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SINGULAR THIRD-ORDER 3-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, we prove existence of infinitely many positive and concave solutions , by means of a simple approach, to 3^{th} order three-point singular boundary value problem

$$\begin{cases} x'''(t) = \alpha(t) f(t, x(t)), & 0 < t < 1, \\ x(0) = x'(\eta) = x''(1) = 0, & (1/2 < \eta < 1). \end{cases}$$

Moreover with respect to multiplicity of solutions, we don't assume any monotonicity on the nonlinearity.

We rely on a combination of the analysis of the corresponding vector field on the phase-space along with Knesser's type properties of the solutions funnel and the well-known Krasnosel'skiĭ's fixed point theorem. *The later is applied on a new very simple cone K, just on the plane R^2.* These extensions justify the efficiency of our new approach compared to the commonly used one, where the cone $K \subset C([0,1], \mathbb{R})$ and the existence of a *positive* Green's function is a necessity.

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1. Introduction

Ma in [17] and latter Webb [21], Kaufmann [12] and Kaufmann and Raffoul [13] proved the existence of a positive solution to the three-point nonlinear boundary-value problem

$$-u''(t) = q(t)f(u(t)), \quad 0 < t < 1,$$
$$u(0) = 0, \quad \alpha u(\eta) = u(1),$$

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where $\alpha > 0$, $0 < \eta < 1$ and $\alpha\eta < 1$. Liu [16] applied a fixed point index method to obtain an interval for the parameter λ, where existence results for

$$-u''(t) = \lambda q(t)\alpha(t) f(u(t)), \quad 0 < t < 1,$$
$$u'(0) = 0, \quad \beta u(\eta) = u(1)$$

are guaranteed.

In the above papers there are no assumptions for singularity of the nonlilearity f at the point $u = 0$. Zhang and Wang [24] and recently Liu [15] obtained some existence results for a singular nonlinear second order 3-point boundary-value problem, where singularity only of $q(t)$ at $t = 0$ or $t = 1$ is permitted. Also recently, using the method of fixed point index, Xu [22] studied the problem

$$-u''(t) = f(u(t)), \quad 0 < t < 1, \quad u(0) = 0, \quad \alpha u(\eta) = u(1),$$

where $f(t, u)$ is allowed to have singularity at $u = 0$. Other applications of Krasnosel'skiĭ's fixed point theorem to semipositone problems can, for example, be found in [2]. Further recently interesting results have been proved in [11], [21] or [15].

Anderson [4] and Anderson and Avery [3], proved that there exist at least three positive solutions to the BVP (1) (below) and the analogous discrete one respectively, by using the Leggett-Williams fixed point theorem. In addition Anderson and Davis [5], Yao in [23] and Haiyan and Liu in [10], using the Krasnosel'skiĭ's fixed point theorem or its extensions showed the existence of multiple solutions to the BVP (1). More similar results can be found in Du et all in [7] and also in Feng and Webb work in [8].

Recently, Du et all [6] via the coincidence degree of Mawhin, proved existence for the BVP

$$\begin{cases} x'''(t) = f(t, x(t), x'(t), x''(t)), \quad 0 < t < 1, \\ x(0) = \alpha x(\xi), \; x''(0) = 0, \; x'(1) = \sum_{j=1}^{m-2} \beta_j x'(\eta_j), \end{cases}$$

at the resonance case. Also in an recent paper, Yongping Sun [20] obtained existence of infinitely many positive solutions to the BVP

$$\begin{cases} u'''(t) = \lambda\alpha(t) f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(\eta) = u''(1) = 0, \; \eta \in (1/2, 1) \end{cases} \tag{1}$$

mainly under sub or superlinearity on the nonlinearity f of the type

$$\begin{cases} \text{There exist positive constants } \theta, \; R \neq r \text{ such that} \\ f(t, x) \leq \frac{r}{\lambda A}, \quad \forall (t, x) \in [0, 1] \times [0, r]; \\ f(t, x) \geq \frac{R}{\lambda B}, \quad \forall (t, x) \in [0, 1] \times [\theta R, R], \\ \text{where } A \text{ and } B \text{ are constants.} \end{cases}$$

Sun, in order to obtain his existence results applied the classical Krasnosel'skii fixed-point theorem on cone expansion-compression type. Furthermore Sun, in order to prove his multiplicity results, assumed monotonicity of the nonlinearity with respect to the second variable.

In [18] and in several references therein, Palamides and Erbe obtained existence results of a monotone positive solution of the singular boundary value problem

$$\frac{1}{p(t)}(p(t)y'(t))' + \text{sign}(1-\alpha)q(t)f(t,y(t),p(t)y'(t)) = 0,$$

$$\begin{cases} \gamma y(0) \pm \delta \lim_{t\to 0+} p(t)y'(t) = 0, \\ \lim_{t\to 1-} p(t)y'(t) - \alpha \lim_{t\to 0+} p(t)y'(t) = 0. \end{cases}$$

Their approach was based on an analysis of the corresponding vector field on the phase-plane along with the shooting technique.

In this work, mainly motivated by the above mentioned papers, especially the ones of Sun [20] and Palamides and Erbe [18], we suppose a sublinearity-type growth rate on $f(t,u)$ at both the origin $u=0$ and $u=+\infty$. The emphasis in this paper is mainly to use as our basis the continuum properties (connectedness and compactness) of the solution funnel (Knesser's theorem), combined with the corresponding vector field's ones. Then the classical Krasnosel'skiĭ's fixed point theorem is applied just on the two-dimensional Euclidean space. This results in the use of quite similar natural assumptions. Furthermore, we eliminate at all the related monotonicity assumption on the nonlinearity in [20]. In this way, we prove existence of infinitely many positive solutions for the boundary value problem

$$\begin{cases} x'''(t) = \alpha(t)\, F(t,x(t)), \quad 0<t<1, \\ x(0) = x'(\eta) = x''(1) = 0. \end{cases} \tag{E}$$

In addition, we don't use the corresponding Green's function, *the needed positivity of which implies the usual restriction* $\eta \in (1/2, 1)$. This is clearly an advantage of our approach. For example, if the boundary conditions in (E) were

$$x(0) = x(1) = x''(\eta) = 0,$$

then the corresponding Green's function is not positive. This causes many difficulties to obtain a positive solution, via the commonly used approach (see for example [19]). Moreover the construction of the Green's kernel to another type of BVP may be difficult or even impossible.

2. Preliminaries

Consider the third-order nonlinear singular boundary value problem (E), where we assume within this paper that $\eta \in (1/2, 1)$ and the continuous functions $\alpha(t)$, $t \in (0,1)$ and $F \in C(\Omega, [0,+\infty))$ are nonnegative,where $\Omega = [0,1] \times [0,+\infty)$.

Then a vector field is defined, with crucial properties for our study. More precisely, considering the (x', x'') phase semi-plane $(x'>0)$, we easily check that $x''' = \alpha(t)\, F(t,x) \geq 0$. Thus, any trajectory $(x'(t), x''(t))$, $t \geq 0$, emanating from any point in the second quadrant:

$$\{(x', x'') : x' > 0,\ x'' < 0\},$$

evolves in a natural way, when $x'(t) > 0$, toward the negative $x''-$semi-axis and then, when $x'(t) \leq 0$ toward the negative $x'-$semi-axis. Thus, assuming a certain growth rate on f (e.g. a sublinearity), we can control the vector field in a way that assures the existence of a trajectory satisfying the given boundary conditions. These properties, which will be referred to as *"the nature of the vector field ", combined with the Krasnosel'skiĭ's theorem, are the main tools that we will employ in our study.*

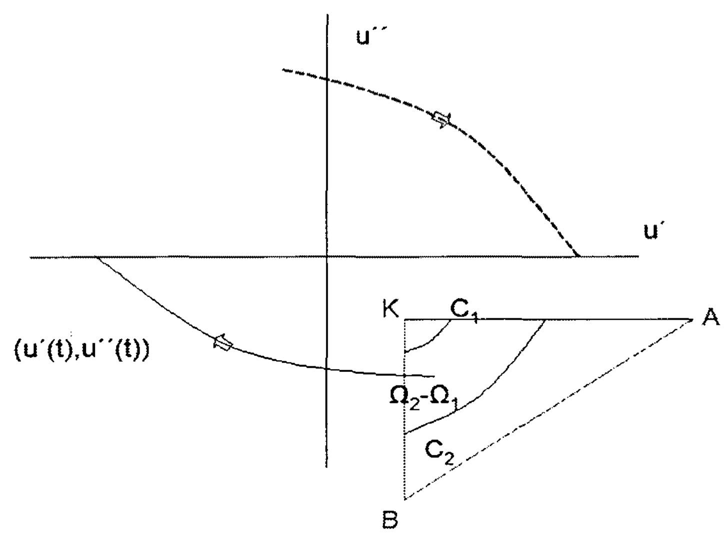

Fig. 1

More precisely we will apply the well known Krasnosel'skiĭ's fixed point theorem in cones (see [9]):

Lemma 1. *Let E be a Banach space and $K^* \subset E$ a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let*

$$T : K^* \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right) \to K^*$$

be a completely continuous operator. We assume furthermore

(A) $||Tu|| \leq ||u||,\ \forall u \in K^* \cap \partial\Omega_1$ *and* $||Tu|| \geq ||u||,\ \forall u \in K^* \cap \partial\Omega_2$ *or*

(B) $||Tu|| \geq ||u||,\ \forall u \in K^* \cap \partial\Omega_2$ *and* $||Tu|| \leq ||u||,\ \forall u \in K^* \cap \partial\Omega_1.$

Then T has a fixed point in $K^* \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$.

3. Existence Results.

Consider the third-order nonlinear singular three-point boundary value problem:

$$u''' = \alpha(t) f(t,u), \quad 0 < t < 1, \tag{2}$$

$$u(0) = u'(\eta) = u''(1) = 0. \tag{3}$$

where f is a continuous extension of F, i.e.

$$f(t,u) = \begin{cases} F(t,u), & u \geq 0 \\ F(t,0), & u < 0. \end{cases}$$

Considering the sign property of F, we conclude that

$$f(t,u) \geq 0, \quad (t,u) \in [0,1] \times \mathbb{R}.$$

Lemma 2. *Assume that a solution $u = u(t)$ of a BVP (2)-(3) satisfies the inequalities*

$$u'(t) > 0, \ 0 \leq t < \eta \text{ and } u''(t) < 0, \ 0 \leq t < 1.$$

Then

$$u(t) \geq 0, \ 0 \leq t \leq 1,$$

provided that $\eta \in \left(\frac{1}{2}, 1\right)$.

Proof. Suppose that there is a $T \in (\eta, 1)$ such that

$$u(t) > 0, \ t \in (0,T), \ u(T) = 0 \text{ and } u(t) < 0, \ t \in (T,1].$$

Consider, two symmetric, with respect to η, partitions

$$\{2\eta - T = r_0 < r_1 < ... < r_k = \eta\} \quad \text{and} \quad \{\eta = t_0 < t_1 < ... < t_k = T\}$$

of $[2\eta - T, \eta]$ and $[\eta, T]$ respectively, i.e.

$$r_k - r_{k-1} = t_1 - t_0, \ r_{k-1} - r_{k-2} = t_2 - t_1, ..., \ r_1 - r_0 = t_k - t_{k-1}.$$

Notice firth that $2\eta - T \geq 0$. By the concavity of $u = u(t)$ and since the map $u = u''(t)$, $t \in [0,1]$, is nondecreasing and negative,

$$u'(r_i) > -u'(t_{k-i}), \quad (i = 0, 1, ..., k-1).$$

Thus

$$-(t_{k-i+1} - t_{k-i})u'(t_{k-i}) < (r_{i+1} - r_i)u'(r_i), \quad (i = 1, 2, ..., k),$$

that is

$$-\sum_{i=1}^{k}(t_{k-i+1} - t_{k-i})u'(t_{k-i}) < \sum_{i=1}^{k}(r_{i+1} - r_i)u'(r_i). \tag{4}$$

In addition, because the map $u' = u'(t)$, $0 \leq r \leq T$ is continuous (and bounded), we can choose the $\max\{r_i - r_{i-1} : i = 1, 2, ..., k\}$ small enough, and provided that $2\eta - T \geq 0$, we obtain

$$\int_0^{\eta} u'(t)\, dt \geq \int_{2\eta - T}^{\eta} u'(t)\, dt > -\int_{\eta}^{T} u'(r)\, dr.$$

Consequently,

$$u(T) = \int_0^{\eta} u'(t)\,dt + \int_{\eta}^{T} u'(r)\,dr > 0,$$

a contradiction. ∎

Remark 1. : *The restriction $\eta \in \left(\frac{1}{2}, 1\right)$ is necessary for the validity of the above Lemma 2. The next counter example is due to the Referee: For $\eta = 1/3$ and $f(t,u) = 1$, the function $u(t) = \left(t^3/6\right) - \left(t^2/2\right) + (5t/18)$ is a solution of the BVP (2)-(3), which satisfies the assumptions of Lemma. But $u(1) = -1/18 < 0$.*

Lemma 3. *Let $u = u(t)$, $t \in [0,1]$ be a solution of the differential equation (2) such that*

$$u(0) = 0,\ u'(0) = u_0' > 0 \ \text{ and } \ u''(0) = u_0'' < 0. \tag{5}$$

Then

$$u(t) \geq 0, \quad t \in [0,1]$$

for any initial value (u_0', u_0'') with $u_0'' \geq -2u_0'$.

Proof. By Taylor's formula

$$u(t) = tu_0' + \frac{t^2}{2}u_0'' + \frac{t^3}{2}\int_0^1 (1-s)^2\,\alpha(st)\,f[st, u(ts)]ds, \ t \in [0,1].$$

and (5), we get $u(t) > 0$ for all t in a (right) neighborhood of $t = 0$. Assuming that there exists a $t^* \in (0,1)$ such that

$$u(t^*) = 0 \ \text{ and } \ u(t) \geq 0, \quad t \in [0, t^*],$$

and provided that $u_0'' \geq -2u_0'$, we get, noticing the sign of the nonlinearity

$$\frac{t^*}{2}\left(2u_0' + t^* u_0''\right) \leq 0 \quad \Leftrightarrow \quad t^* \geq -\frac{2u_0'}{u_0''} \geq 1,$$

a contradiction. ∎

Throughout this paper, we assume that $0 < \theta < 1/2$ and there exist positive constants r_0 and R_0 with

$$R_0 \leq r_0 \leq \frac{R_0}{\eta},$$

such that, for any

$$F(t,x) > \frac{R_0}{A_0}, \ (t,x) \in [0,1] \times [0, r_0] \tag{A_1}$$

and vice verstra there exists an $R_1^* \geq R_0$ such that

$$F(t,x) < \frac{r_0}{B_0}, \quad (t,x) \in [0,1] \times [\theta R_1^*, +\infty). \tag{A_2}$$

where

$$A_0 = \int_0^1 \alpha(s)\,ds > 0 \quad \text{and} \quad B_0 = \int_{\theta}^{1-\theta} \alpha(s)\,ds > 0.$$

Proposition 4. *For every initial value* (u_0', u_0''), *with*

$$u_0'' \leq -r_0 \frac{1+\eta^2}{\eta} < -r_0 \leq -u_0',$$

any solution $u = u(t)$ *of the initial value problem (2)-(5) satisfies*

$$u'(\eta) > 0, \quad u''(1) > 0 \quad \text{and} \quad u(t) \geq 0, \quad t \in [0,1].$$

Proof. We choose (without loss of generality)

$$u_0' = r_0 \quad \text{and} \quad u_0'' = -\frac{R_0}{\eta} \tag{6}$$

and assume that $u''(1) < 0$. By the sign property of f, it follows that $u'''(t) > 0$. Thus, the function $u''(t)$, $t \in [0,1]$, is increasing. Hence, by the Mean Value Theorem,

$$u''(t) = u_0'' + t^* \int_0^t \alpha(s) f[s, u(s)] ds \leq 0, \ 0 \leq t \leq 1.$$

Since the derivative $u'(t)$, $0 \leq t \leq 1$, is nonincreasing, we obtain $u'(t) \leq u_0'$, $0 \leq t \leq 1$, and so $u(t) \leq t u_0' \leq u_0' = r_0$, $t \in [0,1)$. Hence, in view of the assumption (A1), we obtain the contradiction $0 > u''(1) \geq u_0'' + R_0 > u_0'' + \frac{R_0}{\eta} = 0$.

On the other hand, if we assume that $u'(\eta) < 0$, there is a $t^* \in (0,\eta)$ such that $u'(t^*) = 0$. Again by Taylor's formula and the sign of nonlinearity:

$$\begin{aligned} 0 &= u'(t^*) = u_0' + t^* u_0'' + \int_0^{t^*} (t^* - s) \alpha(s) f[s, u(s)] ds \\ &\geq u_0' + t^* u_0'' > u_0' + \eta u_0'' = r_0 - R_0 \geq 0, \end{aligned}$$

a contradiction. ∎

Lemma 5. *Consider a function* $y \in C^{(3)}[(0,1), [0,+\infty)]$ *such that*

$$\begin{cases} y(0) = 0, \ \ y'(0) > 0 \ \ \text{and} \ \ y''(0) < 0 \ \ \text{and} \\ y'''(t) \geq 0, \ 0 < t < 1, \ \ y'(\eta) \leq 0 \ \ \text{and} \ \ y''(1) \leq 0. \end{cases}$$

Then

$$\min_{\theta \leq t \leq 1-\theta} y(t) \geq \theta ||y||.$$

Proof. Since $y'''(t) \geq 0$, the function $y''(t)$ is nondecreasing. So noticing that $y''(1) \leq 0$, this implies that

$$y''(t) \leq 0, \ 0 < t < 1.$$

Now by the concavity of $y(t)$, for any μ, t_1 and t_2 in $[0,1]$, we have

$$y(\mu t_1 + (1-\mu) t_2) \geq \mu y(t_1) + (1-\mu) y(t_2).$$

Also by the assumption $y'(\eta) \leq 0$, there is a $t^* \in (0,\eta)$ such that $y'(t^*) = 0$ and $||y|| = y(t^*)$. Therefore

$$y(t) \geq ||y|| \min_{\theta \leq t \leq 1-\theta} \left\{ \frac{t}{t^*}, \frac{1-t}{1-t^*} \right\} \geq ||y|| \min_{\theta \leq t \leq 1-\theta} \{t, 1-t\} = \theta ||y||.$$

∎

We recall choices (6) and $r_0 < \frac{R_0}{\eta}$ and fix the obtained initial point $K = (u_0', u_0'')$. Consider furthermore the simplex $S = [K, A, B]$, where the vertices $A = (u_A', u_0'')$ and $B = (u_0', 0)$ are chosen so that

$$u_A' + u_0'' = \eta^{-1} R_1^*, \quad \text{i.e.} \quad u_A' = \frac{R_0 + R_1^*}{\eta},$$

and u_B'' will be defined latter (above of Proposition 7).

Proposition 6. *Any solution $u = u(t)$ of (2) emanating from the initial point $A = (u_A', u_0'')$ (we will denote such a choice by $u \in \mathcal{X}(A)$) satisfies*

$$||u|| \geq R_1^* \quad and \quad u''(1) \leq 0.$$

Proof. We will first show that

$$u'(t) > \eta^{-1} R_1^*, \quad 0 \leq t \leq 1.$$

If not, proceeding as in the proof of Proposition 4, we have $u'(t^*) = \eta^{-1} R_1^*$ for some $t^* \in (0, 1]$, and $u'(t) \geq \eta^{-1} R_1^*$, $t \in (0, t^*)$. Then we get the contradiction

$$\begin{aligned} u'(t^*) &= u_A' + t^* u_0'' + t^{*2} \int_0^1 (1-s)\, \alpha(st^*)\, f[st^*, u(st^*)] ds \\ &> u_A' + u_0'' = \eta^{-1} R_1^*. \end{aligned}$$

Hence, given that $u'(t) \leq u_A'$, $0 \leq t \leq 1$, we obtain

$$\eta^{-1} R_1^* \leq u'(t) \leq \frac{R_1 + R_1^*}{\eta} \quad \text{and} \quad u(t) > 0, \quad 0 \leq t \leq 1$$

and this yields

$$||u|| = u(1) = \int_0^1 u'(s)\, ds \geq \eta^{-1} R_1^* > R_1^*.$$

Assuming now that $u''(1) \geq 0$,we have $u''(t^*) = 0$ and $u''(t) < 0$, $t \in (0, 1)$. Hence, the map $u = u(t)$, $0 \leq t \leq t^*$, is concave and thus, by Lemma 5,

$$\min\{u(t) : \theta \leq t \leq \min\{1-\theta, t^*\}\} \geq \theta R_1^*.$$

However, in view of the assumption (A_2) and since $r_0 < \frac{R_0}{\eta}$, we obtain

$$u''(t^*) = u_0'' + \int_0^{t^*} \alpha(s)\, f[s, u(s)] ds \leq u_0'' + t^* r_0 < 0,$$

a contradiction. ∎

Remark 2. *By Propositions 4 and 6, there always exists a point $P_1 \in [K, A]$ such that $u''(t) < 0$, $0 \leq t < 1$ and $u''(1) = 0$, $u \in \mathcal{X}(P_1)$.*

Let us now denote

$$L = \max\{f(t,u) : (t,u) \in [0,1] \times [0,r_0]\}.$$

For any $\varepsilon > 0$ (fixed) we set $u''_B = \min\left\{-\frac{r_0}{\eta} - A_0\eta(L+\varepsilon),\ -LA_0\right\}$ and consider the point

$$B = (r_0, u''_B).$$

Proposition 7. *The derivative of every solution $u = u(t)$ of (2) emanating from any initial point B satisfies*

$$u'(\eta) \le 0, \text{ and } u''(1) \le 0.$$

Proof. Assuming to the contrary that $u''(1) > 0$, we obtain a point $t^* \in (0,1)$ such that

$$u''(t) < 0,\ 0 \le t < t^* \text{ and } u''(t^*) = 0.$$

Since the derivative $u'(t) < 0,\ 0 \le t \le t^*$, is nonincreasing, we obtain $u'(t) \le u'_0,\ 0 \le t \le t^*$.Thus $u(t) \le tu'_0 \le u'_0 = r_0,\ 0 \le t \le t^*$. By the modification f of the nonlinearity $F(t,u)$, it follows that its argument $u = u(t) \ge 0,\ 0 \le t \le 1$. Hence, the choice of u''_B, implies that

$$u''(t^*) = u''_B + \int_0^{t^*} \alpha(s) f[s,u(s)]ds \le u''_B + L\int_0^{t^*} \alpha(s)\,ds < u''_B + LA_0 \le 0,$$

a contradiction. This yields the second required result $u''(1) \le 0$.

Furthermore we may show, similarly to the proof of Proposition 4, that $u(t) \le tu'_0 \le u'_0 = r_0,\ 0 \le t \le 1]$. Hence, by Taylor's formula and the above choice of u''_B,

$$\begin{aligned} u'(\eta) &= u'_0 + \eta u''_B + \eta^2 \int_0^1 (1-s)\,\alpha(s\eta)\, f[s\eta, u(s\eta)]ds \\ &< u'_0 + \eta u''_B + \eta^2 L \int_0^1 (1-s)\,\alpha(s\eta)\,ds < 0. \end{aligned}$$

∎

Remark 3. *By Propositions 4 and 7, there always exist points $P_1^*, P_2^* \in [K,B]$ such that*

$$u'(t) > 0,\ t \in (0,\eta) \text{ and } u'(\eta) = 0,\ u \in \mathcal{X}(P_1^*)$$

and

$$u''(t) < 0,\ t \in [0,1) \text{ and } u''(1) = 0,\ u \in \mathcal{X}(P_2^*).$$

Let $P_1 = (u'_0, u''_1)$ be a point in the face $[K,B]$ such that $u'(\eta) = 0$, for some solution $u = u(t)$ emanating from the initial point P_1. We will denote the latter choice by $u \in \mathcal{X}(P_1)$. Assuming first that $u''(1) \ge 0$, consider the cone

$$K^* = \left\{(u', u'') \in \mathbb{R}^2 :\ u' \ge 0,\ \ u'' \le 0\right\}$$

in $\mathbb{R}^2$ and define the sets (recalling the notation $S = [K,A,B]$ for the triangle, (see Figure)

$$\begin{aligned}
\Omega_1 &= \left\{ \begin{array}{c} P=(u_1',u_1'')\in K^*: P+K\in S, \\ u'(t)>0,\ 0\le t<\eta,\quad u'(\eta)>0,\quad u''(1)>0,\ \forall u\in \mathcal{X}(P+K) \end{array} \right\} \\
C_1 &= \left\{ P=(u_1',u_1'')\in \bar{\Omega}_1:\ \ u'(\eta)=0 \ \text{ and } \ u''(1)\ge 0,\ u\in \mathcal{X}(P+K)\ \right\}.
\end{aligned}$$

and

$$\begin{aligned}
\Omega_2 &= \{ P=(u_1',u_1'')\in K^*:\ \ : P+K\in S,\ u''(1)>0,\ \ \forall u\in \mathcal{X}(P+K)\ \} \\
C_2 &= \left\{ P\in \bar{\Omega}_2: u''(t)<0,\ t\in[0,1),\ \ u''(1)=0,\quad u\in \mathcal{X}(P+K) \right\},
\end{aligned}$$

where we recall once again that

$$K=(u_0',u_0'')=\left(r_0,-\frac{R_0}{\eta}\right) and P_1+K=\left(r_0+u_1',-\frac{R_0}{\eta}+u_1''\right).$$

Similarly whenever $u'(\eta)=0$ implies $u''(1)\le 0,\ u\in \mathcal{X}(P_1)$, we define the sets

$$\begin{aligned}
\Omega_1^* &= \left\{ \begin{array}{c} P=(u_1',u_1'')\in K^*: P+K\in S, \\ u''(t)>0,\ 0\le t\le 1 \ \text{ and } \ u'(\eta)>0,\quad u\in \mathcal{X}(P+K) \end{array} \right\} \\
C_1^* &= \left\{ P\in \bar{\Omega}_1^*:\ \ u''(1)=0 \ \text{ and } \ u'(\eta)\ge 0,\ u\in \mathcal{X}(P+K)\ \right\}.
\end{aligned}$$

and

$$\begin{aligned}
\Omega_2^* &= \left\{ \begin{array}{c} P=(u_1',u_1'')\in K^*:\ \ : P+K\in S, \\ u'(t)>0,\ 0\le t\le \eta,\ \ u\in \mathcal{X}(P+K) \end{array} \right\} \\
C_2^* &= \left\{ P\in \bar{\Omega}_2^*: u'(\eta)=0,\quad u\in \mathcal{X}(P+K) \right\}.
\end{aligned}$$

Remark 4. *By Remark 3, $C_i\ne\varnothing,\ i=1.2$, and $cl\left(\bar{\Omega}_2\backslash\Omega_1\right)\ne\varnothing$. The latter is a compact set, whenever $C_1\cap C_2=\varnothing$.*

Theorem 8. *Under assumptions (A_1) and (A_2), the boundary value problem (E) admits at least one positive and concave solution.*

Proof. We notice first that, if $C_1\cap C_2\ne\varnothing$, the BVP (2)-(3) clearly accepts a solution. Thus we assume $C_1\cap C_2=\varnothing$. Since the set $cl\left(\bar{\Omega}_2\backslash\Omega_1\right)$ is compact, by the upper continuity of solutions with respect to their initial values (Knesser's property of the solutions funnel),

$$C_i=\partial\Omega_i \ \text{ and } \ C_i^*=\partial\Omega_i^*,\ \ (i=1,2),$$

the set

$$\left\{u''(1): u\in cl\left(\bar{\Omega}_2\backslash\Omega_1\right)\right\}$$

is also compact and $\bar{\Omega}_2\backslash\Omega_1\ne\varnothing$. Therefore a constant $\mu>0$ exists, such that

$$\mu u''(1)\le -u''(0),\ \forall u\in cl\left(\bar{\Omega}_2\backslash\Omega_1\right). \tag{7}$$

Now for any point $P=(u_1',u_1'')$, we define the map

$$T: cl\left(\bar{\Omega}_2\backslash\Omega_1\right)\to K^*,\ \ T(P)=\left(-u'(\eta)+u_1',\mu u''(1)+u_1''\right),$$

where the solution $u = u(t)$ has its initial value at the point $P + K$, i.e. $u \in \mathcal{X}(P + K)$. The map T is well defined, that is $T(P) \in K^*$, since $P \in cl\left(\bar{\Omega}_2 \backslash \Omega_1\right)$ implies that $u'(\eta) \leq 0,\ u''(1) \geq 0$ and furthermore (7) stands true.

Similarly, if $P \in C_1$, noticing that $u''(1) = 0$ and $u'(\eta) = 0$, we obtain

$$||T(P)|| = |-u'(\eta) + u_1'| + |\mu u''(1) + u_1''| \leq |u_1'| + |u_1''| = ||P||,$$

due to the facts that $u'(\eta) = 0$ and since by (7)

$$|\mu u''(1) + u_1''| \leq |u_1''| \Leftrightarrow -\mu u''(1) - u_1'' \leq -u_1'' \Leftrightarrow u''(1) \geq 0,$$

where the last inequality is true by the definition of C_1.

Similarly, if $P \in C_2$, noticing that $u'(\eta) \leq 0$ and $u''(1) = 0$, we obtain

$$||T(P)|| = |-u'(\eta) + u_1| + |\mu u''(1) + u_1''| \geq |u_1'| + |u_1''| = ||P||.$$

Finally, by an application of Lemma 1, we obtain a fixed point of T in $K^* \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$, that is a solution of the BVP (2)-(3). Consequently, by the nature of the vector field and noticing Lemma 2, we get $u(t) \geq 0,\ t \in [0,1]$, and this means that $u = u(t)$ is a positive solution of the original boundary value problem (E).

We assume now that $u'(\eta) = 0$, implies $u''(1) \leq 0$. Assume again that $C_1^* \cap C_2^* = \varnothing$. Since the set $cl\left(\bar{\Omega}_2^* \backslash \Omega_1^*\right)$ is compact, by continuity the set

$$\left\{u'(\eta) :\ u \in \mathcal{X}\left(cl\left(\bar{\Omega}_2^* \backslash \Omega_1^*\right)\right)\right\}$$

is also compact. Therefore a constant $\mu > 0$ exists, such that

$$\mu u'(\eta) \leq u'(0), \quad \forall\, u \in \mathcal{X}\left(cl\left(\bar{\Omega}_2^* \backslash \Omega_1^*\right)\right). \tag{8}$$

Now for any point $P = (u_1', u_1'')$, we define the map

$$T : cl\left(\bar{\Omega}_2^* \backslash \Omega_1^*\right) \to K^*, \quad T(P) = \left(-\mu u'(\eta) + u_1', u''(1) + u_1''\right),$$

where the solution $u = u(t)$ has its initial value at the point $P + K$, i.e. $u \in \mathcal{X}(P + K)$. The map T is well defined, that is $T(P) \in K^*$, since $P \in cl\left(\bar{\Omega}_2^* \backslash \Omega_1^*\right)$ implies that $u''(0) < 0,\ u''(1) \leq 0$ and furthermore (8) stands true.

Considering now a point $P \in C_1^*$, we have

$$||T(P)|| = |-\mu u'(\eta) + u_1'| + |u''(1) + u_1''| \leq |u_1'| + |u_1''| = ||P||,$$

because $u''(1) = 0,$ and since by (8)

$$|-\mu u'(\eta) + u_1'| \leq |u_1'| \Leftrightarrow -\mu u'(\eta) + u_1' \leq u_1' \Leftrightarrow u'(\eta) \geq 0,$$

the last inequality being true, by the definition of C_1^*.

Similarly, if $P \in C_2^*$, we obtain

$$||T(P)|| = |-\mu u'(\eta) + u_1'| + |u''(1) + u_1''| \geq |u_1'| + |u_1''| = ||P||,$$

given that $u'(\eta) = 0$ and both $u''(1)$ and u_1'' are negative.

Hence, by another application of Lemma 1, we obtain a fixed point of the map T in $K^* \cap \left(\bar{\Omega}_2^* \backslash \Omega_1^*\right)$, that is a solution of the BVP (2)-(3). Finally, noticing again Lemma 2, we conclude that the obtained solution $u = u(t)$ of (2) is also a positive solution of equation in (E). ∎

Corollary 9. *Suppose that*

$$\lim_{x\to 0+}\min_{0\le t\le 1}\frac{f(t,x)}{x}=\beta>0 \quad and \quad \lim_{x\to +\infty}\max_{0\le t\le 1}\frac{f(t,x)}{x}=0.$$

Then, the BVP (2)-(3) has at least one positive solution.

Proof. Via the first assumption at $x=0$, there exists a positive $R_0\le 1$ and $r_0\in\left(R_0,\frac{R_0}{\eta}\right)$ such that

$$\min_{0\le t\le 1}\frac{f(t,x)}{x}=\beta>\frac{R_0}{A_0},\quad 0\le x\le r_0.$$

Therefore

$$f(t,x)\ge\frac{R_0}{A_0},\ (t,x)\in[0,1]\times[0,r_0],$$

that is the condition (A_1) is fulfilled. Moreover, since $\lim\max_{0\le t\le 1}f(t,x)=0$, for the above obtained r_0, there exists an $R_1^*\ge R_0$ such that

$$f(t,x)\le\frac{r_0}{B_0},\ (t,x)\in[0,1]\times[\theta R_1^*,+\infty),$$

and thus the assumption (A_2) is also fulfilled. Hence, Theorem 8 guarantees the result. ∎

4. Multiplicity Results

Theorem 10. *Suppose that assumptions (A_1) and (A_2) hold true. Then, there exists a sequence $\{u_n\}$ of bounded and positive solutions to the BVP (2)-(3).*

Proof. By the nature of the vector field, for any $u\in\mathcal{X}(B)$ we have $u'(\eta)<0$ and $u''(1)<0$. Hence by the continuity of solutions upon their initial values, we can find a sub-triangle

$$[K^*,A^*,B]\subseteq Int\,[K,A,B]$$

with the face $[K^*,A^*]$ parallel to $[K,A]$ such that

$$u'(\eta)<0 \text{ and } u''(1)<0,\quad u\in\mathcal{X}(P),\quad P\in[K^*,A^*,B]. \tag{9}$$

We set $K^*=(r_0,\hat{u}_0'')$ and consider a new simplex $[K_1,A_1,B_1]$ with

$$K_1=\left(r_1,-\frac{R_1}{\eta}\right),\quad B_1=\left(r_1,u''_{B_1}\right)\ \text{ and}$$
$$A_1=\left(\frac{R_1^*+R_1}{\eta},-\frac{R_1}{\eta}\right)$$

(then, $[K_1,A_1]$ is parallel to $[K,A]$) under the choice

$$r_1\in(0,r_0),\ u''_{B_1}=\min\left\{-r_1\eta^{-1}-A_0\eta\left(L_1+\varepsilon_n\right),\ -L_1B\right\}\quad\text{and}\quad -\frac{R_1}{\eta}>\hat{u}_0'',$$

where now

$$L_1=\max\{f(t,u):(t,u)\in[0,1]\times[0,r_1]\}\le L.$$

In view of assumptions (A_1) and (A_2), we may apply once again the Krasnosel'skiĭ's theorem on the triangle $[K_1, A_1, B_1]$, in order to obtain another positive solution $u = u_1(t)$ of the BVP (2)-(3). By the construction of the triangle $[K_1, A_1, B_1]$ and (9), it is obvious that $u = u(t)$ is different than the solution $u = u(t)$, $0 \le t \le 1$, obtained at Theorem 8.

If we continue this procedure, choosing the sequence $\{r_n\}$ such that $\lim r_n = \lim \varepsilon_n = 0$, we may easily obtain a sequence $\{u_n\}$ of solutions to the BVP (2)-(3). Furthermore, since $u_n''(t) \le 0$, $0 \le t \le 1$,

$$u_n'(t) = u_n'(0) + \int_0^t u''(s)\,ds \le u_n'(0) = \frac{R_n + R_1}{\eta}.$$

Consequently, given that $\{R_n\}$ is decreasing, the sequence $\{u_n\}$ is a bounded one. ∎

5. Discussion

Assuming that both functions $\alpha(t)$ and $f(t,x,y,z)$ are negative, we may easily demonstrate similar existence and multiplicity results. Indeed, considering the (x', x'') phase semi-plane ($x' \le 0$), we easily check that $x''' = \alpha(t) f(t,x) < 0$. Thus, any trajectory $(x'(t), x''(t))$, $t \ge 0$, emanating from any point in the third quarter

$$\{(x', x'') : x' < 0,\ x'' > 0\}$$

evolves in a natural way, when $x'(t) < 0$, toward the positive x''−semi-axis and then, (when $x'(t) \ge 0$) toward the positive x'−semi-axis. This results, under a certain growth rate on f , that we can control the vector field, in a way that assures the existence of a trajectory satisfying the given boundary conditions. We notice that in present situation, the obtaining solution $(x'(t), x''(t))$ is convex, in contrast to the previous case, where it is concave (see Figure).

Furthermore we could easily get analogous results, for the case where the nonlinearity is superlinear.

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