EXISTENCE, UNIQUENESS AND STABILITY OF IMPULSIVE STOCHASTIC PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAYS

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ABSTRACT. This article presents the result on existence, uniqueness and stability of mild solution of impulsive stochastic partial neutral functional differential equations under sufficient condition. The results are obtained by using the method of successive approximation.

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1. Introduction

Neutral differential equations arise in many area of science and engineering, have received much attention in the last decades. The ordinary neutral differential equation is very extensive to study the theory of aeroelasticity, see Kolmanvskii and nosov [10] and the lossless transmission lines [4] and the references therein. Partial neutral differential equations with delays are motivated from stabilization of lumped control systems, theory of heat conduction in materials [7, 8] and the references therein. E.Hernandez and Donal O'Regan [6], studied partial neutral differential equations by defining S-mild solution to the neutral system by assuming some temporal and spatial regularity type condition for the function $t \to g(t, x_t)$ is used to study some neutral system. In [15], Rodkina studied several existence results with an unbounded delay. In [4], Govindan generalized the main result of [15].

Recently impulsive differential equations are also well to model problems see [11, 18]. There is much notice in the field of fixed impulsive type equations [1, 7] and the references therein. The study of impulsive stochastic differential equations (ISDEs) is a new research area. There are few publications in the theory of ISDEs, Jun Yang et al.[9], studied the stability analysis of impulsive stochastic

differential equations with delays. Zhigno Yang et al.[20], studied the exponential p- stability of impulsive stochastic differential equations with delays. In [16, 17], R. Sakthivel and J. Luo studied the existence and asymptotic stability in p-th moment of mild solutions to impulsive stochastic partial differential equations with and without infinite delays through fixed point theory. Motivated by [13, 14], we will generalize the existence and uniqueness of the solution to impulsive stochastic partial neutral functional differential equations (ISNFDEs) under non-Lipschitz condition and Lipschitz condition. Moreover, we study the stability through the continuous dependence on the initial values by means of Corollary of Bihari inequality. Further, we refer [3, 12, 19].

The paper is organized as follows. In section 2, we recall briefly the notations, definitions, lemmas and preliminaries facts which are used throughout this paper. In section 3, we study the existence and uniqueness of ISNFDEs by reducing the linear growth conditions. In section 4, we study the continuous dependence on the initial values. Finally in section 5, an example is presented to illustrate our results

2. Preliminaries

Let X, Y be real separable Hilbert spaces and $L(Y, X)$ be the space of bounded linear operators mapping Y into X. For convenience, we shall use the same notations $\|.\|$ to denote the norms in X, Y and $L(Y, X)$ without any confusion. Let (Ω, B, P) be a complete probability space with an increasing right continuous family $\{B_t\}_{t \geq 0}$ of complete sub σ-algebra of B. Let $\{w(t) : t \geq 0\}$ denote a Y-valued Wiener process defined on the probability space (Ω, B, P) with covariance operator Q, that is $E < w(t), x >_Y < w(s), y >_Y = (t \wedge s) < Qx, y >_Y$, for all $x, y \in Y$, where Q is a positive, self-adjoint, trace class operator on Y. In particular, we denote $w(t)$, $t \geq 0$, a Y- valued Q- Wiener process with respect to $\{B_t\}_{t \geq 0}$.

In order to define stochastic integrals with respect to the Q- Wiener process $w(t)$, we introduce the subspace $Y_0 = Q^{1/2}(Y)$ of Y which, endowed with the inner product $< u, v >_{Y_0} = < Q^{-1/2}u, Q^{-1/2}v >_Y$ is a Hilbert space. We assume that there exists a complete orthonormal system $\{e_i\}_{i \geq 1}$ in Y, a bounded sequence of nonnegative real numbers λ_i such that $Qe_i = \lambda_i e_i, i = 1, 2, \ldots$, and a sequence $\{\beta_i\}_{i \geq 1}$ of independent Brownian motions such that

$$< w(t), e > = \sum_{n=1}^{\infty} \sqrt{\lambda_i} < e_i, e > \beta_i(t), \ e \in Y,$$

and $B_t = B_t^w$, where B_t^w is the sigma algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Y_0, X)$ denote the space of all Hilbert- Schmidt operators from Y_0 into X. It turns out to be a separable Hilbert space equipped with the norm $\|\mu\|_{L_2^0}^2 = tr((\mu Q^{1/2})(\mu Q^{1/2})^*)$ for any $\mu \in L_2^0$. Clearly for any bounded operators $\mu \in L(Y, X)$ this norm reduces to $\|\mu\|_{L_2^0}^2 = tr(\mu Q \mu^*)$.

In this article, we will examine the impulsive stochastic partial neutral functional differential equations of the form

$$
\begin{cases}
d\big(x(t) + g(t, x_t)\big) &= [Ax(t) + f(t, x_t)]dt + a(t, x_t)dw(t), \ t \neq t_k, \ 0 \leq t \leq T, \\
\Delta x(t_k) &= x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \ t = t_k, \ k = 1, 2, \ldots m, \\
x(t) &= \varphi \in D_{B_0}^b((-\infty, 0], X_\eta),
\end{cases}
\tag{1}
$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t) = \{S(t)\}_{t \geq 0}$ with $D(A) \subset X$. If $S(t)$ is uniformly bounded analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A^\eta)$, for $0 < \eta \leq 1$, as a closed linear operator with dense domain $D(-A^\eta)$ in X. If X_η represents the space $D(-A^\eta)$ endowed with norm $\| \cdot \|$, then we have the following properties:

Lemma 1. [12] *Assume that the following conditions hold:*
i : For $0 < \eta < 1$, X_η is a Banach space.
ii : For $0 < \eta \leq \beta < 1$, the embedding $X_\beta \hookrightarrow X_\eta$ is continuous.
iii : There exists a constant $C_\eta > 0$ depending on $0 < \eta \leq 1$ such that

$$
\|-A^\eta S(t)\|^2 \leq \frac{C_\eta}{t^{2\eta}}, \ t > 0.
$$

We now make the system (1) precise : Let $A : X \to X$ be the infinitesimal generator of an analytic semigroup $\{S(t), t \geq 0\}$ defined on X. Let the functions $f : \Re^+ \times D_\eta \to X$; $g : \Re^+ \times D_\eta \to X$; $a : \Re^+ \times D_\eta \to L(Y, X)$, where $\Re^+ = [0, \infty)$, are Borel measurable. Here $D_\eta = D((-\infty, 0], X_\eta)$ denote the family of all right piecewise continuous functions with left-hand limit φ from $(-\infty, 0]$ to X_η. The phase space $D((-\infty, 0], X_\eta)$ is assumed to be equipped with the norm $\|\varphi\|_t = \sup_{-\infty < \theta \leq 0} |\varphi(\theta)|$. We also assume $D_{B_0}^b((-\infty, 0], X_\eta)$ to denote the family of all almost surely bounded, B_0-measurable, D_η- valued random variables. Further, let $\mathcal{B}_T$ is a Banach space of all B_t-adapted process $\varphi(t, w)$ with almost surely continuous in t for fixed $w \in \Omega$ with norm defined for any $\varphi \in \mathcal{B}_T$

$$
\|\varphi\|_{\mathcal{B}_T} = \big(\sup_{0 \leq t \leq T} E\|\varphi\|_t^2 \big)^{1/2}.
$$

Furthermore, the fixed moments of time t_k satisfies $0 < t_1 < \ldots < t_m < T$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Also $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump in the state x at time t_k with I_k determining the size of the jump.

Lemma 2. [2] *Let $T > 0$ and $u_0 \geq 0$, $u(t), v(t)$ be continuous functions on $[0, T]$. Let $K : \Re^+ \to \Re^+$ be a concave continuous and nondecreasing function such that $K(r) > 0$ for $r > 0$. If*

$$
u(t) \leq u_0 + \int_0^t v(s)K(u(s))ds \ \ \text{for all } 0 \leq t \leq T,
$$

then

$$u(t) \leq G^{-1}\Big(G(u_0) + \int_0^t v(s)ds\Big) \quad \text{for all such } t \in [0,T] \text{ that}$$

$$G(u_0) + \int_0^t v(s)ds \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{K(s)}$, $r \geq 0$ and G^{-1} is the inverse function of G. In particular, if, moreover, $u_0 = 0$ and $\int_{0+} \frac{ds}{K(s)} = \infty$, then $u(t) = 0$ for all $0 \leq t \leq T$.

In order to obtain the stability of solutions, we give the extended Bihari inequality

Lemma 3. [13] *Let the assumptions of Lemma 2 hold. If*

$$u(t) \leq u_0 + \int_t^T v(s)K(u(s))ds \text{ for all } 0 \leq t \leq T,$$

then

$$u(t) \leq G^{-1}\Big(G(u_0) + \int_t^T v(s)ds\Big) \quad \text{for all such } t \in [0,T] \text{ that}$$

$$G(u_0) + \int_t^T v(s)ds \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{K(s)}$, $r \geq 0$ and G^{-1} is the inverse function of G.

Corollary 1. [13] *Let the assumptions of Lemma 2 hold and $v(t) \geq 0$ for $t \in [0,T]$. If for all $\epsilon > 0$, there exists $t_1 \geq 0$ such that for $0 \leq u_0 < \epsilon$, $\int_{t_1}^T v(s)ds \leq \int_{u_0}^\epsilon \frac{ds}{K(s)}$ holds. Then for every $t \in [t_1,T]$, the estimate $u(t) \leq \epsilon$ holds.*

Lemma 4. [3] *For any $r \geq 1$ and for arbitrary L_2^0- valued predictable process $\Phi(\cdot)$*

$$\sup_{s \in [0,t]} E\| \int_0^s \Phi(u)dw(u)\|_X^{2r} = (r(2r-1))^r \Big(\int_0^t (E\|\Phi(s)\|_{L_2^0}^{2r})ds \Big)^r.$$

Definition 1. A semigroup $\{S(t),\ t \geq 0\}$ is said to be uniformly bounded if $\|S(t)\| \leq M$ for all $t \geq 0$, where $M \geq 1$ is some constant. If $M = 1$, then the semigroup is said to be contraction semigroup.

Definition 2. A stochastic process $\{x(t), t \in (-\infty, T]\}, (0 < T < \infty)$ is called a mild solution of equation (1), if
  (i) $x(t)$ is B_t- adapted;
  (ii) $x(t)$ satisfies the integral equation

$$x(t) = \begin{cases} \varphi \in D_{B_0}^b((-\infty,0], X_\eta), & t \in (-\infty,0], \\ \\ S(t)\big[\varphi(0) + g(0,\varphi)\big] - g(t,x_t) - \int_0^t AS(t-s)g(s,x_s)ds \\ + \int_0^t S(t-s)f(s,x_s)ds + \int_0^t S(t-s)a(s,x_s)dw(s) & \quad (2) \\ + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), \qquad a.s \qquad t \in [0,T]. \end{cases}$$

3. Existence and uniqueness

In this section, we discuss the existence and uniqueness of mild solution of the system (1). We need the following hypotheses to use in our results.

Hypotheses:

(H_1) : A is the infinitesimal generator of an analytic semigroup $S(t)$, whose domain $D(A)$ is dense in X.

(H_2) : For each $x, y \in D_\eta$ and for all $t \in [0, T]$, such that,

$$\|f(t, x_t) - f(t, y_t)\|^2 \vee \|a(t, x_t) - a(t, y_t)\|^2 \le K(\|x - y\|_t^2),$$

where $K(\cdot)$ is a concave non-decreasing function from $\Re^+$ to $\Re^+$, such that $K(0) = 0$, $K(u) > 0$, for $u > 0$ and $\int_{0+} \frac{du}{K(u)} = \infty$.

(H_3) : Assuming that there exists a positive number $L_g > 0$ such that for any $x, y \in D_\eta$ and for $t \in [0, T]$, we have

$$\|(-A)^\eta g(t, x_t) - (-A)^\eta g(t, y_t)\|^2 \le L_g \|x - y\|_t^2,$$

(H_4) : The function $I_k \in C(X, X)$ and there exists some constant h_k such that

$$\|I_k(x(t_k)) - I_k(y(t_k))\|^2 \le h_k \|x - y\|_t^2, \quad \text{for each } x, y \in D_\eta, \ k = 1, 2 \dots, m.$$

(H_5) : For all $t \in [0, T]$, it follows that $f(t, 0), (-A)^\eta g(t, 0), a(t, 0), I_k(0) \in L^2$, for $k = 1, 2 \dots, m$ such that

$$\|f(t, 0)\|^2 \vee \|(-A)^\eta g(t, 0)\|^2 \vee \|a(t, 0)\|^2 \vee \|I_k(0)\|^2 \le \kappa_0,$$

where $\kappa_0 > 0$ is a constant.

Let us now introduce the successive approximations to equation (2) as follows

$$x^n(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0], \text{ for } n = 0, 1, 2, \dots ; \\ S(t)\varphi(0), & t \in [0, T], \text{ for } n = 0; \\ S(t)\big[\varphi(0) + g(0, \varphi)\big] - g(t, x_t^n) - \displaystyle\int_0^t AS(t-s)g(s, x_s^n)ds \\ \quad + \displaystyle\int_0^t S(t-s)f(s, x_s^{n-1})ds + \int_0^t S(t-s)a(s, x_s^{n-1})dw(s) \\ \quad + \displaystyle\sum_{0 < t_k < t} S(t - t_k)I_k(x^{n-1}(t_k)), \quad a.s \ \ t \in [0, T], \text{ for } n = 1, 2, \dots . \end{cases} \quad (3)$$

with an arbitrary non-negative initial approximation $x^0 \in \mathcal{B}_T$.

Note that the above scheme is not explicit as the function g on the right hand side of equation (3) depends explicitly on x^n. But, this seems to be standard when one considers neutral equations, see [4, 5, 15].

Theorem 1. *Let the assumptions* $(H_1) - (H_5)$ *hold, then the system* (1) *has unique mild solution* $x(t)$ *in* $\mathcal{B}_T$ *provided* $\widetilde{Q} = \max\{Q_1, Q_5\} < 1$ *and*

$$E\{ \sup_{0 \le t \le T} \|x^n(t) - x(t)\|^2 \} \to 0 \quad \text{as } n \to \infty$$

where $\{x^n(t)\}_{n\geq 1}$ are the successive approximations (3).

Proof . Let $x^0 \in \mathcal{B}_T$ be a fixed initial approximation to (3). To begin with under assumptions (H_1) - (H_5), $Q_i > 0, i = 1, \ldots, 12$, are some constants and observing that $\|S(t)\| \leq M$ for some $M \geq 1$ and all $t \in [0, T]$. Then for any $n \geq 1$, we have,

$$E\|x^n(t)\|^2 \leq 6M^2 E \|\varphi(0) + g(0, \varphi)\|^2$$
$$+12\| - A^{-\eta}\|^2 E\left[\|(-A^\eta)g(t, x_t^n) - (-A^\eta)g(t, 0)\|^2 + \|(-A^\eta)g(t, 0)\|^2\right]$$
$$+12TE \int_0^t \| - A^{1-\eta}S(t - s)\|^2 \left[\|(-A^\eta)g(s, x_s^n) - (-A^\eta)g(s, 0)\|^2 + \|(-A^\eta)g(s, 0)\|^2\right]ds$$
$$+12TM^2 E \int_0^t \left[\|f(s, x_s^{n-1}) - f(s, 0)\|^2 + \|f(s, 0)\|^2\right]ds$$
$$+12M^2 E \int_0^t \left[\|a(s, x_s^{n-1}) - a(s, 0)\|^2 + \|a(s, 0)\|^2\right]ds$$
$$+12M^2 mE \sum_{k=1}^m \left[\|I_k(x^{n-1}(t_k)) - I_k(0)\|^2 + \|I_k(0)\|^2\right].$$

From the Lemma 1, (H_3) and (H_5) the following relation holds:

$$E\|(-A)S(t-s)g(s, x^n{}_s)\|^2 = E\|(-A^{1-\eta})S(t-s)(-A^\eta)g(s, x_s)\|^2$$
$$\leq 2\|(-A^{1-\eta})S(t-s)\|^2 \left[E\|(-A^\eta)g(s, x^n{}_s) - (-A^\eta)g(s, 0)\|^2 + E\|(-A^\eta)g(s, 0)\|^2\right]$$
$$\leq \frac{2C_{1-\eta}}{(t-s)^{2(1-\eta)}} \left[L_g E\|x^n\|_s^2 + \kappa_0\right]. \tag{4}$$

Thus from the above,

$$E\|x^n(t)\|^2 \leq 12M^2 \left[E\|\varphi(0)\|^2 + E\|g(0, \varphi)\|^2\right]$$
$$+12\| - A^{-\eta}\|^2 \left[L_g E\|x^n\|_t^2 + \kappa_0\right]$$
$$+12T \int_0^t \frac{2C_{1-\eta}}{(t-s)^{2(1-\eta)}} \left[L_g E\|x^n\|_s^2 + \kappa_0\right]$$
$$+12(T+1)M^2 E \int_0^t \left[K(\|x^{n-1}\|_s^2) + \kappa_0\right]ds$$
$$+12M^2 m \sum_{k=1}^m \left[h_k E\|x^{n-1}\|_t^2 + \kappa_0\right]$$

$$E\|x^n\|_t^2 \leq Q_2 + \frac{12M^2(T+1)}{1 - Q_1} E \int_0^t K(\|x^{n-1}\|_s^2)ds + \frac{12M^2 m \sum_{k=1}^m h_k}{1 - Q_1} \left\{\|x^{n-1}\|_t^2\right\}$$

where, $Q_1 = 12\left(\| - A^{-\eta}\|^2 + 2\frac{C_{1-\eta}}{2\eta-1}T^{2\eta}\right)L_g$.

Given that $K(\cdot)$ is concave and $K(0) = 0$, we can find a pair of positive constants a and b such that $K(u) \leq a + bu$, for all $u \geq 0$, and applying mathematical

induction, we get

$$E\left\|x^n\right\|_t^2 \leq Q_3 + \frac{12M^2(T+1)b}{1-Q_1} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} E\|x^0\|_s^2 ds \qquad (5)$$
$$+ \frac{12M^2 m \sum_{k=1}^m h_k}{1-Q_1} \left\{\frac{T^{n-1}}{(n-1)!} E\left\|x^0\right\|_t^2\right\},$$

where, $Q_3 = Q_2 + \frac{12M^2(T+1)Ta}{1-Q_1}$, we know that,

$$E\left\|x^0(t)\right\|^2 \leq M^2 E\left\|\varphi(0)\right\|^2 = Q_4 < \infty. \qquad (6)$$

Thus, $\sup_{t\in[0,T]} E\left\|x^0\right\|_t^2 < \infty$, then from (5), $\sup_{t\in[0,T]} E\left\|x^n\right\|_t^2 < \infty$, for all $n = 1, 2, \ldots$
and $t \in [0, T]$. This proves the boundedness of $\{x^n\}$.
Let us next show that $\{x^n\}$ is Cauchy in $\mathcal{B}_T$. For this, choose $T_1 \in [0, T)$ such that

$$\frac{5M^2(T_1+1)}{1-Q_5} K\left(\frac{(t-s)^n}{(n)!} Q_9\right) \leq \frac{5M^2(T_1+1)}{1-Q_5} \frac{(t-s)^n}{(n)!} Q_9, \text{ for all } 0 \leq t \leq T_1,$$

consider,

$$E\left\|x^{n+1}(t) - x^n(t)\right\|^2 \leq 5\left(\|-A^{-\eta}\|^2 + \frac{C_{1-\eta}}{2\eta-1} T_1^{2\eta}\right) L_g E\|x^{n+1} - x^n\|_t^2$$
$$+ 5M^2(T_1+1) \int_0^t K(E\|x^n - x^{n-1}\|_s^2) ds$$
$$+ 5M^2 m \sum_{k=1}^m h_k E\|x^n - x^{n-1}\|_t^2.$$

Thus,

$$E\left\|x^{n+1} - x^n\right\|_t^2 \leq \frac{5M^2(T_1+1)}{1-Q_5} \int_0^t K(E\|x^n - x^{n-1}\|_s^2) ds \qquad (7)$$
$$+ \frac{5M^2 m \sum_{k=1}^m h_k}{1-Q_5} E\|x^n - x^{n-1}\|_t^2,$$

where, $Q_5 = 5\left(\|-A^{-\eta}\|^2 + \frac{C_{1-\eta}}{2\eta-1} T_1^{2\eta}\right) L_g$. Moreover,

$$\left\|x^1(t) - x^0(t)\right\|^2 = \|S(t)g(0, \varphi) - \left[g(t, x_t^1) - g(t, x_t^0)\right] - g(t, x_t^0)$$
$$- \int_0^t AS(t-s)\left[g(s, x_s^1) - g(s, x_s^0)\right] ds$$
$$- \int_0^t AS(t-s)g(s, x_s^0) ds + \int_0^t S(t-s)f(s, x_s^0) ds$$
$$+ \int_0^t S(t-s)a(s, x_s^0) dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x^0(t_k))\|^2.$$

Then, we get

$$E\left\|x^1 - x^0\right\|_t^2 \leq Q_6 + \frac{16\Big(\big[\|-A^{-\eta}\|^2 + \frac{C_{1-\eta}}{2\eta-1}T_1^{2\eta}\big]L_g + M^2 m \sum_{k=1}^m h_k\Big)}{1-Q_7}E\|x^0\|_t^2$$

$$+ \frac{16M^2(T_1+1)}{1-Q_7}\int_0^t K(E\|x^0\|_s^2)ds,$$

taking supremum over t, and from (6), we get

$$\sup_{t\in[0,T_1]} E\left\|x^1 - x^0\right\|_t^2 \leq Q_8 + \frac{16M^2(T_1+1)}{1-Q_7}\int_0^t K(Q_4)ds$$

$$\leq Q_9. \tag{8}$$

Thus applying mathematical induction in (7) and from (8),

$$\sup_{t\in[0,T_1]} E\left\|x^{n+1} - x^n\right\|_t^2 \leq \frac{5M^2(T_1+1)}{1-Q_5}\int_0^t K\Big(\frac{(t-s)^n}{n!}\sup_{t\in[0,T_1]}E\|x^1-x^0\|_s^2\Big)ds$$

$$+\frac{5M^2 m \sum_{k=1}^m h_k}{1-Q_5}\Big\{\frac{T_1^n}{n!}\Big\}\sup_{t\in[0,T_1]}E\|x^1-x^0\|_t^2$$

$$\leq \frac{5M^2(T_1+1)}{1-Q_5}\int_0^t K\Big(\frac{(t-s)^n}{n!}Q_9\Big)ds$$

$$+\frac{5M^2 m \sum_{k=1}^m h_k}{1-Q_5}\Big\{\frac{T_1^n}{n!}\Big\}Q_9$$

$$\leq \frac{5M^2(T_1+1)}{1-Q_5}\int_0^t \frac{(t-s)^n}{n!}Q_9\,ds$$

$$+\frac{5M^2 m \sum_{k=1}^m h_k}{1-Q_5}\Big\{\frac{T_1^n}{n!}\Big\}Q_9$$

$$\leq Q_{10}\frac{T_1^n}{n!}, \quad n \geq 0, \ t \in [0,T_1].$$

Note that for any $m > n \geq 1$, we have,

$$\sup_{t\in[0,T_1]} E\left\|x^m(t) - x^n(t)\right\|^2 \leq \sum_{r=n}^{+\infty}\sup_{t\in[0,T_1]} E\left\|x^{r+1} - x^r\right\|_t^2$$

$$\leq \sum_{r=n}^{+\infty}\Big(Q_{10}\frac{T_1^r}{r!}\Big)$$

$$\to 0 \text{ as } n \to \infty. \tag{9}$$

This shows that $\{x^n\}$ is Cauchy in $\mathcal{B}_T$. Then the standard Borel- Cantelli lemma argument can be used to show that $x^n(t) \to x(t)$ uniformly in t on $[0,T_1]$. Hence $x(t)$ is a solutions of (1) in the interval $[0,T_1]$. By Theorem 6 in [13], the existence of solution of (1) on $[0,T]$ can be obtained by iteration.

Now, we prove the uniqueness of the solution (2). Let $x_1, x_2 \in \mathcal{B}_T$ be two solutions to (1) on some interval $(-\infty, T]$. Then, for $t \in (-\infty, 0]$, the uniqueness

is obvious and for $0 \leq t \leq T$, we have

$$
\begin{aligned}
E\|x_1(t) - x_2(t)\|^2 \leq\ & 5\left(\left[\| - A^{-\eta}\|^2 + \frac{C_{1-\eta}}{2\eta - 1}T^{2\eta}\right]L_g + M^2 m \sum_{k=1}^{m} h_k\right)E\|x_1 - x_2\|_t^2 \\
& + 5M^2(T+1)\int_0^t K(E\|x_1 - x_2\|_s^2)ds
\end{aligned}
$$

Thus,

$$
E\|x_1 - x_2\|_t^2 \leq \frac{5M^2(T+1)}{1 - Q_{11}}\int_0^t K(E\|x_1 - x_2\|_s^2)ds,
$$

where,
$$
Q_{11} = 5\left(\left[\| - A^{-\eta}\|^2 + \frac{C_{1-\eta}}{2\eta - 1}T^{2\eta}\right]L_g + M^2 m \sum_{k=1}^{m} h_k\right).
$$

Thus, Bihari inequality yields that

$$
\sup_{t\in[0,T]} E\|x_1 - x_2\|_t^2 = 0,\ 0 \leq t \leq T.
$$

Thus, $x_1(t) = x_2(t)$, for all $0 \leq t \leq T$. Therefore, for all $-\infty < t \leq T$, $x_1(t) = x_2(t)$ a.s. This achieve the proof. □

4. Stability

In this section, we mean in this stability is that small changes in the initial conditions lead to small changes in the solutions over a given finite time interval.

Definition 3. *A mild solution $x(t)$ of the system (1) with initial value ϕ is said to be stable in the mean square if for all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$
E\|x(t) - \hat{x}(t)\|^2 \leq \epsilon \text{ when } E\|\phi - \hat{\phi}\|^2 < \delta, \text{ for all } t \in [0, T]. \tag{10}
$$

where $\hat{x}(t)$ is another mild solution of the system (1) with initial value $\hat{\phi}$.

Theorem 2. *Let $x(t)$ and $y(t)$ be mild solutions of the system (1) with initial values φ_1 and φ_2 respectively. Assume the assumptions of Theorem 1 are satisfied, then the mild solution of the system (1) is stable in the quadratic mean.*

Proof. By the assumptions, $x(t)$ and $y(t)$ are two mild solutions of equations (1) with initial values φ_1 and φ_2 respectively, then for $0 \leq t \leq T$

$$
\begin{aligned}
x(t) - y(t) =\ & S(t)\left(\left[\varphi_1(0) - \varphi_2(0)\right] + \left[g(0, \varphi_1) - g(0, \varphi_2)\right]\right) - \left[g(t, x_t) - g(t, y_t)\right] \\
& - \int_0^t AS(t-s)\left[g(s, x_s) - g(s, y_s)\right]ds + \int_0^t S(t-s)\left[f(s, x_s) - f(s, y_s)\right]ds \\
& + \int_0^t S(t-s)\left[a(s, x_s) - a(s, y_s)\right]dw(s) + \sum_{0<t_k<t} S(t - t_k)\left[I_k(x(t_k)) - I_k(y(t_k))\right].
\end{aligned}
$$

So, estimating as before, we get

$$
\begin{aligned}
E\|x(t) - y(t)\|^2 &\leq 7M^2\big(1 + \| - A^{-\eta}\|^2 L_g\big)E\|\varphi_1 - \varphi_2\|^2 \\
&\quad + 7\Big(\big[\| - A^{-\eta}\|^2 + \frac{C_{1-\eta}}{2\eta - 1}T^{2\eta}\big]L_g + M^2 m \sum_{k=1}^{m} h_k\Big)E\|x - y\|_t^2 \\
&\quad + 7M^2(T + 1)\int_0^t K(E\|x - y\|_s^2)ds,
\end{aligned}
$$

Thus,

$$
\begin{aligned}
E\|x - y\|_t^2 &\leq \frac{7M^2\big(1 + \| - A^{-\eta}\|^2 L_g\big)}{1 - Q_{12}}E\|\varphi_1 - \varphi_2\|^2 \\
&\quad + \frac{7M^2(T + 1)}{1 - Q_{12}}\int_0^t K(E\|x - y\|_s^2)ds,
\end{aligned}
$$

where, $Q_{12} = 7\big(\big[\| - A^{-\eta}\|^2 + \frac{C_{1-\eta}}{2\eta-1}T^{2\eta}\big]L_g + M^2 m \sum_{k=1}^m h_k\big)$.

Let $K_1(u) = \frac{7M^2(T+1)}{1-Q_{12}}K(u)$, for K is a concave increasing function from $\Re^+$ to $\Re^+$ such that $K(0) = 0$, $K(u) > 0$ for $u > 0$ and $\int_{0+}\frac{du}{K(u)} = +\infty$. So, $K_1(u)$ is obvious a concave function from $\Re^+$ to $\Re^+$ such that $K_1(0) = 0$, $K_1(u) \geq K(u)$, for $0 \leq u \leq 1$ and $\int_{0+}\frac{du}{K_1(u)} = +\infty$. So for any $\epsilon > 0$, $\epsilon_1 \triangleq \frac{1}{2}\epsilon$, so we have $\lim_{s \to 0}\int_s^{\epsilon_1}\frac{du}{K_1(u)} = \infty$. So, there is a positive constant $\delta < \epsilon_1$, such that $\int_\delta^{\epsilon_1}\frac{du}{K_1(u)} \geq T$.

From Corollary 1, let

$$
\begin{aligned}
u_0 &= \frac{7M^2\big(1 + \|(-A^{-\eta})\|^2 L_g\big)}{1 - Q_{12}}E\|\varphi_1 - \varphi_2\|^2, \\
u(t) &= E\|x - y\|_t^2, \ v(t) = 1,
\end{aligned}
$$

when $u_0 \leq \delta \leq \epsilon_1$, we have

$$
\int_{u_0}^{\epsilon_1}\frac{du}{K_1(u)} \geq \int_\delta^{\epsilon_1}\frac{du}{K_1(u)} \geq T = \int_0^T v(s)ds.
$$

So, for any $t \in [0, T]$, the estimate $u(t) \leq \epsilon_1$ holds. This completes the proof. $\square$

Remark 1. If $m = 0$ in (1), then the system behave as stochastic partial neutral functional differential equations with infinite delays of the form

$$
\begin{cases}
d\big(x(t) + g(t, x_t)\big) &= [Ax(t) + f(t, x_t)]dt + a(t, x_t)dw(t), \ 0 \leq t \leq T, \\
x(t) &= \varphi \in D_{B_0}^b((-\infty, 0], X_\eta),
\end{cases}
\tag{11}
$$

by applying Theorem 1 under the hypotheses $(H_1) - (H_3), (H_5)$ the system (11) guarantees the existence and uniqueness of the mild solution.

Remark 2. The system (11) satisfies the Remark 1. Then by Theorem 2, the mild solution of the system (11) is stable in the quadratic mean.

5. An example

We conclude this work with an example of the form

$$d\left[u(t,x) + \int_0^\pi b(y,x)u(t sint,y)dy\right] = \left[\frac{\partial^2}{\partial x^2}u(t,x) + H(t,u(t sint,x))\right]dt \qquad (12)$$

$$+ \sigma\, G(t,u(t sint,x))d\beta(t), \qquad t \neq t_k,\ 0 \leq t \leq T,$$

$$u(t_k^+) - u(t_k^-) = (1+b_k)u(x(t_k)),\ t = t_k,\ k = 1,2,\ldots m,$$

$$u(t,0) = u(t,\pi) = 0$$

$$u(t,x) = \Phi(t,x),\ 0 \leq x \leq \pi,\ -\infty < t \leq 0.$$

Let $X = L^2([0,\pi])$ and $Y = R^1$, the real number σ is magnitude of continuous noise, $\beta(t)$ is a standard one dimension Brownian motion, $\Phi \in D_{B_0}^b((-\infty,0],X)$, $b_k \geq 0$ for $k = 1,2,\ldots,m$ and $\sum_{k=1}^m b_k < \infty$.

Define A an operator on X by $Au = \frac{\partial^2 u}{\partial x^2}$ with the domain

$$D(A) = \left\{u \in X \middle| u \text{ and } \frac{\partial u}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 u}{\partial x^2} \in X,\right.$$

$$\left. u(0) = u(\pi) = 0. \right\}$$

It is well known that A generates a strongly continuous semigroup $S(t)$ which is compact, analytic and self adjoint. Moreover, the operator A can be expressed as

$$Au = \sum_{n=1}^\infty n^2 < u,u_n > u_n,\ u \in D(A),$$

where $u_n(\zeta) = (\frac{2}{\pi})^{\frac{1}{2}}\sin(n\zeta)$, $n = 1,2,\ldots$, is the orthonormal set of eigenvectors of A. Then the operator $(-A)^{\frac{1}{2}}$ is given by

$$(-A)^{\frac{1}{2}}u = \sum_{n=1}^\infty n < u,u_n > u_n \text{ on the space}$$

$$D[(-A)^{\frac{1}{2}}] = \left\{u \in X; \sum_{n=1}^\infty n < u,u_n > u_n \in X\right\}.$$

This satisfies $\|S(t)\| \leq 1$, $t \geq 0$, and hence is a contraction semigroup. In particular,

$$\|(-A)^{-\frac{1}{2}}\| = \frac{1}{\Gamma\frac{1}{2}}\int_0^\infty t^{\frac{1}{2}-1}\|S(t)\|dt \leq 1.$$

We assume that the following condition hold:
(i): The function b is measurable and

$$\int_0^\pi \int_0^\pi b^2(y,x)dydx < \infty.$$

(ii): The function $\frac{\partial}{\partial t}b(y,x)$ is measurable $b(y,0) = b(y,\pi) = 0$ and let

$$L_g = \Big[\int_0^\pi \int_0^\pi \big(\frac{\partial}{\partial t}b(y,x)\big)^2 dydx \Big]^{\frac{1}{2}} < \infty.$$

Assuming that conditions (i) and (ii) are verified, then the problem (12) can be modeled as the abstract impulsive stochastic partial neutral functional differential equation (1). Define now

$$g(t,x_t) = \int_0^\pi b(y,x)u(tsint,y)dy, \ \ f(t,x_t) = H(t,u(tsint,x)),$$

$$a(t,x_t) = \sigma G(t,u(tsint,x)) \text{ and } I_k(x(t_k)) = (1+b_k)u(x(t_k)) \text{ for } k = 1,2,\ldots m.$$

The next results a consequence of Theorem 1 and Theorem 2 respectively.

Proposition 1. *Assume that the hypotheses $(H_1) - (H_5)$ hold. Then there exists a unique mild solution u of the system (12) provided*

$$\widetilde{Q} = \max\{Q_1, Q_5\} < 1.$$

is satisfied.

Proposition 2. *Assume that the conditions of Proposition 1 hold. Then the mild solution u of the system (12) is stable in the quadratic mean.*

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