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POSITIVE SOLUTION FOR FOURTH-ORDER FOUR-POINT STURM-LIOUVILLE BOUNDARY VALUE PROBLEM

JIAN-PING SUN* AND XIAO-YUN WANG

ABSTRACT. This paper is concerned with the following fourth-order four-point Sturm-Liouville boundary value problem

$$\begin{aligned} &u^{(4)}(t) = f(t, u(t), u''(t)), \ 0 \le t \le 1, \\ &\alpha u(0) - \beta u'(0) = \gamma u(1) + \delta u'(1) = 0, \\ &au''(\xi_1) - bu'''(\xi_1) = cu''(\xi_2) + du'''(\xi_2) = 0. \end{aligned}$$

Some sufficient conditions are obtained for the existence of at least one positive solution to the above boundary value problem by using the well-known Guo-Krasnoselskii fixed point theorem.

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1. Introduction

Boundary value problems (BVPs for short) of fourth-order ordinary differential equations arise from a variety of different areas of applied mathematics and physics. Fourth-order two-point BVPs have received much attention from many authors. One may see [1]-[3], [6], [7], [9]-[14] and the references therein for related results. Recently, an increasing interest in studying the existence of solutions and positive solutions for fourth-order four-point BVPs is observed; see for example [4], [8] and [15]. In particular, the authors in [15] studied the following fourth-order four-point BVP

$$\begin{aligned} &u^{(4)}(t) = f(t, u(t), u''(t)), \ 0 \le t \le 1, \\ &u(0) = u(1) = 0, \\ &au''(\xi_1) - bu'''(\xi_1) = 0, \ cu''(\xi_2) + du'''(\xi_2) = 0, \end{aligned} \tag{1}$$

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where $0 \le \xi_1 < \xi_2 \le 1$, a, b, c, d are nonnegative constants, $ad+bc+ac(\xi_2-\xi_1) > 0$, $-a\xi_1+b \ge 0$, $c(\xi_2-1)+d \ge 0$ and $f \in C([0,1] \times [0,+\infty) \times (-\infty, 0], [0,+\infty))$. The existence of at least one positive solution to the BVP (1) was proved by using the Guo-Krasnoselskii fixed point theorem under the assumption that:

(1) f was superlinear, i.e., $\max f_0 = 0$ and $\min f_\infty = +\infty$; or

(2) f was suberlinear, i.e., $\min f_0 = +\infty$ and $\max f_\infty = 0$, where

$$\max f_0 = \lim_{-y \to 0^+} \max_{t\in[0,1]} \sup_{x\in[0,+\infty)} \frac{f(t,x,y)}{-y},$$

$$\min f_\infty = \lim_{-y \to +\infty} \min_{t\in[0,1]} \inf_{x\in[0,+\infty]} \frac{f(t,x,y)}{-y},$$

$$\min f_0 = \lim_{-y \to 0^+} \min_{t\in[0,1]} \inf_{x\in[0,+\infty)} \frac{f(t,x,y)}{-y},$$

$$\max f_\infty = \lim_{-y \to +\infty} \max_{t\in[0,1]} \sup_{x\in[0,+\infty)} \frac{f(t,x,y)}{-y}.$$

However, roughly speaking, these conditions imposed on f require that $f(t,x,y)$ is bounded in x, which is a very strong assumption.

In this paper we will investigate the following more general fourth-order four-point Sturm-Liouville BVP

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)), \ 0 \le t \le 1, \\ \alpha u(0) - \beta u'(0) = \gamma u(1) + \delta u'(1) = 0, \\ a u''(\xi_1) - b u'''(\xi_1) = c u''(\xi_2) + d u'''(\xi_2) = 0. \end{cases} \tag{2}$$

Throughout this paper, we always assume that $0 \le \xi_1 < \xi_2 \le 1$, $\alpha, \beta, \gamma, \delta, a, b, c, d$ are nonnegative constants, $\rho_1 := \alpha\gamma+\alpha\delta+\gamma\beta > 0$, $\rho_2 := ad+bc+ac(\xi_2-\xi_1) > 0$, $-a\xi_1+b \ge 0$, $c(\xi_2-1)+d \ge 0$ and $f \in C([0,1] \times [0,+\infty) \times (-\infty,0], [0,+\infty))$. Define $\eta_1 = \xi_1 + \frac{1}{4}(\xi_2-\xi_1)$ and $\eta_2 = \xi_2 - \frac{1}{4}(\xi_2-\xi_1)$. By modifying the definitions of $\max f_0, \min f_\infty, \min f_0$ and $\max f_\infty$ as follows:

$$f^0 = \limsup_{x+|y|\to 0^+} \max_{t\in[\xi_1,\xi_2]} \frac{f(t,x,y)}{x+|y|}, \quad f_\infty = \liminf_{x+|y|\to +\infty} \min_{t\in[\eta_1,\eta_2]} \frac{f(t,x,y)}{x+|y|},$$

$$f_0 = \liminf_{x+|y|\to 0^+} \min_{t\in[\eta_1,\eta_2]} \frac{f(t,x,y)}{x+|y|}, \quad f^\infty = \limsup_{x+|y|\to +\infty} \max_{t\in[\xi_1,\xi_2]} \frac{f(t,x,y)}{x+|y|},$$

we obtain the existence of at least one positive solution for the BVP (2). Our main tool is the following well-known Guo-Krasnoselskii fixed point theorem [5].

Theorem 1. *Let X be a Banach space and K be a cone in X. Assume Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let*

$$T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(1) $\|Tu\| \le \|u\|, \forall u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|, \forall u \in K \cap \partial\Omega_2$

or

(2) $\|Tu\| \geq \|u\|, \forall u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|, \forall u \in K \cap \partial\Omega_2$.
Then T *has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Preliminary lemmas

In this section, we give some lemmas which will be employed to obtain the existence of positive solution for the BVP (2). Denote by $G_1(t,s)$ the Green function of the BVP

$$\begin{array}{l} -u''(t) = 0, \quad 0 \leq t \leq 1, \\ \alpha u(0) - \beta u'(0) = \gamma u(1) + \delta u'(1) = 0. \end{array}$$

Then it is well known that $G_1(t,s)$ can be written as

$$G_1(t,s) = \frac{1}{\rho_1} \begin{cases} (\alpha s + \beta)(\gamma + \delta - \gamma t), & 0 \leq s \leq t \leq 1, \\ (\alpha t + \beta)(\gamma + \delta - \gamma s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Let $X = C[0,1]$. Then $(X, \|\cdot\|)$ is a Banach Space, here $\|\cdot\|$ is defined as usual by the sup norm. Set

$$K = \{v \in X \mid v(t) \geq 0 \text{ for } t \in [0,1]\} \text{ and } P = \{v \in K \mid \min_{t \in [\eta_1, \eta_2]} v(t) \geq \frac{1}{4}\|v\|\}.$$

Then it is easy to know that K and P are cones in X. Now we define an integral operator $S : K \to X$ by

$$(Sv)(t) = \int_0^1 G_1(t,s) v(s)\, ds, \ t \in [0,1]. \tag{3}$$

It is obvious that

$$\|Sv\| = \max_{t \in [0,1]} \int_0^1 G_1(t,s) v(s)\, ds \leq \max_{t \in [0,1]} \int_0^1 G_1(t,s) \|v\|\, ds = \Gamma \|v\|, \tag{4}$$

where $\Gamma = \max_{t \in [0,1]} \int_0^1 G_1(t,s)\, ds > 0$.

Lemma 1. *If* v *is a positive solution of the following BVP*

$$\begin{array}{l} v''(t) + f(t, (Sv)(t), -v(t)) = 0, \ 0 \leq t \leq 1, \\ av(\xi_1) - bv'(\xi_1) = cv(\xi_2) + dv'(\xi_2) = 0, \end{array} \tag{5}$$

then $u = Sv$ *is a positive solution of the BVP* (2).

Proof. Since the proof is easy, we omit it. □

Now we denote by $G_2(t,s)$ the Green function of the BVP

$$\begin{array}{l} -v''(t) = 0, \ 0 \leq t \leq 1, \\ av(\xi_1) - bv'(\xi_1) = cv(\xi_2) + dv'(\xi_2) = 0. \end{array}$$

It is well known that

$$G_2(t,s) = \frac{1}{\rho_2} \begin{cases} (a(s - \xi_1) + b)(c(\xi_2 - t) + d), & s \leq t, \ \xi_1 \leq s \leq \xi_2, \\ (a(t - \xi_1) + b)(c(\xi_2 - s) + d), & t \leq s, \ \xi_1 \leq s \leq \xi_2. \end{cases}$$

For $G_2(t,s)$, we need the following results whose proof can be found in [15].

Lemma 2. *$0 \leq G_2(t,s) \leq G_2(s,s)$ for $(t,s) \in [0,1] \times [\xi_1, \xi_2]$ and $\frac{1}{4}G_2(s,s) \leq G_2(t,s)$ for $(t,s) \in [\eta_1, \eta_2] \times [\xi_1, \xi_2]$.*

Define an operator $T : P \to X$ by

$$(Tv)(t) = \int_{\xi_1}^{\xi_2} G_2(t,s) f(s, (Sv)(s), -v(s))\, ds, \ t \in [0,1]. \tag{6}$$

Obviously, if v is a fixed point of T in P, then v is a positive solution of the BVP (5).

Lemma 3. *$T : P \to P$ is completely continuous.*

Proof. Let $v \in P$. Then it follows form Lemma 2 that

$$\begin{aligned} 0 \leq (Tv)(t) &= \int_{\xi_1}^{\xi_2} G_2(t,s) f(s, (Sv)(s), -v(s))\, ds \\ &\leq \int_{\xi_1}^{\xi_2} G_2(s,s) f(s, (Sv)(s), -v(s))\, ds, \ t \in [0,1], \end{aligned}$$

and so,

$$\|Tv\| \leq \int_{\xi_1}^{\xi_2} G_2(s,s) f(s, (Sv)(s), -v(s))\, ds,$$

which together with Lemma 2 implies that

$$\begin{aligned} \min_{t \in [\eta_1, \eta_2]} (Tv)(t) &= \min_{t \in [\eta_1, \eta_2]} \int_{\xi_1}^{\xi_2} G_2(t,s) f(s, (Sv)(s), -v(s))\, ds \\ &\geq \frac{1}{4} \int_{\xi_1}^{\xi_2} G_2(s,s) f(s, (Sv)(s), -v(s))\, ds \\ &\geq \frac{1}{4} \|Tv\|, \end{aligned}$$

which shows that $T(P) \subset P$. Furthermore, it is easy to prove that $T : P \to P$ is completely continuous by an application of the Arzela-Ascoli theorem. □

3. Main results

Theorem 2. *Suppose that f is superlinear, i.e., $f^0 = 0$ and $f_\infty = +\infty$. Then the BVP (2) has at least one positive solution.*

Proof. Since $f^0 = 0$, we may choose $h_1 > 0$ so that

$$f(t,x,y) \leq \epsilon(x + |y|) \ \text{for } t \in [\xi_1, \xi_2] \ \text{and} \ (x + |y|) \in [0, h_1], \tag{7}$$

where $\epsilon > 0$ satisfies

$$\epsilon(1 + \Gamma) \int_{\xi_1}^{\xi_2} G_2(s,s)\, ds \leq 1. \tag{8}$$

Let $\Omega_1 = \left\{ v \in X | \, \|v\| < \frac{h_1}{1+\Gamma} \right\}$. Then for any $v \in P \cap \partial\Omega_1$, it follows from Lemma 2, (4), (7) and (8) that

$$\begin{aligned}
(Tv)(t) &= \int_{\xi_1}^{\xi_2} G_2(t,s) f(s,(Sv)(s), -v(s))\, ds \\
&\leq \int_{\xi_1}^{\xi_2} G_2(s,s) f(s,(Sv)(s), -v(s))\, ds \\
&\leq \epsilon \int_{\xi_1}^{\xi_2} G_2(s,s) ((Sv)(s) + |-v(s)|)\, ds \\
&\leq \epsilon (1+\Gamma) \int_{\xi_1}^{\xi_2} G_2(s,s) \|v\|\, ds, \\
&\leq \|v\|, \ t \in [0,1],
\end{aligned}$$

which shows that

$$\|Tv\| \leq \|v\| \ \text{ for } v \in P \cap \partial\Omega_1. \tag{9}$$

On the other hand, since $f_\infty = +\infty$, there exists $h_2 > \frac{h_1}{1+\Gamma}$ such that

$$f(t,x,y) \geq \epsilon^* (x+|y|) \ \text{ for } t \in [\eta_1, \eta_2] \ \text{ and } \ (x+|y|) \in \left[\frac{1}{4}h_2, +\infty\right), \tag{10}$$

where $\epsilon^* > 0$ satisfies

$$\frac{1}{4}\epsilon^* \int_{\eta_1}^{\eta_2} G_2(\eta_1, s)\, ds \geq 1. \tag{11}$$

Let $\Omega_2 = \{v \in X | \, \|v\| < h_2\}$. Then for any $v \in P \cap \partial\Omega_2$, $\min\limits_{t\in[\eta_1,\eta_2]} v(t) \geq \frac{1}{4}\|v\| = \frac{1}{4}h_2$. In view of (10) and (11), we have

$$\begin{aligned}
(Tv)(\eta_1) &= \int_{\xi_1}^{\xi_2} G_2(\eta_1,s) f(s,(Sv)(s), -v(s))\, ds \\
&\geq \int_{\eta_1}^{\eta_2} G_2(\eta_1,s) f(s,(Sv)(s), -v(s))\, ds \\
&\geq \epsilon^* \int_{\eta_1}^{\eta_2} G_2(\eta_1,s) ((Sv)(s) + |-v(s)|)\, ds \\
&\geq \epsilon^* \int_{\eta_1}^{\eta_2} G_2(\eta_1,s) |-v(s)|\, ds, \\
&\geq \frac{1}{4}\epsilon^* \int_{\eta_1}^{\eta_2} G_2(\eta_1,s) \|v\|\, ds \\
&\geq \|v\|,
\end{aligned}$$

which implies that

$$\|Tv\| \geq \|v\| \ \text{ for } v \in P \cap \partial\Omega_2. \tag{12}$$

Therefore, it follows from (9), (12) and Theorem 1 that the operator T has one fixed point $v \in P \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$, which means that the BVP (2) has at least one positive solution. □

Theorem 3. *Suppose that f is sublinear, i.e., $f_0 = +\infty$ and $f^\infty = 0$. Then the BVP (2) has at least one positive solution.*

Proof. Since $f_0 = +\infty$, there exists $h_3 > 0$ such that

$$f(t, x, y) \geq \epsilon (x + |y|) \text{ for } t \in [\eta_1, \eta_2] \text{ and } (x + |y|) \in [0, h_3], \tag{13}$$

where $\epsilon > 0$ satisfies

$$\frac{1}{4}\epsilon \int_{\eta_1}^{\eta_2} G_2(\eta_1, s)\, ds \geq 1. \tag{14}$$

Let $\Omega_3 = \left\{v \in X |\, \|v\| < \frac{h_3}{1+\Gamma}\right\}$. Then for any $v \in P \cap \partial\Omega_3$, in view of (13) and (14), we have

$$\begin{aligned}
(Tv)(\eta_1) &= \int_{\xi_1}^{\xi_2} G_2(\eta_1, s) f(s, (Sv)(s), -v(s))\, ds \\
&\geq \int_{\eta_1}^{\eta_2} G_2(\eta_1, s) f(s, (Sv)(s), -v(s))\, ds \\
&\geq \epsilon \int_{\eta_1}^{\eta_2} G_2(\eta_1, s)\left((Sv)(s) + |-v(s)|\right) ds \\
&\geq \epsilon \int_{\eta_1}^{\eta_2} G_2(\eta_1, s) |-v(s)|\, ds, \\
&\geq \frac{1}{4}\epsilon \int_{\eta_1}^{\eta_2} G_2(\eta_1, s) \|v\|\, ds \\
&\geq \|v\|,
\end{aligned}$$

which implies that

$$\|Tv\| \geq \|v\| \text{ for } v \in P \cap \partial\Omega_3. \tag{15}$$

On the other hand, since $f^\infty = 0$, we may choose $M > 0$ so that

$$f(t, x, y) \leq \epsilon^* (x + |y|) \text{ for } t \in [\xi_1, \xi_2] \text{ and } (x + |y|) \in [M, +\infty), \tag{16}$$

where $\epsilon^* > 0$ satisfies

$$\epsilon^* (1 + \Gamma) \int_{\xi_1}^{\xi_2} G_2(s, s)\, ds \leq \frac{1}{2}. \tag{17}$$

Let

$$M^* = \max\left\{f(t, x, y) : t \in [\xi_1, \xi_2],\, x \in [0, M] \text{ and } y \in [-M, 0]\right\}.$$

Then it is easy to see that

$$f(t, x, y) \leq \epsilon^*(x + |y|) + M^* \text{ for } t \in [\xi_1, \xi_2],\ x \in [0, +\infty) \text{ and } y \in (-\infty, 0]. \tag{18}$$

Set

$$h_4 > \max\left\{\frac{h_3}{1+\Gamma}, 2M^* \int_{\xi_1}^{\xi_2} G_2(s,s)\, ds\right\}. \tag{19}$$

Let $\Omega_4 = \{v \in X \mid \|v\| < h_4\}$. Then for any $v \in P \cap \partial\Omega_4$, by Lemma 2, (4), (17), (18) and (19), we know that

$$\begin{aligned}(Tv)(t) &= \int_{\xi_1}^{\xi_2} G_2(t,s) f(s,(Sv)(s),-v(s))\, ds \\ &\leq \int_{\xi_1}^{\xi_2} G_2(s,s) f(s,(Sv)(s),-v(s))\, ds \\ &\leq \int_{\xi_1}^{\xi_2} G_2(s,s)\left[\epsilon^*(1+\Gamma)\|v\| + M^*\right] ds \\ &\leq \frac{1}{2}\|v\| + \frac{1}{2}h_4, \\ &= \|v\|,\ t \in [0,1],\end{aligned}$$

which shows that

$$\|Tv\| \leq \|v\| \text{ for } v \in P \cap \partial\Omega_4. \tag{20}$$

Therefore, it follows from (15), (20) and Theorem 1 that the operator T has one fixed point $v \in P \cap (\overline{\Omega}_4 \setminus \Omega_3)$, which means that the BVP (2) has at least one positive solution. □

4. An example

In this section, an example is given to illustrate the main results of this paper.

Example 1. Consider the BVP

$$\begin{aligned}&u^{(4)}(t) = t\left(u(t) - u''(t)\right)^2,\ 0 \leq t \leq 1, \\ &u(0) - \tfrac{1}{2}u'(0) = u(1) + \tfrac{1}{2}u'(1) = 0, \\ &u''\left(\tfrac{1}{4}\right) - \tfrac{1}{2}u'''\left(\tfrac{1}{4}\right) = u''\left(\tfrac{3}{4}\right) + \tfrac{1}{2}u'''\left(\tfrac{3}{4}\right) = 0.\end{aligned} \tag{21}$$

Since $f(t,x,y) = t(x-y)^2$, a simple computation shows that $f^0 = 0$ and $f_\infty = +\infty$. It follows from Theorem 2 that the BVP (21) has at least one positive solution.

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Jian-Ping Sun received his Ph.D at Lanzhou University in 2007. Since 2007, he has been a professor at Lanzhou University of Technology. In 2008, he obtained the support from the National Natural Science Foundation of China. He is currently a member of the editorial board of International Journal of Differential Equations. His research interests focus on the boundary value problems of nonlinear dynamic equations.

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou 730050, People's Republic of China
e-mail: jpsun@lut.cn

Xiao-Yun Wang is now a graduate of Lanzhou University of Technology. Her research interests focus on the boundary value problems of nonlinear differential equations.

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou 730050, People's Republic of China
e-mail: wangxy1984@mail2.lut.cn