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# MODIFIED KRASNOSELSKI-MANN ITERATIONS FOR NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : K \to K$ be a nonexpansive mapping with a nonempty fixed point set $Fix(T)$. Let $f : K \to K$ be a contraction mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \tag{0.1}$$

$$\sum_{n=0}^{\infty} \alpha_n = +\infty, \tag{0.2}$$

$$0 < a \leq \beta_n \leq b < 1 \text{ for all } n \geq 0. \tag{0.3}$$

Then it is proved that the modified Krasnoselski-Mann iterative sequence $\{x_n\}$ given by

$$\begin{cases} x_0 \in K, \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \ n \geq 0 \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, \ n \geq 0, \end{cases} \tag{0.4}$$

converges strongly to a point $p \in Fix(T)$ which satisfies the variational inequality

$$\langle p - f(p), p - z\rangle \leq 0, \ z \in Fix(T). \tag{0.5}$$

This result improves and extends the corresponding results of Yao et al[Y. Yao, H. Zhou, Y. C. Liou, *Strong convergence of a modified Krasnoselski-Mann iterative algorithm for non-expansive mappings,* J Appl Math Comput (2009)29:383-389].

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## 1. Introduction

Let $K$ be a nonempty closed convex subset of a Hilbert space $H$. Recall that a mapping $f : K \to K$ is said to be a *contraction mapping* if there exists a constant $\lambda \in [0,1)$ such that

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$$\|f(x) - f(y)\| \leq \lambda \|x - y\|$$

for each $x, y \in K$. Throughout the paper we use $\Pi_K$ to denote the collection of all contraction self-mappings of $K$; that is,

$$\Pi_K = \{f : \ f : K \to K \ a \ contraction\}.$$

By Banach contraction mapping principle, each $f \in \Pi_K$ has a unique fixed point in $K$, and for each $x \in K$ the Picard iterative sequence $\{T^n x\}$ converges strongly to the fixed point.

A mapping $T : K \to K$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for each $x, y \in K$. We use $Fix(T)$ to denote the set of fixed points of $T$; that is, $Fix(T) = \{x \in K : T(x) = x\}$. Throughout this section, $f$ and $T$ denote contraction and nonexpansive self-mappings of a nonempty closed convex subset $K$ of a real Hilbert space $H$, respectively.

A wide variety of problems can be solved by finding a fixed point of a particular operator, and algorithms for finding such points play a prominent role in a number of applications. In particular, construction of fixed points of nonexpansive mappings and its applications are in the center stage of nonlinear analysis. In general, the Picard iteration $\{T^n x\}$ may not behave well for nonexpansive mappings.

In 1953, Mann[6] introduced iterative algorithm given by

$$\begin{cases} x_0 \in K \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \ n \geq 0. \end{cases} \tag{1.1}$$

In the literature, the iterative algorithm defined in (1.1) is referred as Krasnoselski-Mann iteration (or Mann iteration).

Many well-known algorithms in signal processing and image reconstruction are iterative in nature. For instance, Byrne [2] shown the application of Mann iteration to signal processing and image reconstruction. Also, the projection onto convex sets methods and iterative optimization procedures, such as entropy or likelihood maximization, are the primary examples. In this line, H. K. Xu[9] applied Krasnoselski-Mann iteration to solve quadratic optimization.

Almost all of the results in the literature on the Mann iterative algorithm for nonexpansive mapping have only weak convergence even in a Hilbert space. To obtain strong convergence, in 2009, Yao et al[11] proposed the modified Krasnoselski-Mann iterative algorithm defined as

$$\begin{cases} x_0 \in K \\ y_n = (1 - \alpha_n) x_n, \ n \geq 0 \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n T y_n, \ n \geq 0, \end{cases} \tag{1.2}$$

and proved the following

**Theorem 1.1.** ( [11], Theorem 3.1) *Let $H$ be a real Hilbert space. Let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences* *in* $[0,1]$ *such that*

$$(YZL1)\ \lim_{n\to\infty}\alpha_n = 0;\quad (YZL2)\ \sum_{n=0}^{\infty}\alpha_n = \infty;$$
$$(YZL3)\ \beta_n \in [a,b] \subseteq (0,1)\ \textit{for all}\ n \geq 0.$$

*Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *defined in (1.2) converge strongly to a fixed point of* $T$.

The purpose of this paper is to propose the generalized form of the above iteration, and to prove strong convergence of the proposed iteration to a fixed point of $T$ which satisfies certain variational inequality.

## 2. Preliminaries

Let $H$ be a real Hilbert space with inner product denoted by $\langle\ ,\ \rangle$ and norm by $\|\ ,\ \|$, and $x, y \in H$. Then

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2\\ &= \|x\|^2 + [2\langle x,y\rangle + 2\|y\|^2] - \|y\|^2\\ &= \|x\|^2 + 2\langle y, x+y\rangle - \|y\|^2.\end{aligned}$$

Thus we have the following

**Lemma 2.1.** *Let* $H$ *be a real Hilbert space and* $x, y \in H$. *Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

Let $K$ be a nonempty closed convex subset of a Hilbert space $H$. A mapping $T : K \to H$ is said to be demiclosed if for any sequence $\{x_n\} \subseteq K$ which converges weakly to $x_0 \in K$, the strong convergence of the sequence $\{Tx_n\}$ to $y_0 \in H$ implies $Tx_0 = y_0$.

Below, Lemma 2.2 and Lemma 2.3 were proved in 1967 by Z. Opial[7].

**Lemma 2.2.** ([7], Lemma 1) *If in a Hilbert space* $H$ *the sequence* $\{x_n\}$ *is weakly convergent to* $x_0$, *then for any* $x \neq x_0$,

$$\limsup_{n\to\infty}\|x_n - x_0\| < \limsup_{n\to\infty}\|x_n - x\|.$$

Nowadays Lemma 2.2 is referred as Opial's condition.

**Lemma 2.3.** ([7], Lemma 2) *Let* $K$ *be a nonempty closed convex subset of a Hilbert space* $H$. *Then for every nonexpansive mapping* $T : K \to H$, *the mapping* $I - T$ *is demiclosed.*

Recall that the metric projection $P_K$ from a Hilbert space $H$ to a nonempty closed convex subset $K$ of $H$ is defined as follows: Given $x \in H$, $P_K(x)$ is the only point in $K$ with the property

$$\|x - P_K(x)\| = \inf\{\|x - y\| : y \in K\}.$$

The following lemma characterizes $P_K$.

**Lemma 2.4.** ([5], pp.132-136) *Let $K$ be a closed convex subset of a real Hilbert space $H$. Given $x \in H$ and $y \in K$. Then $y = P_K(x)$ if and only if for all $z \in K$*

$$\langle y - x, y - z\rangle \leq 0.$$

Let $f \in \Pi_K$ and let $T$ be a nonexpansive self-mapping of $K$. Then for each $t \in (0,1)$, a mapping $T_t : K \to K$ defined by

$$T_t x = tf(x) + (1-t)Tx, \ x \in K,$$

is a contraction; indeed for $x, y \in K$ we have

$$\begin{aligned} \|T_t x - T_t y\| &\leq t\|f(x) - f(y)\| + (1-t)\|Tx - Ty\| \\ &\leq (1 - t(1-\lambda))\|x - y\|, \end{aligned}$$

where $\lambda \in [0,1)$ is the Lipschitz constant of $f$.

Let $y_t \in K$ be the unique fixed point of $T_t$; that is,

$$y_t = tf(y_t) + (1-t)Ty_t. \tag{2.1}$$

In 2004, H. K. Xu [10] proved

**Lemma 2.5.** [10] *Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T$ be a nonexpansive self-mapping of $K$ with $Fix(T) \neq \emptyset$ and $f \in \Pi_K$. Let $\{y_t\}$ be given by (2.1). Then we have*

(1) $\lim_{t\to 0} y_t = \bar{x}$ *exists;*
(2) $\bar{x} = P_S f(\bar{x})$, *where $S = Fix(T)$, or equivalently, $\bar{x}$ is the unique solution in $Fix(T)$ to the variational inequality*

$$\langle (I - f)\bar{x}, \bar{x} - z\rangle \leq 0, \ z \in Fix(T).$$

**Lemma 2.6.** ([8]) *Let $\{\lambda_n\}$ be a subset of a nonnegative real numbers such that*

(1) $\lambda_{n+1} \leq (1-\alpha_n)\lambda_n + \alpha_n\beta_n$,
(2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$, *and*
(3) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n\beta_n| < +\infty$.

*where $\{\alpha_n\} \subseteq (0,1)$ and $\{\beta_n\}$ is a real sequence. Then $\lim_{n\to\infty} \lambda_n = 0$.*

## 3. Main results

Let $H$ be a real Hilbert space with inner product denoted by $\langle\ ,\ \rangle$ and norm by $\|\ ,\ \|$ and $K$ a nonempty closed convex subset of $H$. We use $\Pi_K$ to denote the set of all contraction mappings of $K$ into itself. The following lemma plays key role in our proof of the main result of the paper.

**Lemma 3.1.** *Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$ and suppose $T : K \to K$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$. Then there is a unique mapping $\Delta : \Pi_K \to Fix(T)$ such that*

$$\limsup_{n\to\infty} \langle (I - f)\Delta(f), \Delta(f) - x_n\rangle \leq 0,$$

*for any given $f \in \Pi_K$ and a bounded approximate fixed point sequence $\{x_n\}$ of $T$ in $K$.*

*Proof.* Let $f \in \Pi_K$ and let $\{x_n\}$ be a bounded approximate fixed point sequence of $T$ in $K$; that is, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. By Lemma 2.5, there exists a unique continuous path $\{y_t\}$ in $K$ such that

$$y_t = tf(y_t) + (1-t)Ty_t,$$

for all $t \in (0,1)$ and $\lim_{t\to 0} y_t = q \in Fix(T)$. Define $\Delta(f) = q$.

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (I-f)q, q-x_n\rangle = \lim_{j\to\infty} \langle (I-f)q, q-x_{n_j}\rangle.$$

Without loss of generality, we assume that $\{x_{n_j}\}$ is weakly convergent to $\bar{x} \in K$. Since $I-T$ is demiclosed by Lemma 2.3, $\bar{x} \in Fix(T)$.

Thus, we get

$$\begin{aligned}\limsup_{n\to\infty} \langle (I-f)q, q-x_n\rangle &= \lim_{j\to\infty} \langle (I-f)q, q-x_{n_j}\rangle \\ &= \langle (I-f)q, q-\bar{x}\rangle.\end{aligned}$$

Again by Lemma 2.5, we have $\langle (I-f)q, q-\bar{x}\rangle \le 0$. Therefore,

$$\limsup_{n\to\infty} \langle (I-f)q, q-x_n\rangle \le 0.$$

This completes the proof. □

Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$, $T: K \to K$ a nonexpansive mapping and $f \in \Pi_K$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$. Consider the iterative sequences $\{x_n\}$ and $\{y_n\}$ in $K$ defined by

$$\begin{cases} x_0 \in K \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \ n \ge 0 \\ x_{n+1} = (1-\beta_n)y_n + \beta_n Ty_n, \ n \ge 0. \end{cases} \tag{3.1}$$

**Theorem 3.2.** *Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$, $T: K \to K$ a nonexpansive mapping with $Fix(T) \neq \emptyset$ and $f: K \to K$ a contraction mapping with Lipschitz constant $\lambda \in [0,1)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ such that*

$$\lim_{n\to\infty} \alpha_n = 0, \tag{3.2}$$

$$\sum_{n=0}^{\infty} \alpha_n = +\infty, \tag{3.3}$$

$$0 < a \le \beta_n \le b < 1 \text{ for some constants } a, b. \tag{3.4}$$

*Then for any initial point $x_0 \in K$, the sequences $\{x_n\}$ and $\{y_n\}$ defined in (3.1) converge strongly to a point $p \in Fix(T)$ which satisfies the variational inequality*

$$\langle p - f(p), p - z\rangle \le 0, \ \forall z \in Fix(T).$$

*Proof.* Let $z \in Fix(T)$ be fixed. It follows from (3.1) that

$$\begin{aligned}
\|x_{n+1} - z\| &\le (1-\beta_n)\|y_n - z\| + \beta_n \|Ty_n - z\| \\
&\le \|y_n - z\| \\
&\le \alpha_n \|f(x_n) - z\| + (1 - \alpha_n \|x_n - z\| \\
&\le \alpha_n \|f(x_n) - f(z)\| + \alpha_n \|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\
&\le \alpha_n \lambda \|x_n - z\| + \alpha_n \|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\
&= \alpha_n \|f(z) - z\| + [1 - \alpha_n(1-\lambda)]\|x_n - z\| \\
&\le \max\{\frac{1}{1-\lambda}\|f(z) - z\|,\ \|x_n - z\|\}.
\end{aligned}$$

By induction, we obtain

$$\|x_{n+1} - z\| \le \max\{\frac{1}{1-\lambda}\|f(z) - z\|,\ \|x_0 - z\|\}.$$

Therefore, the sequence $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{Tx_n\}$, $\{y_n\}$ and $\{Ty_n\}$.

Now for each $x \in K$ and $z \in Fix(T)$ we obtain

$$\begin{aligned}
\|Tx - x\|^2 &= \|(Tx - z) + (z - x)\|^2 \\
&= \|Tx - z\|^2 + 2\langle Tx - z, z - x\rangle + \|z - x\|^2 \\
&\le 2\|z - x\|^2 + 2\langle Tx - z, z - x\rangle \\
&\le 2\|z - x\|^2 + 2\langle Tx - x, z - x\rangle + 2\langle x - z, z - x\rangle \\
&= 2\|z - x\|^2 + 2\langle Tx - x, z - x\rangle - 2\|z - x\|^2 \\
&= 2\langle x - z, x - Tx\rangle.
\end{aligned}$$

Hence for every $x \in K$ and $z \in Fix(T)$, we have

$$\|Tx - x\|^2 \le 2\langle x - z, x - Tx\rangle. \tag{3.5}$$

From (3.1), we note that

$$y_n - Ty_n = \frac{1}{\beta_n}(y_n - x_{n+1}). \tag{3.6}$$

By Lemma 2.5, there is a unique point $\Delta(f) = p \in Fix(T)$ which satisfies the variational inequality

$$\langle p - f(p), p - z\rangle \le 0,\ \forall z \in Fix(T).$$

We need to show that both sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the point $p$.

From (3.1) and (3.5), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\beta_n)y_n+\beta_n Ty_n-p\|^2\\ &= \|(y_n-p)-\beta_n(y_n-Ty_n)\|^2\\ &= \|y_n-p\|^2-2\beta_n\langle y_n-p,y_n-Ty_n\rangle+\beta_n^2\|y_n-Ty_n\|^2\\ &\le \|y_n-p\|^2-\beta_n\|y_n-Ty_n\|^2+\beta_n^2\|y_n-Ty_n\|^2\\ &= \|y_n-p\|^2-\beta_n(1-\beta_n)\|y_n-Ty_n\|^2;\end{aligned}$$

so that from (3.6) we get

$$\|x_{n+1}-p\|^2 \le \|y_n-p\|^2-\frac{1-\beta_n}{\beta_n}\|y_n-x_{n+1}\|^2. \tag{3.7}$$

It follows from (3.4) that

$$\mu=\frac{1-b}{b}\le\frac{1-\beta_n}{\beta_n}. \tag{3.8}$$

Therefore, by Lemma 2.1, (3.1), (3.7) and (3.8), we get

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\le \|y_n-p\|^2-\mu\|y_n-x_{n+1}\|^2\\ &= \|\alpha_n(f(x_n)-f(p))+\alpha_n(f(p)-p)+(1-\alpha_n)(x_n-p)\|^2\\ &\quad -\mu\|\alpha_n(f(x_n)-x_n)+(x_n-x_{n+1})\|^2\\ &= \alpha_n^2\|f(x_n)-f(p)\|^2+2\alpha_n(1-\alpha_n)\langle f(p)-p,x_n-p\rangle\\ &\quad +(1-\alpha_n)^2\|x_n-p\|^2-\mu\|\alpha_n(f(x_n)-x_n)+(x_n-x_{n+1})\|^2\\ &\le [1-\alpha_n(1-\lambda)]\|x_n-p\|^2+2\alpha_n(1-\alpha_n)\langle f(p)-p,y_n-p\rangle\\ &\quad -\mu\alpha_n^2\|f(x_n)-x_n\|^2-2\mu\alpha_n\langle f(x_n)-x_n,x_n-x_{n+1}\rangle-\mu\|x_n-x_{n+1}\|^2\\ &\le \|x_n-p\|^2+\alpha_n[-\|x_n-p\|^2+2\langle f(p)-p,x_n-p\rangle-\mu\|f(x_n)-x_n\|^2\\ &\quad -2\mu\langle f(x_n)-x_n,x_n-x_{n+1}\rangle]-\mu\|x_n-x_{n+1}\|^2\\ &\le \|x_n-p\|^2+\alpha_n M-\mu\|x_n-x_{n+1}\|^2,\end{aligned}$$

where $M=\sup\{2\langle f(p)-p,x_n-p\rangle-\mu\|f(x_n)-x_n\|^2-\|x_n-p\|^2-2\mu\langle f(x_n)-x_n,x_n-x_{n+1}\rangle : n\ge 0\}$. Therefore, we have

$$\|x_{n+1}-p\|^2-\|x_n-p\|^2+\mu\|x_n-x_{n+1}\|^2\le\alpha_n M. \tag{3.9}$$

Now we consider two cases.

**Case 1.** $\{\|x_n-p\|\}$ is a monotonically decreasing sequence.

Then $\lim_{n\to\infty}\|x_n-p\|$ exists. Hence from (3.9) we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.10}$$

It is not difficult to see from (3.1) and (3.2) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.11}$$

From (3.10) and (3.11), we can easily get that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.12}$$

It follows from (3.6) and (3.12) that

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0. \tag{3.13}$$

Since

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \\ &\le 2\|x_n - y_n\| + \|y_n - Ty_n\|,\end{aligned}$$

hence from (3.11) and (3.13), we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.14}$$

By Lemma 3.1 and (3.13), we have

$$\limsup_{n\to\infty} \langle p - f(p), p - y_n\rangle \le 0. \tag{3.15}$$

Now from Lemma 2.1 we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \|y_n - p\|^2 \\ &= \|\alpha_n(f(x_n) - f(p)) + \alpha_n(f(p) - p) + (1-\alpha_n)(x_n - p)\|^2 \\ &\le \|\alpha_n(f(x_n) - f(p)) + (1-\alpha_n)(x_n - p)\|^2 + 2\alpha_n\langle f(p) - p, y_n - p\rangle \\ &\le [\alpha_n \|f(x_n) - f(p)\| + (1-\alpha_n)\|x_n - p\|]^2 + 2\alpha_n\langle f(p) - p, y_n - p\rangle \\ &\le [1 - \alpha_n(1-\lambda)]^2 \|x_n - p\|^2 + 2\alpha_n\langle f(p) - p, y_n - p\rangle;\end{aligned}$$

so that

$$\|x_{n+1} - p\|^2 \le [1 - \alpha_n(1-\lambda)]\|x_n - p\|^2 + 2\alpha_n\langle f(p) - p, y_n - p\rangle. \tag{3.16}$$

Now using (3.3), (3.15), (3.16) and applying Lemma 2.6, we get

$$\lim_{n\to\infty} \|x_n - p\|^2 = 0.$$

Therefore, the sequence $\{x_n\}$ converges strongly to $p$. Consequently, $\{y_n\}$ also converges strongly to $p$.

**Case 2.** $\{\|x_n - p\|\}$ is not a monotonically decreasing sequence.

Let $\Gamma_n = \|x_n - p\|^2$, $n \ge 0$. Since $\{\|x_n - p\|\}$ is not a monotonically decreasing sequence, there exists a nonnegative integer $n_0$ such that $\Gamma_{n_0} \le \Gamma_{n_0+1}$. Define a sequence $\{\Phi_n\}_{n=n_0}^{\infty}$ of nonnegative integers as follows

$$\Phi_n = \max\{j : j \le n,\ \Gamma_j \le \Gamma_{j+1}\},\ n \ge n_0.$$

It is clear from the definition that $\{\Phi_n\}_{n=n_0}^{\infty}$ is an increasing sequence. If we assume that $\Phi_n \to N < +\infty$ as $n \to \infty$, then $\{\|x_n - p\|\}_{n=N}^{\infty}$ is a monotonically decreasing sequence and we have nothing to prove as the conclusion follows from Case 1. Thus we assume $\Phi_n \to +\infty$ as $n \to \infty$. Since $\Gamma_{\Phi_n} \le \Gamma_{\Phi_n+1}$ for each $n \ge n_0$, we get from (3.9) that

$$\mu \|x_{\Phi_n+1} - x_{\Phi_n}\|^2 \le \Gamma_{\Phi_n+1} - \Gamma_{\Phi_n} + \mu \|x_{\Phi_n+1} - x_{\Phi_n}\|^2 \le \alpha_{\Phi_n} M.$$

Thus we have

$$\lim_{n\to\infty} \|x_{\Phi_n+1} - x_{\Phi_n}\| = 0. \tag{3.17}$$

Since $\Phi_n \to +\infty$ as $n \to \infty$, we also have

$$\lim_{n\to\infty} \|x_{\Phi_n} - y_{\Phi_n}\| = 0. \tag{3.18}$$

It follows from (3.6), (3.17) and (3.18) that

$$\lim_{n\to\infty} \|Ty_{\Phi_n} - y_{\Phi_n}\| = 0. \tag{3.19}$$

From (3.18) and (3.19), we get

$$\lim_{n\to\infty} \|Tx_{\Phi_n} - x_{\Phi_n}\| = 0. \tag{3.20}$$

By Lemma 3.1 and (3.19), we obtain

$$\lim_{n\to\infty} \langle p - f(p), p - y_{\Phi_n}\rangle \le 0. \tag{3.21}$$

By using Lemma 2.1, for each positive integer $n \ge n_0$ we obtain

$$\begin{aligned}
0 \le \Gamma_{\Phi_n+1} - \Gamma_{\Phi_n} &= \|x_{\Phi_n+1} - p\|^2 - \|x_{\Phi_n} - p\|^2 \\
&\le \|y_{\Phi_n} - p\|^2 - \|x_{\Phi_n} - p\|^2 \\
&\le [1 - \alpha_{\Phi_n}(1-\lambda)]^2 \|x_{\Phi_n} - p\|^2 + 2\alpha_{\Phi_n}\langle f(p) - p, y_{\Phi_n} - p\rangle - \|x_{\Phi_n} - p\|^2 \\
&\le [1 - \alpha_{\Phi_n}(1-\lambda)] \|x_{\Phi_n} - p\|^2 + 2\alpha_{\Phi_n}\langle f(p) - p, y_{\Phi_n} - p\rangle - \|x_{\Phi_n} - p\|^2 \\
&= -\alpha_{\Phi_n}(1-\lambda) \|x_{\Phi_n} - p\|^2 + 2\alpha_{\Phi_n}\langle f(p) - p, y_{\Phi_n} - p\rangle;
\end{aligned}$$

so that

$$\|x_{\Phi_n} - p\|^2 \le \frac{2}{1-\lambda}\langle f(p) - p, y_{\Phi_n} - p\rangle. \tag{3.22}$$

Hence, we deduce from (3.21) and (3.22) that

$$\lim_{n\to\infty} \Gamma_{\Phi_n} = \lim_{n\to\infty} \|x_{\Phi_n} - p\|^2 = 0; \tag{3.23}$$

so that

$$\lim_{n\to\infty} \|y_{\Phi_n} - p\|^2 = 0. \tag{3.24}$$

As a consequence of (3.24), we get

$$\lim_{n\to\infty} \Gamma_{\Phi_n+1} = \lim_{n\to\infty} \|x_{\Phi_n+1} - p\|^2 = 0. \tag{3.25}$$

It is easy to check that for $n \ge n_0$, $\Gamma_j > \Gamma_{j+1}$ for all $\Phi_n + 1 \le j \le n$ if $\Phi_n < n$. Hence, $\Gamma_n \le \Gamma_{\Phi_n+1}$ whenever $\Phi_n < n$. It is also trivial that $\Gamma_n \le \Gamma_{\Phi_n+1}$ if $\Phi_n = n$. Therefore, for each $n \ge n_0$ we have

$$0 \le \Gamma_n \le \Gamma_{\Phi_n+1}. \tag{3.26}$$

It follows from (3.25) and (3.26) that

$$\lim_{n\to\infty} \Gamma_n = 0. \tag{3.27}$$

Therefore, $\{x_n\}$ converges strongly to $p$. Consequently, $\{y_n\}$ also converges strongly to $p$. This completes the proof. □

**Remark 3.3.** If $f \equiv u \in K$, then Theorem 3.2 reduces to the following corollary.

**Corollary 3.4.** *Let $K$, $H$, and $T : K \to K$ be as in Theorem 3.2. Let $u \in K$ be fixed. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ satisfying the control conditions (3.2), (3.3) and (3.4). Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by*

$$\begin{cases} x_0 \in K \\ y_n = \alpha_n u + (1-\alpha_n)x_n, \ n \geq 0 \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, \ n \geq 0. \end{cases} \tag{3.28}$$

*converge strongly to a point $p \in Fix(T)$ which is nearest to $u$.*

**Remark 3.5.** If u=O and K=H, then Corollary 3.4 reduces to Theorem 3.1 of Yao et al[11]. Therefore, Theorem 3.2 is an extension of main results of Yao et al[11].

## References

1. F. E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces* Bull. Amer. Math. Soc.74(1968), 660-665.
2. C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction,* Inverse Problems 20 (2004) 103120.
3. S. S. Chang, *On Chidume's open questions and approximate solutions of multivalued strongly accretive mapping equations in Banach spaces,* J. Math. Anal. Appl. 216(1997) 94-111.
4. J. Diestel, *Geometry of Banach Spaces - Selected Topics,* (Lecture Notes in Mathematics; 485), Springer-Verlag Berlin. Heidelberg. New York 1975.
5. M. A. Khamsi, W. A. Kirk, *An Introuction to Metric Spaces and Fixed Point Theory,* John Wiley & Sons, Inc. New York/ Chichester/ Weinheim/ Brisbane/ Sigapore/ Toronto, 2001.
6. W. R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc. 4(1953)506-510
7. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings* Bull. Amer. Math. Soc. 73(1967), 591-597.
8. H. K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc. 2(2002) 240-256.
9. H. K. Xu, *An Iterative Approach to Quadratic Optimization,* J. Opt. Theory Appl. 116(2003) 659-678.
10. H. K. Xu, *Viscosity approximation methods for nonexpansive mappings,* J. Math. Anal. Appl. 298(2004) 279-291.
11. Y. Yao, H. Zhou, Y. C. Liou,*Strong convergence of a modified Krasnoselski-Mann iterative algorithm for non-expansive mappings,* J Appl Math Comput (2009)29:383-389, DOI 10.1007/s12190-008-0139-z.

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