\beta(x) + (1-\lambda)\beta(y)$$

for all $x,\ y \in P$ and $\lambda \in [0,1]$.

Let γ and θ be nonnegative continuous convex functionals on P, α a nonnegative continuous concave functional on P, and φ a nonnegative continuous functional on P. Then for positive real numbers a, b, c and d, we define the following convex sets:

$$\begin{aligned}
&P(\gamma, d) = \{z \in P \mid \gamma(z) < d\},\\
&P(\gamma, \alpha, b, d) = \{z \in P \mid b \leq \alpha(z),\ \gamma(z) < d\},\\
&P(\gamma, \theta, \alpha, b, c, d) = \{z \in P \mid b \leq \alpha(z),\ \theta(z) \leq c,\ \gamma(z) < d\},\\
&R(\gamma, \varphi, a, d) = \{z \in P \mid a \leq \varphi(z),\ \gamma(z) < d\}.
\end{aligned}$$

The following well-known fixed-point theorem due to Avery and Peterson [6] is fundamental to search for triple positive solutions of the problem (1.1).

Lemma 2.1.[6] *Let E be a real Banach space and $K \subset E$ be a cone in E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and φ be a nonnegative continuous* *functional on* P *satisfying* $\varphi(\mu z) \le \mu\varphi(z)$, *for* $0 \le \mu \le 1$, *such that for some positive numbers* M *and* d,

$$\alpha(z) \le \varphi(z) \quad and \quad \|z\| \le M\gamma(z), \tag{2.1}$$

for all $z \in \overline{P(\gamma, d)}$. *Suppose* $T : \overline{P(r, d)} \longrightarrow \overline{P(r, d)}$ *is a completely continuous operator and there exist positive numbers* a, b *and* c *with* $a < b$ *such that*

(B_1) $\{z \in P(\gamma, \theta, \alpha, b, c, d) | \alpha(z) > b\} \ne \phi$ *and* $\alpha(Tz) > b$ *for* $z \in P(\gamma, \theta, \alpha, b, c, d)$;

(B_2) $\alpha(Tz) > b$ *for* $z \in P(\gamma, \alpha, b, d)$ *with* $\theta(Tz) > c$;

(B_3) $0 \notin R(\gamma, \varphi, a, d)$ *and* $\varphi(Tz) < a$ *for* $z \in R(\gamma, \varphi, a, d)$ *with* $\varphi(z) = a$.

Then T *has at least three fixed points* $z_1, z_2, z_3 \in \overline{P(\gamma, d)}$, *such that*

$\gamma(z_i) \le d$, *for* $i = 1, 2, 3$; $\quad b < \alpha(z_1)$;

$a < \varphi(z_2)$ *with* $\alpha(z_2) < b$; $\quad \varphi(z_3) < a$.

We need some preliminary results before proving our main results. First, Green's function for the associated linear BVP is constructed.

Lemma 2.2. *Let* $0 < \nu < 1$, $g \in C[0, 1]$, *then the following boundary value problem*

$$z''(t) + g(t) = 0, \quad 0 < t < 1, \tag{2.2}$$

$$z(0) = \nu z(1) \ge 0, \quad z'(\eta) = 0, \tag{2.3}$$

has a unique solution $z(t) = \displaystyle\int_0^1 G(t, s)g(s)ds$, where

$$G(t, s) = \frac{1}{1-\nu}\begin{cases} s, & s \le t,\ s \le \eta, \\ \nu s + (1-\nu)t, & t \le s \le \eta, \\ s - \nu - (1-\nu)t, & \eta \le s \le t, \\ \nu s - \nu, & \eta \le s,\ t \le s, \end{cases}$$

Proof. It follows from (2.2) that

$$\begin{aligned} z(t) &= z(0) + z'(0)t - \int_0^t ds \int_0^s g(r)dr, \qquad z'(0) = \int_0^\eta g(s)ds, \\ z(0) &= \frac{\nu}{1-\nu} z'(0) - \frac{\nu}{1-\nu}\int_0^1 ds \int_0^s g(r)dr \\ &= \frac{\nu}{1-\nu}\int_0^\eta g(s)ds - \frac{\nu}{1-\nu}\int_0^1 ds\int_0^s g(r)dr \end{aligned}$$

So

$$\begin{aligned}
z(t) &= \frac{\nu}{1-\nu}\int_0^\eta g(s)ds - \frac{\nu}{1-\nu}\int_0^1 ds\int_0^s g(r)dr \\
&\quad + \int_0^\eta tg(s)ds - \int_0^t ds\int_0^s g(r)dr \\
&= \frac{\nu+(1-\nu)t}{1-\nu}\int_0^\eta g(s)ds - \frac{\nu}{1-\nu}\int_0^1 ds\int_0^s g(r)dr - \int_0^t ds\int_0^s g(r)dr \\
&= \frac{\nu+(1-\nu)t}{1-\nu}\int_0^\eta g(s)ds - \frac{\nu}{1-\nu}\int_0^1 dr\int_r^1 g(s)ds - \int_0^t dr\int_r^t g(s)ds \\
&= -\int_0^t (t-s)g(s)ds - \frac{\nu}{1-\nu}\int_0^1 (1-s)g(s)ds + \frac{\nu+(1-\nu)t}{1-\nu}\int_0^\eta g(s)ds
\end{aligned}$$

If $t \le \eta$, then

$$z(t) = \int_0^t \frac{s}{1-\nu}g(s)ds + \int_t^\eta \frac{\nu s+(1-\nu)t}{1-\nu}g(s)ds - \int_\eta^1 \frac{\nu(1-s)}{1-\nu}g(s)ds,$$

If $t > \eta$, then

$$z(t) = \int_0^\eta \frac{s}{1-\nu}g(s)ds + \int_\eta^t \frac{s-\nu-(1-\nu)t}{1-\nu}g(s)ds - \int_t^1 \frac{\nu(1-s)}{1-\nu}g(s)ds.$$

Therefore, the problem (2.2) and (2.3) has a unique solution

$$\begin{aligned}
z(t) &= -\int_0^t (t-s)g(s)ds - \frac{\nu}{1-\nu}\int_0^1 (1-s)g(s)ds + \frac{\nu+(1-\nu)t}{1-\nu}\int_0^\eta g(s)ds \\
&= \begin{cases} \int_0^t \frac{s}{1-\nu}g(s)ds + \int_t^\eta \frac{\nu s+(1-\nu)t}{1-\nu}g(s)ds - \int_\eta^1 \frac{\nu(1-s)}{1-\nu}g(s)ds, & t \le \eta, \\ \int_0^\eta \frac{s}{1-\nu}g(s)ds + \int_\eta^t \frac{s-\nu-(1-\nu)t}{1-\nu}g(s)ds - \int_t^1 \frac{\nu(1-s)}{1-\nu}g(s)ds, & t > \eta \end{cases} \\
&= \int_0^1 G(t,s)g(s)ds.
\end{aligned}$$

This completes the proof. □

Lemma 2.3. *Suppose that* $0 \le \nu < 1$, *then* $| G(t,s) | \le \frac{1}{1-\nu}$, *for* $(t,s) \in [0,1]\times[0,1]$.

From Lemma 2.2 we know that if $0 \le \nu < 1$ and $0 < \eta < 1$, then for $g \in C[0,1]$ and $g(t) \ge 0$, the unique solution $z(t)$ of the BVP (2.2) and (2.3) is nonnegative and satisfies $0 \le z(t) \le \frac{1}{1-\nu}\int_0^1 g(s)ds$.

In what follows, we shall consider the Banach space $C[0,1]$ equipped with the ordering $x \le y$ if $x(t) \le y(t)$ for all $t \in [0,1]$, and the maximum norm

$$\|z\| = \max\left\{\max_{0\le t\le 1}|z(t)|,\ \max_{0\le t\le 1}|z'(t)|\right\}.$$

Denote the cone K by

$K = \{z \in C[0,1] \mid z(t) \geq 0, z(0) = \nu z(1) \geq 0,\ z'(\eta) = 0,\ z \text{ is concave on } [0,1]\}$

Define the integral operator $T : K \longrightarrow C[0,1]$ by

$$Tz(t) = \int_0^1 G(t,s)a(s)f(s,z(s),z'(s))ds.$$

It follows from Lemma 2.1, the problem (1.1) has a positive solution $z^* = z^*(t)$ if and only if z^* is a fixed point of T.

Lemma 2.4. *Every function $z \in P$ is differential a.e. (almost everywhere) on $(0,1)$ and satisfies*

$$z(t) \geq \|z\|t(1-t) \quad \text{on} \quad [0,1],$$
$$|z'(t)| \leq \frac{z(t)}{t(1-t)} \quad \text{a.e. on } (0,1).$$

Proof. Since $z(t)$ is continuous on $[0,1]$, let $z(t_0) = \max\limits_{0\leq t\leq 1} z(t)$, for $t_0 \in (0,1)$, then $\|z\| = z(t_0)$. It follows from concavity of $z(t)$ on $[0,1]$ that

$$z(t) \geq t\|z\| \quad \text{for} \quad z \in [0,t_0],$$

$$z(t) \geq (1-t)\|z\| \quad \text{for} \quad z \in [t_0,1].$$

Therefore $z(t) \geq t(1-t)\|z\|$.

On the other hand, by mean value theorem, we have

$$|z'(t)| \leq \frac{z(t)}{t} \quad \text{for} \quad z \in [0,t_0],$$
$$|z'(t)| \leq \frac{z(t)}{1-t} \quad \text{for} \quad z \in [t_0,1].$$

Therefore $|z'(t)| \leq \dfrac{z(t)}{t(1-t)}$. This completes the proof. □

We adopt the following assumptions:

(H_1) $a \in C((0,1),[0,+\infty))$ with $0 < \int_0^1 a(s)ds < +\infty$;

(H_2) $f \in C([0,1] \times [0,+\infty) \times R, [0,+\infty))$.

Lemma 2.5. *Assume that (H_1) and (H_2) hold. Then $T : K \longrightarrow K$ is completely continuous.*

Proof. From the fact that $-z'' = a(t)f(t,z(t),z'(t)) \geq 0$, $z(t) \geq 0$ and Lemma 2.1. we know that z is concave on $[0,1]$. By Lemma 2.2, we know that $TK \subset K$.

For $n \geq 2$, define

$$a_n(t) = \begin{cases} \inf\left\{a(t),\, a(\frac{1}{n}),\, a(\eta)\right\}, & 0 < t < \frac{1}{n}, \\ a(t), & \frac{1}{n} \leq t \leq \frac{n-1}{n}, \\ \inf\left\{a(t),\, a(\frac{n-1}{n}),\, a(\eta)\right\}, & 1-\frac{1}{n} < t < 1. \end{cases}$$

Then $a_n : [0,1] \longrightarrow [0,+\infty)$ is continuous, and $a_n(t) \le a(t),\ t \in (0,1)$.

Define

$$T_n z(t) = \int_0^1 G(t,s)a_n(s)f(s,z(s),z'(s))ds, \quad n \ge 2.$$

Thus $T_n : K \longrightarrow K$. Obviously, for each $n \ge 2$, T_n is compact on K. Denote $B_\tau = \{z \in K : \|z\| \le \tau\}$ for all $\tau > 0$, then T_n is uniformly approximate to T. In fact, for fixed $\tau > 0$ and $z \in B_\tau$ for any $t \in [0,1]$, we have

$$\begin{aligned}|T_n z(t) - Tz(t)| &= \left| \int_0^1 G(t,s)[a(s) - a_n(s)]f(s,z(s),\ z'(s))ds \right| \\ &\le \frac{1}{1-\nu}\Big[\int_0^{\frac{1}{n}} |a(s) - a_n(s)| f(s,z(s),z'(s))ds \\ &+ \int_{\frac{n-1}{n}}^1 |a(s) - a_n(s)| f(s,z(s),z'(s))ds \Big] \\ &\le \frac{c_0}{1-\nu}\left[\int_0^{\frac{1}{n}} |a(s) - a_n(s)|ds + \int_{\frac{n-1}{n}}^1 |a(s) - a_n(s)|ds \right]\end{aligned}$$

where $c_0 = \max\{f(t,z(t),z'(t))|(t,\ z,\ z') \in [0,1] \times [0,\ \tau] \times [-\tau,\tau]\}$. From (H_1), we know $|a(s) - a_n(s)| \in L^1(0,\ 1)$. Notice that $0 \le a_n(s) \le a(s)$, so (2.4) implies that $\lim_{n\to\infty} \|T_n z - Tz\| = 0$ for $z \in B_\tau$. Therefore T is compact. On the other hand, by Lebesgue Control Convergence theorem, we easily see $T : K \longrightarrow K$ is continuous. So T is completely continuous. The proof is complete. □

3. Main results

In this section we shall impose growth conditions on f which allow us to apply Lemma 2.1 to establish the existence of triple positive solutions of the problem (1.1). Let the nonnegative continuous convex functional θ, γ, the nonnegative continuous concave functional α, and the nonnegative continuous functional φ be defined on the cone K by

$$\gamma(z) = \max_{0\le t\le 1} |z'(t)|, \quad \varphi(z) = \theta(z) = \max_{0\le t\le 1} |z(t)|, \quad \alpha(z) = \min_{0\le t\le 1} |z(t)|, \text{ for } z \in K.$$

Lemma 3.1. $\max_{0\le t\le 1} |z(t)| \le \frac{1}{1-\nu} \max_{0\le t\le 1} |z'(t)| \le \frac{1-\eta}{1-\nu} \max_{0\le t\le 1} |z''(t)|$, for $z \in K$.

Proof. Notice that for all $z \in K$, we obtain

$$\begin{aligned} z(t) &= z(0) + \int_0^t z'(s)ds = \nu z(1) + \int_0^t z'(s)ds \\ &\le \nu \max_{0\le t\le 1} |z(t)| + t \max_{0\le t\le 1} |z'(t)| \le \nu \max_{0\le t\le 1} |z(t)| + \max_{0\le t\le 1} |z'(t)| \end{aligned}$$

$$\begin{aligned} z'(t) &= z'(\eta) + \int_\eta^t z''(s)ds \le \max_{0\le t\le 1} |z''(t)|(t-\eta) \\ &\le (1-\eta) \max_{0\le t\le 1} |z''(t)|. \end{aligned}$$

Therefore

$$\max_{0\le t\le 1} |z(t)| \le \frac{1}{1-\nu} \max_{0\le t\le 1} |z'(t)| \le \frac{1-\eta}{1-\nu} \max_{0\le t\le 1} |z''(t)|. \tag{3.1}$$

Consequently, combining with the concavity of z, the functionals defined above satisfy

$$\begin{aligned} &\nu\theta(z) \le \alpha(z) \le \theta(z) = \varphi(z), \\ &\|z\| = \max\{\theta(z),\, \gamma(z)\} \le \frac{1}{1-\nu}\gamma(z). \end{aligned} \tag{3.2}$$

for all $z \in \overline{P(\gamma, d)}$. Therefore, condition (2.1) is satisfied. □

For convenience, in what follows, we denote constants by

$$B = \min\left\{\int_0^1 G(0,s)a(s)ds,\ \int_0^1 G(1,s)a(s)ds\right\},\quad M = \int_0^1 a(s)ds,\quad N = \frac{M}{1-\nu}.$$

Now we present our main result and proof.

Theorem 3.1. *Suppose that* (H_1) *and* (H_2) *hold. Assume that there exists* $0 < a < b \le \frac{d}{1-\nu}$ *such that*

(A_1) $f(t,u,v) \le \frac{d}{M}$ *for* $(t,u,v) \in [0,1] \times [0, \frac{d}{1-\nu}] \times [-d, d]$,

(A_2) $f(t,u,v) > \frac{b}{B}$ *for* $(t,u,v) \in [0,1] \times [b, \frac{b}{\nu}] \times [-d, d]$,

(A_3) $f(t,u,v) \le \frac{a}{N}$ *for* $(t,u,v) \in [0,1] \times [0, a] \times [-d, d]$.

Then the problem (1.1) *has at least three positive solutions* z_1, z_2 *and* z_3 *satisfying*

$$\max_{0\le t\le 1} |z_i'(t)| \le d \ \ for\ \ i = 1,2,3; \qquad b < \min_{0\le t\le 1} |z_1(t)|, \quad \max_{0\le t\le 1} |z_1(t)| \le \frac{d}{1-\nu}$$

$$a < \max_{0\le t\le 1} |z_2(t)| \le \tfrac{b}{\nu} \ \ with\ \ \min_{0\le t\le 1} |z_2(t)| < b;$$

and $\max_{0\le t\le 1} |z_3(t)| < a$.

Proof. We now show that all the conditions of Lemma 2.1 are satisfied. If $z \in \overline{P(\gamma, d)}$, then $\gamma(z) = \max_{0\le t\le 1} |z'(t)| \le d$. From (3.1), one has $\max_{0\le t\le 1} |z'(t)| \le d$; $\max_{0\le t\le 1} |z(t)| \le \frac{d}{1-\nu}$, then assumption (A_1) implies $f(t,\, z(t),\, z'(t)) \le \frac{d}{M}$.

On the other hand, for all $z \in K \subset P$, we have $Tz \in K \subset P$, then Tz is concave on [0, 1] and

$$\max_{t\in[0,1]} |(Tz)'(t)| = \max\left\{|(Tz)'(0)|,\ |(Tz)'(1)|\right\}.$$

Thus, for any $t \in [0, 1]$, we obtain

$$\gamma(Tz) = \max_{0\le t\le 1} |(Tz)'(t)|$$
$$= \max_{0\le t\le 1} \left\{ \int_0^t a(s)f(s, z(s), z'(s))ds, \int_t^1 a(s)f(s, z(s), z'(s))ds \right\}$$
$$\le \frac{d}{M} \int_0^1 a(s)ds = d.$$

Hence, $T : \overline{P(\gamma, d)} \longrightarrow \overline{P(\gamma, d)}$.

To check condition (B_1) of Lemma 2.1, we choose $z(t) = \frac{b}{\nu}$, $t \in [0, 1]$. It is easy to see that $z(t) = \frac{b}{\nu} \in P(\gamma, \theta, \alpha, b, \frac{b}{\nu}, d)$ and $\alpha(z) = \alpha(\frac{b}{\nu}) > b$, and so

$$\left\{ z \in P(\gamma, \theta, \alpha, b, \frac{b}{\nu}, d) \mid \alpha(z) > b \right\} \neq \phi.$$

Hence, for $z \in P(\gamma, \theta, \alpha, b, \frac{b}{\nu}, d)$, there is $b \le z(t) \le \frac{b}{\nu}$, $|z'(t)| \le d$ for $t \in [0, 1]$. Thus, it follows from condition (A_2) of this theorem, we have $f(t, z(t), z'(t)) > \frac{b}{B}$ for $t \in [0, 1]$, this together with the conditions of α and the cone K, there are the following two cases to distinguish (i) $\alpha(Tz) = Tz(0)$ and (ii) $\alpha(Tz) = Tz(1)$.

In case (i), we have

$$\alpha(Tz) = Tz(0) = \int_0^1 G(0, s)a(s)f(s, z(s), z'(s))ds > \frac{b}{B} \int_0^1 G(0, s)a(s)ds \ge b.$$

In case (ii), we have

$$\alpha(Tz) = Tz(1) = \int_0^1 G(1, s)a(s)f(s, z(s), z'(s))ds > \frac{b}{B} \int_0^1 G(1, s)a(s)ds \ge b,$$

i.e., $\alpha(Tz) > b$ for all $\{z \in P(\gamma, \theta, \alpha, b, \frac{b}{\nu}, d)\}$. This show that condition (B_1) of Theorem 2.1 is satisfied.

Secondly, from (3.2), we have $\alpha(Tz) \ge \nu\theta(Tz) > \nu \cdot \dfrac{b}{\nu} = b$, for all $z \in P(\gamma, \alpha, b, d)$ with $\theta(Tz) > \frac{b}{\nu}$. Thus, condition (B_2) of Lemma 2.1 is satisfied.

Finally, we show that condition (B_3) of Lemma 2.1 also holds. Obviously, $\varphi(0) = 0 < a$, we have $0 \notin R(\gamma, \varphi, a, d)$. Suppose that $z \in R(\gamma, \varphi, a, d)$ with $\varphi(z) = a$. Then by the condition (A_3) of this theorem, we have

$$\varphi(z) = \max_{0\le t\le 1} |Tz(t)| = \max_{0\le t\le 1} \int_0^1 |G(t, s)| a(s)f(s, z(s), z'(s))ds$$
$$< \frac{1}{1-\nu} \cdot \frac{a}{N} \int_0^1 a(s)ds = a.$$

So, the condition (B_3) of Lemma 2.1 is also satisfied. Therefore, an application of Lemma 2.1 implies the boundary value problem (1.1) has at least three positive solutions z_1, z_2 and z_3 such that

$$\max_{0\le t\le 1} |z_i'(t)| \le d \quad \text{for} \quad i = 1, 2, 3; \quad b < \min_{0\le t\le 1} |z_1(t)|, \quad \max_{0\le t\le 1} |z_1(t)| \le \frac{d}{1-\nu},$$

$$a < \max_{0\le t\le 1} |z_2(t)| \le \frac{b}{\nu} \quad \text{with} \quad \min_{0\le t\le 1} |z_2(t)| < b$$

and $\max_{0\le t\le 1} |z_3(t)| < a$. This completes the proof. □

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