J. Appl. Math. & Informatics Vol. **28**(*2010*), *No. 3 - 4, pp. 723 - 737*
Website: http://www.kcam.biz

FRAMES AND SAMPLING THEOREMS IN MULTIWAVELET SUBSPACES

ZHANWEI LIU*, GUOCHANG WU AND XIAOHUI YANG

ABSTRACT. In this paper, we investigate the sampling theorem for frame in multiwavelet subspaces. By the frame satisfying some special conditions, we obtain its dual frame with explicit expression. Then, we give an equivalent condition for the sampling theorem to hold in multiwavelet subspaces. Finally, a sufficient condition under which the sampling theorem holds is established. Some typical examples illustrate our results.

AMS Mathematics Subject Classification : 40A05,41A30,41A58
Key words and phrases : Frame; regular sampling; multiwavelet; sampling theorem; shift-invariant subspace.

1. Introduction

At the present time the sampling theorem plays a crucial role in signal processing and communication, as it establishes an equivalence between discrete signals and analogue (continuous) signals. For a band-limited signal, the classical Shannon sampling theorem provides an exact representation by its uniform samples with a sampling rate higher than its Nyquist rate. But there exist several problems. Firstly, real world signals or images are never exactly band-limited. Secondly, there is no such device as an ideal (anti-aliasing or reconstruction) low-pass filter. Thirdly, Shannons reconstruction formula is rarely used in practice (especially with images) because of the slow decay of the sinc function. Therefore, this classical Shannon sampling theorem has been generalized to many other forms.

Extensions of Shannons sampling theorem to scalar wavelets can be found in [1]-[5], but a scalar wavelet cannot have the orthogonality, compact support, and symmetry at the same time (except the Haar wavelet). It is a disadvantage for signal processing. Meanwhile, multiwavelets have attracted much attention in the research community, since multiwavelet has more desired properties than any

Received September 10, 2009. Revised September 23, 2009. Accepted October 26, 2009.
*Corresponding author. Research was supported by the NSF of China (Grant No.60802061).

scalar wavelet function, such as orthogonality, short compact support, symmetry, high approximation order and so on. The first orthogonal multiwavelet with symmetry, approximation order, and compact support was presented by Geronimo et al. [6]. In addition, the sampling theorems for multiwavelet subspaces were studied in[7]-[10]. The authors of [7] and [9] presented the construction of compactly supported orthogonal multiscaling functions that are continuously differentiable and cardinal. The scaling functions thereby support a Shannonlike sampling theorem. However, the multiwavelets of [7] and [9] do not have symmetry. It is not good for digital signal processing and image compression. They also did not study the sampling theorem for frame in multiwavelet subspaces, which is very important in application.

In our paper, we provide the dual frame of the frame in multiwavelet subspaces under some special conditions, and show its formula in frequency space. Then, we give an equivalent condition for the frame multiwavelet sampling theorem to hold. Finally, a sufficient condition for the frame multiwavelet sampling theorem is presented.

This paper is organized as follows. In Section 2 contains some definitions in this correspondence. Also, we review some relative notations. In the next Section, we study the dual frame for a frame in multiwavelet subspaces. In Section 4, we discuss general uniform sampling and establish the sampling theorem for frame in multiwavelet subspaces. In Section 5, some examples are given to prove our results.

2. Preliminary

We now introduce some notations used in this correspondence. The shift-invariant closed subspace V_0 generated by $\{\phi_1, \phi_2, \cdots, \phi_r\}$

$$V_0 = \overline{span\{\phi_i(\cdot - k) : 1 \leq i \leq r, k \in Z\}} \subset L^2(R).$$

For a function $f \in L^2(R)$, we denote by $\langle f \rangle$ the minimal closed shift invariant subspace that contains f.

Let $\mathbf{f} = [f_1, f_2, \cdots, f_r]^T$ denote vector(we denote vectors and matrices in this paper in boldface). The integration $\int_R \mathbf{f}(x)dx$ is defined as

$$\int_R \mathbf{f}(x)dx = [\int_R f_1(x)dx, \int_R f_2(x)dx, \cdots, \int_R f_r(x)dx]^T.$$

The Fourier transform of vector $\mathbf{f}$ is defined by

$$\widehat{\mathbf{f}}(\omega) = \int_R \mathbf{f}(x)e^{-ix\omega}dx.$$

The inverse Fourier transform of vector $\mathbf{f}$ is written by

$$\check{\mathbf{f}}(x) = \int_R \mathbf{f}(\omega)e^{ix\omega}d\omega.$$

$Z_f(x,\omega) = \sum_{n\in Z} f(x+n)e^{-in\omega}$ is the Zak transform of function f. The Zak transform of vector **f** is defined by

$$Z_{\mathbf{f}(x,\omega)} = [Z_{f_1}(x,\omega), Z_{f_2}(x,\omega), \cdots, Z_{f_r}(x,\omega)]^T.$$

A collection of elements $\{\phi_i(\cdot - k) : 1 \le i \le r, k \in Z\}$ in a Hilbert space $H \subset L^2(R)$ is called a *frame* if there exist constants A and B, $0 < A \le B < \infty$, such that

$$A\|f\|^2 \le \sum_{k\in Z}\sum_{i=1}^{r} |\langle f, \phi_i(\cdot - k)\rangle|^2 \le B\|f\|^2, \quad \forall f \in H.$$

If $\{\phi_i(\cdot - k) : 1 \le i \le r, k \in Z\}$ is a frame for H, then there exists a dual frame $\{\widetilde{\phi}_k\}_{k\in Z}$ for $\{\phi_i(\cdot - k) : 1 \le i \le r, k \in Z\}$[13, Theorem 5.6.5].

Let $\Phi = [\phi_1, \phi_2, \cdots, \phi_r]^T$. For $f \in H$,we can write

$$E_f = \{\omega \in R | G_f(\omega) > 0\}, \quad G_f(\omega) = (\sum_{k\in Z} |\widehat{f}(\omega + 2\pi k)|^2)^{\frac{1}{2}},$$

$$\chi_{E_f} = \begin{cases} 1, & t \in E_f \\ 0, & t \notin E_f. \end{cases}$$

We define operator $T : l^2(Z) \to H$ by

$$T\{c_k\}_{k\in Z} = \sum_{k\in Z} c_k f_k.$$

The adjoint operator $T^* : H \to l^2(Z)$ of T is called the analysis operator and satisfies

$$T^* f = \{\langle f, f_k\rangle\}_{k\in Z}.$$

To each frame $\{f_k\}_{k\in Z}$ there corresponds a bounded positive invertible operator $S = TT^*$, called the frame operator, from H into itself, which satisfies

$$Sf = \sum_{k\in Z} \langle f, f_k\rangle f_k.$$

In this case, $\{S^{-1}f_k\}_{k\in Z}$ is called the canonical dual frame for $\{f_k\}_{k\in Z}$. The canonical dual frame gives the reconstruction formula

$$f = \sum_{k\in Z} \langle f, S^{-1}f_k\rangle f_k = \sum_{k\in Z} \langle f, f_k\rangle S^{-1}f_k.$$

Let $A \subset R$, $Max\{A\}$ means the largest number in subset A, $Min\{A\}$ means the smallest number in subset A.

3. Dual frame for multiwavelet subspaces

The main purpose of this section is to study the dual frame of a frame in multiwavelet subspaces.

Theorem 1. *Let $\{\phi_i(\cdot - k) : 1 \le i \le r, k \in Z\}$ be a frame for V_0 with bounds A, B and $|E_{\phi_i} \cap E_{\phi_j}| = 0$ for all $i, j \in \{1, 2, \cdots, r\}, i \ne j$. If the functions $\widetilde{\phi}_i(\omega) \in L^2(R)(i = 1, 2, \ldots, r)$ are defined by*

$$\widehat{\widetilde{\phi}}_i(\omega) = \begin{cases} \dfrac{\widehat{\phi}_i(\omega)}{\sum\limits_{n\in Z} |\widehat{\phi}_i(\omega + 2n\pi)|^2}, & \omega \in E_{\phi_i} \\ 0, & \omega \notin E_{\phi_i}, \end{cases} \tag{1}$$

then $\{\widetilde{\phi}_i(\cdot - k) : 1 \le i \le r, k \in Z\}$ is a dual frame of the frame $\{\phi_i(\cdot - k) : 1 \le i \le r, k \in Z\}$.

Proof. Suppose that the functions $\widetilde{\phi}_i(\omega) \in L^2(R)(i = 1, 2, \ldots, r)$ are defined by (1). Clearly, $\frac{1}{\sum_{n\in Z} |\widehat{\phi}_i(\omega+2n\pi)|^2}$ is 2π-periodic, and its restriction to $[0, 2\pi]$ belongs to $L^2(0, 2\pi)$. Then, by [13, Lemma 7.3.2], the functions $\{\widetilde{\phi}_i : 1 \le i \le r\}$ belong to V_0. Using the definition of the frame operator, properties of the Fourier transform, and [13, Lemma 7.2.1], we get

$$\begin{aligned} \widehat{S\widetilde{\phi}_i} &= \sum_{j=1}^{r} \sum_{k\in Z} \langle \widetilde{\phi}_i, T_k\phi_j\rangle \widehat{T_k\phi_j} \\ &= \sum_{j=1}^{r} \sum_{k\in Z} \langle \widehat{\widetilde{\phi}}_i, \widehat{T_k\phi_j}\rangle \widehat{T_k\phi_j} \\ &= \sum_{j=1}^{r} \left(\sum_{k\in Z} \langle \widehat{\widetilde{\phi}}_i, e^{-ikx}\widehat{\phi_j}\rangle e^{-ikx} \right) \widehat{\phi_j}. \end{aligned} \tag{2}$$

Now, using the definition of $\widetilde{\phi}_i(\omega)$, we have

$$\begin{aligned} \langle \widehat{\widetilde{\phi}}_i, e^{-ik\omega}\widehat{\phi_j}\rangle &= \int_{-\infty}^{\infty} \widehat{\widetilde{\phi}}_i(\omega)\overline{\widehat{\phi_j}(\omega)} e^{ik\omega} d\omega \\ &= \int_0^{2\pi} \sum_{n\in Z} (\widehat{\widetilde{\phi}}_i(\omega + 2n\pi)\overline{\widehat{\phi_j}(\omega + 2n\pi)}) e^{ik\omega} d\omega \\ &= \int_0^{2\pi} \sum_{n\in Z} \left(\frac{\widehat{\phi}_i(\omega + 2n\pi)}{\sum\limits_{n\in Z} |\widehat{\phi}_i(\omega + 2n\pi)|^2} \overline{\widehat{\phi_j}(\omega + 2n\pi)} \right) e^{ik\omega} d\omega \\ &= \int_0^{2\pi} \left(\frac{\sum\limits_{n\in Z} \widehat{\phi}_i(\omega + 2n\pi)\overline{\widehat{\phi_j}(\omega + 2n\pi)}}{\sum\limits_{n\in Z} |\widehat{\phi}_i(\omega + 2n\pi)|^2} \right) e^{ik\omega} d\omega. \end{aligned}$$

If $i \ne j$, by $\left|E_{\phi_i} \cap E_{\phi_j}\right| = 0$, it follows

$$\int_0^{2\pi} \sum_{n\in Z} \widehat{\phi}_i(\omega + 2n\pi)\overline{\widehat{\phi_j}(\omega + 2n\pi)} = 0.$$

Hence, we can obtain

$$\int_0^{2\pi} \left(\frac{\sum_{n\in Z} \widehat{\phi}_i(\omega + 2n\pi)\overline{\widehat{\phi_j}(\omega + 2n\pi)}}{\sum_{n\in Z} |\widehat{\phi}_i(\omega + 2n\pi)|^2} \right) e^{ik\omega} d\omega = 0.$$

If $i = j$, then

$$\int_0^{2\pi} \left(\frac{\sum_{n\in Z} \widehat{\phi}_i(\omega + 2n\pi)\overline{\widehat{\phi_j}(\omega + 2n\pi)}}{\sum_{n\in Z} |\widehat{\phi}_i(\omega + 2n\pi)|^2} \right) e^{ik\omega} d\omega = \int_0^{2\pi} \chi_{E_{\phi_i}\cap[0,2\pi]}(\omega)\overline{e^{-ik\omega}} d\omega,$$

which is the $-k$-th Fourier coefficient for the function $\chi_{E_{\phi_i}\cap[0,2\pi]}$ in $L^2(0, 2\pi)$.

From above results, obviously, the equation

$$\sum_{k\in Z} \langle \widehat{\widetilde{\phi}}_i, e^{-ikx}\widehat{\phi_j} \rangle e^{-ikx} = \chi_{E_{\phi_i}\cap[0,2\pi]}$$

holds on $[0, 2\pi]$. Since $\chi_{E_{\phi_i}}$ is 2π-periodic, it follows that

$$\sum_{k\in Z} \langle \widehat{\widetilde{\phi}}_i, e^{-ikx}\widehat{\phi_j} \rangle e^{-ikx} = \chi_{E_{\phi_i}} \ \textit{on } R.$$

Noting that $\chi_{E_{\phi_i}} \neq 0$ if $\widehat{\phi}_i \neq 0$, (2) now implies that $\widehat{S\widetilde{\phi}_i} = \chi_{E_{\phi_i}}\widehat{\phi}_i = \widehat{\phi}_i$. Hence we have $S\widetilde{\phi}_i = \phi_i$. Again by the definition of operator S on V_0, the proof is completed. □

4. Multiwavelet sampling theorem for frame

The main purpose of this section is to study the sampling theorem for frame in multiwavelet subspace. At first, we will prove a lemma.

Lemma 2. Let $\{\phi_i(\cdot - k) : 1 \le i \le r, k \in Z\}$ *be a frame for* V_0. *If* $|E_{\phi_i} \cap E_{\phi_j}| = 0, \forall i, j \in \{1, 2, \cdots, r\}$, *then* $\{\phi_i(\cdot - k)\}_{k\in Z}$ *is the frame for the subspace* $\langle \phi_i \rangle$.

Proof. Suppose that the function set F_i is defined by $F_i = \{\widehat{f_i} = \widehat{f}\chi_{E_{\phi_i}} : f \in V_0\}$. Let $\{f_i^l : l \in Z\}$ be the Cauchy sequence in F_i, then there must exists a function $f \in V_0$ such that $lim_{l\to\infty} f_i^l = f$.

By the definition of set F_i, for $\forall f_i^l \in F_i$ and $\forall g \in (V_0 \backslash F_i)$, we have $\langle f_i^l, g\rangle = 0$. So, it is easy to see

$$\langle f, g\rangle = \langle lim_{l\to\infty} f_i^l, g\rangle = 0.$$

Again by $V_0 = F_i \cup (V_0 \backslash F_i)$, it follows $f \in F_i$. Hence, F_i is a closed set.

Suppose that $\{\phi_i(\cdot - k) : 1 \le i \le r, k \in Z\}$ is a frame for the subspace V_0, then there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i=1}^{r}\sum_{k\in Z} |\langle f, \phi_i(x-k)\rangle|^2 \le B\|f\|^2.$$

Clearly, if $f \in F_i$, then $\langle f, \phi_j(x-k)\rangle = 0 (j \ne i)$. Hence, we have

$$A\|f\|^2 \le \sum_{k\in Z} |\langle f, \phi_i(x-k)\rangle|^2 \le B\|f\|^2, \forall f \in F_i.$$

By the definition of Frame and $\phi_i \in F_i$, $\{\phi_i(\cdot - k)\}_{k\in Z}$ is a frame for the closed subspace F_i. Then we get the desired result. □

Now, we introduce the main results in this section.

Theorem 3. *Let $\{\phi_i(\cdot - k) : 1 \le i \le r, k \in Z\}$ be a frame for V_0 with bounds A and B. Suppose that $\phi_i (1 \le i \le r)$ are continuous functions, $|E_{\phi_i} \cap E_{\phi_j}| = 0, i \ne j$, and $sup_{t\in R} \sum_{i=1}^{r}\sum_{k\in Z} |\phi_i(x-k)|^2 < +\infty$. Then the following two assertions are equivalent:*

(a) There exists a frame $\{s_i(\cdot - k) : 1 \le i \le r, k \in Z\}$ for V_0 satisfying $|(E_{s_i}\backslash E_{\phi_i}) \cup (E_{s_j}\backslash E_{\phi_j})| = 0, i \ne j$ such that

$$f(x) = \sum_{n\in Z}\sum_{m=1}^{r} f(n) s_m(x-n), \ \forall f \in V_0,$$

where the convergence is in $L^2(R)$.

(b) For all $i \in \{1, 2, \cdots, r\}$, there exist constants A_i and B_i, $0 < A_i \le B_i < +\infty$ such that

$$A_i \chi_{E_{\phi_i}} \le \left|\widehat{\phi_i^*}(\omega)\right| \le B_i \chi_{E_{\phi_i}}, \ a.e\ \omega \in R,$$

where $\widehat{\phi_i^*}(\omega) = \sum_{n\in Z} \phi_i(n) e^{-in\omega}$

Proof. Assume that (a) holds. From Lemma 2, clearly, $\{\phi_i(\cdot - k)\}_{k\in Z}$ is a frame for $\langle \phi_i\rangle$. Again by Lemma 2, $\{s_i(\cdot - k)\}_{k\in Z}$ is a frame for $\langle s_i\rangle$ too.

Because of $|(E_{s_i}\backslash E_{\phi_i}) \cup (E_{s_j}\backslash E_{\phi_j})| = 0, i \ne j$, we have $E_{s_i} \subseteq E_{\phi_i}$. Noticing that $|E_{\phi_i} \cap E_{\phi_j}| = 0 (i \ne j)$, then

$$s_i = \sum_{n\in Z} \langle s_i, \widetilde{\phi_i}\rangle \phi_i(x-n)$$

holds, where the function $\widetilde{\phi_i}$ is the dual frame for $\{\phi_i(\cdot - k) : 1 \le i \le r, k \in Z\}$. Thus, $s_i \in \langle \phi_i\rangle$.

From above results, according to Theorem 1 and Theorem 3 in [14], we get $\langle \phi_i\rangle = \langle s_i\rangle$. So, by Theorem 1 in [11], there exist constants A_i and B_i, $0 < A_i \le B_i < +\infty$ such that

$$A_i \chi_{E_{\phi_i}}(\omega) \le |\widehat{\phi_i^*}(\omega)| \le B_i \chi_{E_{\phi_i}}(\omega), \ a.e.\ \omega \in R. \tag{3}$$

Assume that (b) holds. From the argument of Theorem 1 in [11], we have $\widehat{\phi_i^*}(\omega) \neq 0 (\forall \omega \in E_{\phi_i})$. By the definition of $\widehat{\phi_i^*}(\omega)$ and E_{ϕ_i}, obviously,

$$(\frac{1}{\widehat{\phi_i^*}(\omega)})\chi_{E_{\phi_i}}(\omega) \in L^2[0, 2\pi].$$

Hence, there exists a sequence $\{c_k\}_{k\in Z} \in l^2$ such that

$$(\frac{1}{\widehat{\phi_i^*}(\omega)})\chi_{E_{\phi_i}}(\omega) = \sum_{k\in Z} c_k e^{-ik\omega}.$$

Let

$$\widehat{s}_i(\omega) = \begin{cases} \frac{\widehat{\phi}_i(\omega)}{\widehat{\phi_i^*}(\omega)}, & \omega \in E_{\phi_i} \\ 0, & \omega \notin E_{\phi_i}, \end{cases}$$

then

$$\sum_{n\in Z} |\widehat{s}_i(\omega + 2n\pi)|^2 = \frac{\sum_{n\in Z} \left|\widehat{\phi}_i(\omega + 2n\pi)\right|^2}{\left|\widehat{\phi_i^*}(\omega)\right|^2}. \tag{4}$$

By (3) and (4), notice that $\{\phi_i(\cdot - k)\}_{k\in Z}$ is a frame for the subspace $\langle\phi_i\rangle$, from [12, Proposition 3.1], there exist constants $0 < C_i \leq D_i < \infty$ such that

$$\begin{aligned} \frac{C_i}{B_i^2}\chi_{E_{s_i}}(\omega) &\leq \sum_{n\in Z} |\widehat{s}_i(\omega + 2n\pi)|^2 \\ &= \frac{\sum_{n\in Z} \left|\widehat{\phi}_i(\omega+2n\pi)\right|^2}{\left|\widehat{\phi_i^*}(\omega)\right|^2} \\ &\leq \frac{D_i}{A_i^2}\chi_{E_{s_i}}(\omega). \end{aligned}$$

Again by [12, Proposition 3.1], $\{s_i(\cdot - k)\}_{k\in Z}$ is a frame for the shift-invariance subspace $\langle s_i\rangle$. Using the definition of function s_i, it is easy to see that $s_i \in \langle\phi_i\rangle$ and $\phi_i \in \langle s_i\rangle$. So $\{s_i(\cdot - k)\}_{k\in Z}$ is a frame for the shift-invariance subspace $\langle\phi_i\rangle$ too. From [11, Theorem 1], we have $E_{s_i} = E_{\phi_i}$. By $\left|E_{\phi_i} \cap E_{\phi_j}\right| = 0, i \neq j$, then $|(E_{s_i} \backslash E_{\phi_i}) \cup (E_{s_j} \backslash E_{\phi_j})| = 0$ and

$$\widehat{s}_i(\omega) = (\sum_{j=1}^{r} \widehat{s}_j(\omega))\chi_{E_{\phi_i}}.$$

Because of $\{\phi_i(-k) : 1 \leq i \leq r, k \in Z\}$ being a frame of the subspace V_0, it follows that for $\forall f \in V_0$ there exist sequences $\{a_k^i\}_{k\in Z} \in l^2(1 \leq i \leq r)$ such that

$$f(x) = \sum_{i=1}^{r} \sum_{k\in Z} a_k^i \phi_i(x - k).$$

Let $f^i(x) = \sum_{k\in Z} a_k^i \phi_i(x - k)$, by $\left|E_{\phi_i} \cap E_{\phi_j}\right| = 0, i \neq j$, then $\langle\phi_i, \phi_j\rangle = 0$. Hence $\langle f^i, f^j\rangle = 0, i \neq j$.

Suppose that $V_0^i = \overline{span}\{\phi_i(x-k)\}_{k\in Z}$, from above results, clearly, $V_0 = \sum_{i=1}^{r} \oplus V_0^i$. Therefore, for all $f \in V_0$, we have $f(x) = \sum_{i=1}^{r} f^i(x)$. Hence,

$$\begin{aligned}
\widehat{f}(\omega) &= \sum_{i=1}^{r} \widehat{\phi}_i(\omega) \sum_{k\in Z} a_k^i e^{-ik\omega} \\
&= \sum_{i=1}^{r} \sum_{k\in Z} a_k^i e^{-ik\omega} \chi_{E_{\phi_i}} \widehat{\phi_i^*}(\omega) \widehat{s}_i(\omega) \\
&= \sum_{i=1}^{r} \sum_{k\in Z} a_k^i e^{-ik\omega} \chi_{E_{\phi_i}} \widehat{\phi_i^*}(\omega) (\sum_{j=1}^{r} \widehat{s}_j(\omega)) \chi_{E_{\phi_i}} \\
&= \sum_{j=1}^{r} \sum_{i=1}^{r} \sum_{k\in Z} a_k^i e^{-ik\omega} \widehat{\phi_i^*}(\omega) \chi_{E_{\phi_i}} \widehat{s}_j(\omega) \\
&= \sum_{j=1}^{r} \sum_{i=1}^{r} \sum_{k\in Z} a_k^i \sum_{n\in Z} \phi_i(n) e^{-i(n+k)\omega} \widehat{s}_j(\omega) \\
&= \sum_{j=1}^{r} \sum_{n'\in Z} \sum_{i=1}^{r} \sum_{k\in Z} a_k^i \phi_i(n'-k) e^{-i(n')\omega} \widehat{s}_j(\omega) \\
&= \sum_{j=1}^{r} \sum_{n'\in Z} f(n') e^{-i(n')\omega} \widehat{s}_j(\omega)
\end{aligned}$$

holds. Taking the Fourier inverse transform on both sides of above equation, we can obtain

$$f(x) = \sum_{j=1}^{r} \sum_{n\in Z} f(n) s_i(x-n).$$

This completes the proof. □

From the argument of Theorem 3, we can get a useful corollary:

Corollary 4. *Let $\{\phi_i(\cdot - k) : 1 \le i \le r, k \in Z\}$ be a frame for V_0 with bounds A and B. Suppose that $\phi_i (1 \le i \le r)$ are continuous functions, $|E_{\phi_i} \cap E_{\phi_j}| = 0, i \ne j$, and $sup_{t\in R} \sum_{i=1}^{r} \sum_{k\in Z} |\phi_i(x-k)|^2 < +\infty$. For all $i \in \{1, 2, \cdots, r\}$, if there exist constants A_i and B_i, $0 < A_i \le B_i < +\infty$ such that*

$$A_i \chi_{E_{\phi_i}} \le \left| \widehat{\phi_i^*}(\omega) \right| \le B_i \chi_{E_{\phi_i}},$$

then the condition (a) in Theorem 3 holds. In this case,

$$\widehat{s}_i(\omega) = \begin{cases} \frac{\widehat{\phi}_i(\omega)}{\widehat{\phi_i^*}(\omega)}, & \omega \in E_{\phi_i} \\ 0, & \omega \ne E_{\phi_i} \end{cases}, 1 \le i \le r,$$

where $\widehat{\phi_i^*}(\omega) = \sum_{n\in Z} \phi_i(n) e^{-in\omega}$.

In the following, we give another main result in this section.

Theorem 5. *Let $\{\phi_i(\cdot - k) : 1 \leq i \leq r, k \in Z\}$ be a frame for V_0 with bounds A and B. Assume that $a_m \in [0,1)$ are constants, $a_i \neq a_j$ for $i \neq j, i, j = 1, 2, \ldots, r$, $\phi_i(1 \leq i \leq r)$ are continuous and $sup_{x\in R} \sum_{i=1}^{r} \sum_{k\in Z} |\phi_i(x-k)|^2 < +\infty$. Then, there exists a frame $\{s_m(\cdot - n) : 1 \leq m \leq r, n \in Z\}$ for V_0 such that*

$$f(x) = \sum_{n\in Z} \sum_{m=1}^{r} f(n + a_m) s_m(x - n), \ \forall f \in V_0$$

holds, where the convergence is both in $L^2(R)$ and uniform on R if there exists a bounded invertible matrix $P_\Phi(\omega)$ for $0 \leq \omega \leq 2\pi$ such that $\check{P}_\Phi^{-1}(\omega)\Phi(\omega) \subset V_0$, where

$$P_\Phi(\omega) = [Z_{\phi_1}(\omega), Z_{\phi_2}(\omega), \cdots, Z_{\phi_r}(\omega)],$$

$$Z_{\phi_l}(\omega) = [\sum_{n\in Z} \phi_l(n + a_1)e^{-in\omega}, \sum_{n\in Z} \phi_l(n + a_2)e^{-in\omega}, \cdots, \sum_{n\in Z} \phi_l(n + a_r)e^{-in\omega}]^T.$$

Proof. Suppose that there exists a bounded invertible matrix $P_\Phi(\omega)$ for $0 \leq \omega \leq 2\pi$ such that $\check{P}_\Phi^{-1}(\omega)\Phi(\omega) \in V_0$ holds, where $P_\Phi^{-1}(\omega)$ is defined by

$$P^{-1}(\omega) = [Z'_{\phi_1}(\omega), \ Z'_{\phi_2}(\omega), \cdots, Z'_{\phi_r}(\omega)],$$

$$Z'_{\phi_l}(\omega) = [\sum_{n\in Z} c^l_{n,1} e^{-in\omega}, \sum_{n\in Z} c^l_{n,2} e^{-in\omega}, \cdots, \sum_{n\in Z} c^l_{n,r} e^{-in\omega}]^T.$$

Let

$$\widehat{S}(\omega) = [\widehat{s}_1(\omega), \widehat{s}_2(\omega), \cdots, \widehat{s}_r(\omega)]^T = P_\Phi^{-1}(\omega)\widehat{\Phi}(\omega), \tag{7}$$

it is easy to know that

$$[\widehat{s}_1(\omega), \cdots, \widehat{s}_r(\omega)]^T = [\sum_{n\in Z} \sum_{l=1}^{r} c^l_{n,1} e^{-in\omega} \widehat{\phi}_l(\omega), \cdots, \sum_{n\in Z} \sum_{l=1}^{r} c^l_{n,r} e^{-in\omega} \widehat{\phi}_l(\omega)]^T.$$

Therefore,

$$\widehat{s}_j(\omega) = \sum_{n\in Z} \sum_{l=1}^{r} c^l_{n,j} e^{-in\omega} \widehat{\phi}_l(\omega), 1 \leq j \leq r.$$

Taking the Fourier inverse transform on both sides of this equation, we obtain

$$s_j(x) = \sum_{n\in Z} \sum_{l=1}^{r} c^l_{n,j} \phi_l(x - n), 1 \leq j \leq r.$$

Since $\{c^l_{n,j}\}_{n\in Z} \in l^2(1 \leq l, j \leq r)$, then there exist constants $0 < C^l_j(1 \leq l, j \leq r)$ such that

$$\sum_{n\in Z} \left|c^l_{n,j}\right|^2 \leq C^l_j, 1 \leq l, j \leq r.$$

From above results, by [11, Lemma 2], we have

$$\begin{aligned}\sum_{k\in Z}|s_j(x+k)|^2 &= \sum_{k\in Z}\left|\sum_{n\in Z}\sum_{l=1}^{r}c^l_{n,j}\phi_l(x+k-n)\right|^2\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{n\in Z}\sum_{l=1}^{r}c^l_{n,j}e^{-i(rn+l)\omega}\right|^2\\ &\quad \left|\sum_{k\in Z}\sum_{l=1}^{r}\phi_l(x+k)e^{-i(rk+l)\omega}\right|^2 d\omega\\ &\le (\sum_{n\in Z}\sum_{l=1}^{r}|c^l_{n,j}|^2)(\sum_{k\in Z}\sum_{l=1}^{r}|\phi_l(x+k)|^2)\\ &\le Max\{C^l_j : 1\le l\le r\}(\sum_{k\in Z}\sum_{l=1}^{r}|\phi_l(x+k)|^2),\end{aligned}$$

then, we get

$$\begin{aligned}\sum_{j=1}^{r}\sum_{k\in Z}|s_j(x+k)|^2 &= \sum_{j=1}^{r}\sum_{k\in Z}\left|\sum_{n\in Z}\sum_{l=1}^{r}c^l_{n,j}\phi_l(x+k-n)\right|^2\\ &\le Max\{C^l_j : 1\le l,j\le r\}\}(\sum_{k\in Z}\sum_{l=1}^{r}|\phi_l(x+k)|^2).\end{aligned}\tag{8}$$

By (7), it is easy to check that

$$\phi_l(x)=\sum_{m=1}^{r}\sum_{k\in Z}\phi_l(a_m+k)s_m(x-k), 1\le l\le r.$$

Let

$$\widetilde{V_0}=\{g(x)\in V_0 : g(x)=\sum_{l=1}^{r}\sum_{k\in Z}g(a_m+k)s_m(x-k), 1\le m\le r\},$$

clearly, $\phi_l(x)\in\widetilde{V_0}$, $\widetilde{V_0}$ is a linear space and for any $g\in\widetilde{V_0}$, $n\in Z$, $g(\cdot-n)\in\widetilde{V_0}$ holds.

In the following, we will prove $\widetilde{V_0}=V_0$. For all $f\in V_0$, since $\{\phi_i(\cdot-k) : 1\le i\le r, k\in Z\}$ is a frame for V_0, then we have $f(x)=\sum_{l=1}^{r}\sum_{k\in Z}c^l_k\phi_l(x-k)$, where $\{c^l_k\}_{k\in Z}\in l^2(1\le l\le r)$.
Define

$$f_n(x)=\sum_{l=1}^{r}\sum_{k=-n}^{n}c^l_k\phi_l(x-k),$$

Notice that

$$\sum_{l=1}^{r}\sum_{k\in Z}|c^l_k|^2<\infty.$$

Then for $\forall\delta > 0$, there exists a constant number $a_1 > 0 (a_1 \in Z)$ such that $\forall b_1 > a_1, b_1 \in Z$,

$$\sum_{l=1}^{r}\sum_{k=a_1}^{b_1}|c_k^l|^2 + \sum_{l=1}^{r}\sum_{k=-b_1}^{-a_1}|c_k^l|^2 < \delta^{\frac{1}{2}}.$$

By

$$sup_{x\in R}\sum_{l=1}^{r}\sum_{k\in Z}|\phi_l(x-k)|^2 < \infty,$$

similar to the above argument, there exists a number $a_2 > 0 (a_2 \in Z)$ such that $\forall b_2 > a_2, b_2 \in Z$,

$$sup_{x\in R}\sum_{l=1}^{r}\sum_{k=a_2}^{b_2}|\phi_l(x-k)|^2 + sup_{x\in R}\sum_{l=1}^{r}\sum_{k=-b_2}^{-a_2}|\phi_l(x-k)|^2 < \delta^{\frac{1}{2}}.$$

Then, for above $\delta > 0$, there exists a number $n = Max\{a_1, a_2\}$ such that for all $m > n, m \in Z$,

$$\begin{aligned}
|f_m - f_n|^2 &= |\sum_{l=1}^{r}\sum_{k=n}^{m} c_k^l\phi_l(x-k) + \sum_{l=1}^{r}\sum_{k=-m}^{-n} c_k^l\phi_l(x-k)|^2 \\
&\le (\sum_{l=1}^{r}\sum_{k=n}^{m}|c_k^l|^2 + \sum_{l=1}^{r}\sum_{k=-m}^{-n}|c_k^l|^2) \\
&\quad (\sum_{l=1}^{r}\sum_{k=n}^{m}|\phi_l(x-k)|^2 + \sum_{l=1}^{r}\sum_{k=-m}^{-n}|\phi_l(x-k)|^2) \\
&\le (sup_{x\in R}\sum_{l=1}^{r}\sum_{k=n}^{m}|\phi_l(x-k)|^2 + sup_{x\in R}\sum_{l=1}^{r}\sum_{k=-m}^{-n}|\phi_l(x-k)|^2) \\
&\quad (\sum_{l=1}^{r}\sum_{k=n}^{m}|c_k^l|^2 + \sum_{l=1}^{r}\sum_{k=-m}^{-n}|c_k^l|^2) \\
&\le \delta.
\end{aligned}$$

By the definition of function f_n, clearly, $\lim\limits_{n\to\infty} f_n = f$. From above results, it is easy to check that the functions $\{f_n\}_{n\in Z}$ converges uniformly to f on R. So, for $\forall x \in R$, we have $\lim\limits_{n\to\infty} f_n(x) = f(x)$.

Again by the definition of function f_n, we have $f_n \in \widetilde{V}_0$. Hence

$$f_n(x) = \sum_{m=1}^{r}\sum_{k\in Z} f_n(a_m + k)s_m(x-k).$$

Notice that $\forall x \in R$, $\lim_{n\to\infty} f_n(x) = f(x)$, then

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) \\ &= \lim_{n\to\infty} \sum_{m=1}^{r} \sum_{k\in Z} f_n(a_m+k)s_m(x-k) \\ &= \sum_{m=1}^{r} \sum_{k\in Z} \lim_{n\to\infty} f_n(a_m+k)s_m(x-k) \\ &= \sum_{m=1}^{r} \sum_{k\in Z} f(a_m+k)s_m(x-k). \end{aligned}$$

Thus, $f \in \widetilde{V}_0$. By (8), notice that the function f is continuous, then

$$\sum_{m=1}^{r} \sum_{k\in Z} f(a_m+k)s_m(x-k)$$

converges uniformly to $f(x)$ on R . □

5. Some examples

In the following, we present some examples to show the reconstruction using Theorem 5, and we analyze and compare them with other results in the literature. In the first examples, we reconstruct a typical multiwavelets using the result of Theorem 5. The second example gives a case in which a signal cannot be reconstructed according to our sampling theorem.

Example 6. Geronimo Hardin Masopust(GHM)'s Multiwavelet[6]:

$$H_0 = \begin{pmatrix} \frac{3}{5} & \frac{4\sqrt{2}}{5} \\ -\frac{\sqrt{2}}{20} & -\frac{3}{10} \end{pmatrix}, \qquad H_1 = \begin{pmatrix} \frac{3}{5} & 0 \\ \frac{9\sqrt{2}}{20} & 1 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 0 & 0 \\ \frac{9\sqrt{2}}{20} & -\frac{3}{10} \end{pmatrix}, \qquad H_3 = \begin{pmatrix} 0 & 0 \\ -\frac{\sqrt{2}}{20} & 0 \end{pmatrix}.$$

It is easy to check that GHM's multiwavelet has orthogonality, compactness, symmetry, and approximation order of 3, but it does not have the general cardinal property. It cannot reconstruct a signal according to Selenick's sampling theorem when $a_1 = 0, a_2 = \frac{1}{2}$, but our sampling theorem holds. We have

$$P_\Phi(\omega) = \begin{pmatrix} 0 & \frac{4\sqrt{2}}{5} \\ e^{-i\omega} & -\frac{3}{10} - \frac{3}{10}e^{-i\omega} \end{pmatrix}.$$

Since

$$P_\Phi^{-1}(\omega) = \begin{pmatrix} \frac{3\sqrt{2}}{16}e^{i\omega} + \frac{3\sqrt{2}}{16} & e^{i\omega} \\ \frac{5}{4\sqrt{2}} & 0 \end{pmatrix},$$

by theorem 5, $[\widehat{s}_1, \widehat{s}_2, \cdots, \widehat{s}_r]^T = P_\Phi^{-1}(\omega)\widehat{\Phi}$, then

$$\widehat{s}_1 = \frac{3\sqrt{2}}{16}e^{i\omega}\widehat{\phi}_1 + \frac{3\sqrt{2}}{16}\widehat{\phi}_1 + e^{i\omega}\phi_2, \qquad \widehat{s}_2 = \frac{5}{4\sqrt{2}}\widehat{\phi}_1.$$

Taking the Fourier invertible transform, we have

$$s_1(x) = \frac{3\sqrt{2}}{16}\phi_1(x) + \frac{3\sqrt{2}}{16}\phi_1(x) + \phi_2(x+1),$$

$$s_2(x) = \frac{5\sqrt{2}}{8}\phi_1(x).$$

So, we can obtain

$$\begin{aligned} f(x) &= \textstyle\sum_{n\in Z} f(n)\left[\frac{3\sqrt{2}}{16}\phi_1(x-n) + \frac{3\sqrt{2}}{16}\phi_1(x+1-n) + \phi_2(x+1-n)\right] \\ &\quad + \textstyle\sum_{n\in Z} f(n+\frac{1}{2})\frac{5\sqrt{2}}{8}\phi_1(x-n). \end{aligned}$$

Example 7 Chui and Lian's Multiwavelet[15]:

$$H_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{7}}{4} & -\frac{\sqrt{7}}{4} \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

$$H_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{7}}{4} & -\frac{\sqrt{7}}{4} \end{pmatrix}.$$

This multiwavelet has orthogonality, compact support, and symmetry/antisymmetry but does not have the general cardinal property. It cannot reconstruct a signal according to Selenick's sampling theorem. When $a_1 = 0, a_2 = \frac{1}{4}$, we have

$$P_\Phi(\omega) = \begin{pmatrix} e^{-i\omega} & \frac{1}{4} - \frac{\sqrt{7}}{8} + (\frac{3}{4} + \frac{\sqrt{7}}{8})e^{-i\omega} \\ 0 & -\frac{\sqrt{7}}{8} + \frac{7}{16} + (\frac{\sqrt{7}}{4} + \frac{7}{16})e^{-i\omega} \end{pmatrix}.$$

Since

$$P_\Phi^{-1}(\omega) = \begin{pmatrix} e^{i\omega} & \frac{-(\frac{3}{4}+\frac{\sqrt{7}}{8})+(-\frac{1}{4}+\frac{\sqrt{7}}{8})e^{-i\omega}}{-\frac{\sqrt{7}}{8}+\frac{7}{16}+(\frac{\sqrt{7}}{4}+\frac{7}{16})e^{-i\omega}} \\ 0 & \frac{1}{-\frac{\sqrt{7}}{8}+\frac{7}{16}+(\frac{\sqrt{7}}{4}+\frac{7}{16})e^{-i\omega}} \end{pmatrix},$$

by theorem 5, $[\widehat{s}_1, \widehat{s}_2, \cdots, \widehat{s}_r]^T = P_\Phi^{-1}(\omega)\widehat{\Phi}$, then

$$\widehat{s}_1(\omega) = e^{i\omega}\widehat{\phi}_1(\omega) + \frac{-(\frac{3}{4}+\frac{\sqrt{7}}{8}) + (-\frac{1}{4}+\frac{\sqrt{7}}{8})e^{-i\omega}}{-\frac{\sqrt{7}}{8}+\frac{7}{16}+(\frac{\sqrt{7}}{4}+\frac{7}{16})e^{-i\omega}}\widehat{\phi}_2(\omega);$$

$$\widehat{s}_2(\omega) = \frac{1}{-\frac{\sqrt{7}}{8}+\frac{7}{16}+(\frac{\sqrt{7}}{4}+\frac{7}{16})e^{-i\omega}}\phi_2(\omega).$$

When $a_1 = 0, a_2 = \frac{1}{2}$, we have

$$P_\Phi(\omega) = \begin{pmatrix} e^{-i\omega} & \frac{1}{2} + \frac{1}{2}e^{-i\omega} \\ 0 & -\frac{\sqrt{7}}{4} + \frac{7}{4}e^{-i\omega} \end{pmatrix}.$$

Obviously, when $\omega = 0$,

$$P_\Phi(\omega) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

The matrix $P_\Phi(0)$ does not exist the invertible matrix $P_\Phi^{-1}(0)$, then it does not satisfy the condition of Theorem 5. Hence, the sampling function cannot be reconstructed for $a_1 = 0, a_2 = \frac{1}{2}$.

6. Conclusion

We study the sampling theorem for frame in multiwavelet subspaces. Multiscaling functions satisfying sampling theorem have orthogonality, regularity, short compact support, symmetry, and high approximation order. It is not possible in the scalar wavelet case. In our paper, for a frame in multiwavelet subspaces, we present its dual frame with explicit formula. Moreover, we give an equivalent condition for the sampling theorem to hold in multiwavelet subspaces, and establish a sufficient condition under which the sampling theorem holds.

References

1. G. G. Walter, *A sampling theorem for wavelet subspaces,* IEEE Trans. Inform. Theory, 38(2)(1992), pp. 881-884.
2. X. G. Xia and Z. Zhang , *On sampling theorem , wavelet and wavelet transforms,* IEEE Trans. Signal Processing, 41(12)1993, pp. 3524-3535.
3. A. J. E. M. Janssen, *The Zak transform and sampling theorems for wavelet subspaces* , IEEE Trans. Signal Processing , 41(12)1993, pp. 3360-3364.
4. Y. Liu, *Irregular sampling for spline wavelet subspaces,* IEEE Trans. Inform. Theory, 42(2)(1996), pp. 623-627.
5. W. Chen, S. Itoh, J. Shiki. *A sampling theorem for shift-invariant spaces*, IEEE Trans Signal Processing, 46(3)(1998), pp. 2802-2810.
6. J. S. Geronimo, D. P. Hardin, and P. R. Massopust, *Fractal functions and wavelet expansions based on several scaling functions*, J. Approx. Theory, 78(3)(1994), pp. 373-401.
7. I. W. Selesnick, *Interpolating multiwavelet bases and the sampling theorem,* IEEE Trans. Signal Processing, 47(6)(1999), pp. 1615-1621.
8. W. C. Sun and X. W. Zhou, *Sampling theorem for multiwavelet subspaces*, Chin. Sci. Bull., 44(14)(1999), pp. 1283-1286.
9. D. Zhou, *Interpolatory orthogonal multiwavelets and refinable functions*, IEEE Trans. Signal Processing, 50(3)(2002), 520-527, 2002.
10. C. Zhao, P. Zhao *Sampling Theorem and Irregular Sampling Theorem for Multiwavelet subspaces*, IEEE Trans Signal Processing, 53(3)(2005), pp. 705-713.
11. X. W. Zhou, W. C. Sun, *On the Sampling Theorem for Wavelet Subspaces,* J. Fourier. Anal. Appl, 5(4)(1999), pp. 347-354.
12. X. W. Zhou, W. C. Sun, *Sampling Theorem for Wavelet Subspaces: Error Estimate and Irregular Sampling,* IEEE Trans Signal Processing, 48(1)(2000), pp. 223-226.
13. O. Christensen, *An Introduction to Frames and Riesz bases,* Birkhäuser, Boston, MA, 2003.
14. C. Blanco, C. Cabrelli and S. Heineken, *Functions in Sampling Spaces,* Sampling Theory In Signal and Image Processing, 5(3)2006, pp. 275-295.

15. C. K. Chui and J. Lian, *A study of orthonormal multi-wavelets,* Appl. Numer. Math., 20(3)1996, pp. 273-298.

Zhanwei Liu received his Ph.D at University of Information Engineering under the direction of Prof. Guo-en Hu. Since 2005 he has been at the Zhenzhou University as a lecturer. His research interests focus on wavelet theory and Sampling theorem.

School of Information Engineering, Zhengzhou University, Zhengzhou 450001 P.R. China
e-mail: changgengliu@163.com

Guochang Wu received his Ph.D at Xi'an Jiaotong University under the direction of Prof. Zheng-xing Cheng. Since 2008 he has been at the Henan University of Finance and Economics as a lecturer. His research interests focus on wavelet theory and Sampling theorem.

College of Information, Henan University of Finance and Economics, Zhengzhou, 450002 P.R. China
e-mail: archang-0111@163.com

Xiaohui Yang received her Ph.D at Xidian University under the direction of Prof. Licheng Jiao. Since 2007 she has been at the Henan University as a lecturer. Her research interests focus on Image processing and Sampling theorem.

Institute of Applied Mathematics, School of Mathematics and Information Sciences, Henan University, Kaifeng, 475004 P.R. China
e-mail: xhyang_lc@163.com