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ON THE GENERALIZED SOR-LIKE METHODS FOR SADDLE POINT PROBLEMS †

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ABSTRACT. In this paper, the generalized SOR-like methods are presented for solving the saddle point problems. Based on the SOR-like methods, we introduce the uncertain parameters and the preconditioned matrixes in the splitting form of the coefficient matrix. The necessary and sufficient conditions for guaranteeing its convergence are derived by giving the restrictions imposed on the parameters. Finally, numerical experiments show that this methods are more effective by choosing the proper values of parameters.

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1. Introduction

The abstract saddle point problem is of the form

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \tag{1}$$

where $A \in R^{m\times m}$ is a symmetric and positive definite matrix, $B \in R^{m\times n} (m \geq n)$ is a matrix of full column rank, and B^T is the transpose of matrix B, $b \in R^m$ and $q \in R^n$ are two given vectors. This class of problems appears in many different fields of scientific computing and engineering applications, such as the constrained optimization [1-6], the finite element method or the finite volume method for solving the Navier-Stokes equations [7-9], the hybrid finite element approximations of second-order elliptic problems and elasticity problems, and the constrained least squares problems and the generalized least squares problems, etc. There have been lots of iterative methods for solving the augmented system

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(1). Among them, the preconditioned iterative methods were provided firstly by Santos and co-works. Several variants of the SOR method and preconditioned conjugate gradient methods were proposed for solving general augmented system by Yuan and co-workers. The preconditioned MINRES method, the QMR method, the preconditioned GMRES method, the SOR-like methods were investigated respectively for solving the augmented system arising from finite element approximations to the Stokes equations. Particularly, the SOR-like method is a simple non-stationary iterative method with no preconditioner. Recently, an iterative method with variable relaxation parameters, the generalized successive overrelaxation methods, the generalized AOR method, the parameterized inexact Uzawa methods, the fast Uzawa algorithms and the generalized symmetric SOR method were presented by some scholars for sloving the augmented system and the generalized saddle point problems, respectively [1,2,5-7,11-18]. Moreover, Feng and co-workers applied the modified homotopy perturbation method to the saddle point problem (1) in [10].

In this paper, we focus on the generalized SOR-like methods for solving the augmented systems. Based on the SOR-like methods, we introduce the uncertain parameters and the preconditioned matrixes in the splitting form of the coefficient matrix. According to the sign of all eigenvalues μ of $Q^{-1}B^TA^{-1}B$, two special cases are considered for solving problem (1). The necessary and sufficient conditions for guaranteeing its convergence are derived by giving the restrictions imposed on the parameters. Finally, the promising characteristic of the proposed algorithms are illustrated by two numerical examples.

The outline of this paper is as follows. In Section 2, the generalized SOR-like methods with the uncertain parameters are given concretely. Moreover, The necessary and sufficient conditions for guaranteeing its convergence are derived in detail. In Section 3, we make some special choices for Q and give numerical experiments for our algorithms. The numerical experiments show that our methods work well for problem (1) arising from the real problems.

2. The generalized SOR-like method

For the coefficient matrix of the augmented system (1), we consider the following splitting:

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} = D - L - U \tag{2}$$

where

$$D = \begin{pmatrix} A & 0 \\ 0 & hQ \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ -B^T & \alpha Q \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -B \\ 0 & \beta Q \end{pmatrix}. \tag{3}$$

Here $Q \in R^{n\times n}$ is a given matrix and needs to be non-singular and "easy" to invert, $h \neq 0, \alpha, \beta$ are three real parameters and satisfy $h = \alpha + \beta$.

Denote $(x^{(k)}, y^{(k)})^T$ be the kth approximation of solution (1), we obtain the generalized SOR-like methods as follows:

$$(D - rL)\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \omega \begin{pmatrix} b \\ q \end{pmatrix} + [(1-\omega)D + (\omega - r)L + \omega U]\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix}, \quad (4)$$

where r and ω are two relaxation parameters. When $r = \omega, \alpha = 0$ and $\beta = 1$, we get the well-known SOR-like methods that firstly proposed by Gloub and co-works [12]. When $h = 1$, we get the generalized AOR method that provided by Shao and co-works [15].

From (3), we have

$$(D - rL) = \begin{pmatrix} A & 0 \\ 0 & hQ \end{pmatrix} - r\begin{pmatrix} 0 & 0 \\ -B^T & \alpha Q \end{pmatrix} = \begin{pmatrix} A & 0 \\ rB^T & (h-\alpha r)Q \end{pmatrix}. \quad (5)$$

Obviously, $\det(D - rL) = \det(A)\det((h-\alpha r)Q) = (h-\alpha r)^n \det(A)\det(Q) \neq 0$ if and only if $h - \alpha r \neq 0$. Hence,

$$(D - rL)^{-1} = \begin{pmatrix} A^{-1} & 0 \\ \frac{r}{\alpha r - h}Q^{-1}B^T A^{-1} & \frac{1}{h-\alpha r}Q^{-1} \end{pmatrix} \quad (6)$$

exists if and only if $h - \alpha r \neq 0$. From (3) and (5), we obtain the generalized SOR-like iteration matrix as follows:

$$M_{\omega,r,h} = \begin{pmatrix} (1-\omega)I_m & -\omega A^{-1}B \\ \frac{\omega(r-1)}{h-\alpha r}Q^{-1}B^T & I_n + \frac{\omega r}{h-\alpha r}Q^{-1}B^T A^{-1}B \end{pmatrix} \quad (7)$$

where $I_m \in R^{m\times m}$ and $I_n \in R^{n\times n}$ are the m-by-m and the n-by-n identity matrices, respectively. If $\omega = 0$, then we have

$$M_{0,r,h} = \mathrm{diag}(I_m, I_n). \quad (8)$$

So the generalized SOR-like method is divergent for all the values of parameters r and h. In the following, we assume that $\omega \neq 0$. Suppose that λ is an eigenvalue of $M_{\omega,r,h}$ and whose eigenvector is $(u, v)^T$, then we have

$$M_{\omega,r,h}(u, v)^T = \lambda(u, v)^T. \quad (9)$$

Hence, it follows from (9) that

$$(1 - \omega - \lambda)u = \omega A^{-1}Bv, \quad (r - \omega - r\lambda)Q^{-1}B^T u = (\lambda - 1)(h - \alpha r)v. \quad (10)$$

Next we study the convergence of generalized SOR-like method for problem (1).

Lemma 1. *Suppose that λ is an eigenvalue of $M_{\omega,r,h}$, then $\lambda \neq 1$.*

Proof. If $\lambda = 1$ and whose eigenvector is $(u, v)^T$, then from (10) we have

$$u = -A^{-1}Bv, \qquad Q^{-1}B^T u = 0, \quad (11)$$

since $\omega \neq 0$. So we obtain $Q^{-1}B^T A^{-1}Bv = 0$. By the nonsingularity of $Q^{-1}B^T A^{-1}B$, we get $v = 0$ and $u = 0$. This is contrary to the definition of eigenvector, thus $\lambda \neq 1$. □

Lemma 2. *Assume that $r = 1$, then the algebraic multiplicity of the eigenvalue $\lambda = 1 - \omega$ is at least m for matrix $M_{\omega,r,h}$; if $r \neq 1$ and $m > n$, then the algebraic multiplicity of the eigenvalue $\lambda = 1 - \omega$ is at least $m - n$ for matrix $M_{\omega,r,h}$; if $r \neq 1$ and $m = n$, then $\lambda = 1 - \omega$ is not an eigenvalue of $M_{\omega,r,h}$.*

Proof. If $r = 1$, then we can obtain the algebraic multiplicity of the eigenvalue $\lambda = 1-\omega$ is at least m from (7). If $r \neq 1$, $\lambda = 1-\omega$ is an eigenvalue of $M_{\omega,r,h}$ and whose eigenvector is $(u,v)^T$, then we have $A^{-1}Bv = 0$, and $Q^{-1}B^Tu = \frac{h-\alpha r}{1-r}v$. Since B is a matrix of full column rank, we get $v = 0$ and $Q^{-1}B^Tu = 0$.

Furthermore, if $m > n$, then the system $Q^{-1}B^Tu = 0$ exists $m - n$ nonzero solutions. Hence the algebraic multiplicity of the eigenvalue $\lambda = 1 - \omega$ is at least $m - n$. If $m = n$, then the system $Q^{-1}B^Tu = 0$ only exists zero solution. Thus $\lambda = 1 - \omega$ is not an eigenvalue of $M_{\omega,r,h}$. □

From Lemma 1 and Lemma 2, it is clear that if $m > n$, then $\lambda = 1 - \omega$ is an eigenvalue of $M_{\omega,r,h}$; if $\lambda \neq 1-\omega$ is an eigenvalue of $M_{\omega,r,h}$ and whose eigenvector is $(u,v)^T$, then $v \neq 0$.

Theorem 1. *Suppose that μ is an eigenvalue of $Q^{-1}B^TA^{-1}B$. If λ satisfies*

$$(\lambda - 1)(1 - \omega - \lambda)(h - \alpha r) = \omega\mu(r - \omega - r\lambda), \tag{12}$$

then λ is an eigenvalue of $M_{\omega,r,h}$. Conversely, if λ is an eigenvalue of $M_{\omega,r,h}$ such that $\lambda \neq 1$ and $\lambda \neq 1 - \omega$ and μ satisfies (12), then μ is an eigenvalue of $Q^{-1}B^TA^{-1}B$.

Proof. According to (10), we have

$$\omega(r - \omega - r\lambda)Q^{-1}B^TA^{-1}Bv = (\lambda - 1)(1 - \omega - \lambda)(h - \alpha r)v,$$

since $\lambda \neq 1 - \omega$. Assume that μ is an eigenvalue of $Q^{-1}B^TA^{-1}B$. Then we get (12). By using Lemma 1 and Lemma 2, we have $\omega(r - \omega - r\lambda) \neq 0$. Reversing the process, we can prove our second assertion. □

Corollary 1. *Suppose that $\rho(M_{\omega,r,h})$ is the spectral radius of $M_{\omega,r,h}$ and $m > n$, then $\rho(M_{\omega,r,h}) \geq |1 - \omega|$.*

For our next result, we first quote the following useful result [16]:

Lemma 3.[Young] *Both roots of the real quadratic equation $x^2 - bx + c = 0$ are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

Theorem 2. *Suppose that B has full rank and A is symmetric and positive definite. Assume that all eigenvalues μ of $Q^{-1}B^TA^{-1}B$ are real. Then, if $\mu < 0$, the generalized SOR-like method is convergent if and only if*

$$h - \alpha r > 0, \quad \frac{r - \omega}{h - \alpha\gamma} > \frac{1}{\mu_n} \quad \text{and} \quad \frac{2r - \omega}{h - \alpha r} < -\frac{4 - 2\omega}{\omega\mu_n}, \tag{13}$$

where μ_n is the smallest eigenvalue of $Q^{-1}B^TA^{-1}B$.

Proof. It follows from Theorem 1 that

$$\lambda^2 - (2 - \omega + \frac{\omega r \mu}{h - \alpha r})\lambda + (1 - \omega + \frac{\omega(r-\omega)\mu}{h - \alpha\gamma}) = 0.$$

By Lemma 3, $|\lambda| < 1$ if and only if

$$|2 - \omega + \frac{\omega r \mu}{h - \alpha r}| < 2 - \omega + \frac{\omega(r-\omega)\mu}{h - \alpha\gamma}, \tag{14}$$

$$|1 - \omega + \frac{\omega(r-\omega)\mu}{h - \alpha\gamma}| < 1. \tag{15}$$

From (14) and (15), we have

$$\omega - \frac{\omega(r-\omega)\mu}{h-\alpha\gamma} > 0, \qquad 2 - \omega + \frac{\omega(r-\omega)\mu}{h-\alpha\gamma} > 0,$$
$$-\frac{\omega^2\mu}{h-\alpha r} > 0, \qquad 4 - 2\omega + \frac{\omega(2r-\omega)\mu}{h-\alpha r} > 0. \tag{16}$$

If $h - \alpha\gamma > 0$, then (16) holds for all $\mu < 0$. So we have

$$(r-\omega)\mu < h - \alpha r, \qquad (2-\omega)(h-\alpha r) > -\omega(r - \frac{\omega}{2})\mu. \tag{17}$$

Hence the following inequalities hold

$$h - \alpha r > 0, \quad \frac{r-\omega}{h-\alpha\gamma} > \frac{1}{\mu} \quad \text{and} \quad \frac{2r-\omega}{h-\alpha r} < -\frac{4-2\omega}{\omega\mu}, \tag{18}$$

for all eigenvalues μ of $Q^{-1}B^TA^{-1}B$. □

Theorem 3. *Suppose that B has full rank and A is symmetric and positive definite. Assume that all eigenvalues μ of $Q^{-1}B^TAB$ are real. Then, if $\mu > 0$, the generalized SOR-like method is convergent if and only if*

$$h - \alpha r < 0, \quad \frac{r-\omega}{h-\alpha\gamma} < \frac{1}{\mu_1} \quad \textit{and} \quad \frac{2r-\omega}{h-\alpha r} > -\frac{4-2\omega}{\omega\mu_1}, \tag{19}$$

where μ_1 is the first eigenvalue of $Q^{-1}B^TA^{-1}B$.

Proof. From (14)-(16), if $h - \alpha\gamma < 0$, then (16) holds for all $\mu > 0$. So we have

$$(r-\omega)\mu > h - \alpha r, \qquad (2-\omega)(h-\alpha r) < -\omega(r - \frac{\omega}{2})\mu. \tag{20}$$

Hence the following inequalities hold

$$h - \alpha r < 0, \quad \frac{r-\omega}{h-\alpha\gamma} < \frac{1}{\mu} \quad \text{and} \quad \frac{2r-\omega}{h-\alpha r} > -\frac{4-2\omega}{\omega\mu}, \tag{21}$$

for all eigenvalues μ of $Q^{-1}B^TA^{-1}B$. □

Corollary 2. *Suppose that B has full rank and A is symmetric and positive definite. Assume that all eigenvalues μ of $Q^{-1}B^TA^{-1}B$ are real. Then, the generalized SOR-like method is convergent for all $\omega = r$ and $\alpha = 0$ such that*

$$0 < \omega < \frac{4}{1+\sqrt{1+4\rho/|h|}}. \tag{22}$$

where ρ is the spectral radius of $Q^{-1}B^TA^{-1}B$ and $|h|$ is the absolute value of h.

Proof. Let $r = \omega$ and $\alpha = 0$ in (18) and (21) respectively, then we have

$$0 < \omega < -\frac{(4-2\omega)h}{\omega\mu_n}, \tag{23}$$

for all $h > 0$ and $\mu < 0$. And

$$0 < \omega < -\frac{(4-2\omega)h}{\omega\mu_1}, \tag{24}$$

for all $h < 0$ and $\mu > 0$.

From (23), we have $\mu_n\omega^2 - 2h\omega + 4h > 0$. Hence we obtain

$$0 < \omega < \frac{4}{1+\sqrt{1-4\mu_n/h}}, \quad \text{where } \mu_n < 0, h > 0. \tag{25}$$

On the other hand, from (24) we have $\mu_1\omega^2 - 2h\omega + 4h < 0$. Hence we obtain

$$0 < \omega < \frac{4}{1+\sqrt{1-4\mu_1/h}}, \quad \text{where } \mu_1 > 0, h < 0. \tag{26}$$

In terms of (25) and (26), we obtain the following result:

$$0 < \omega < \frac{4}{1+\sqrt{1+4\rho/|h|}} < 2, \text{ where } \rho > 0. \tag{27}$$

□

It is noticed that the condition of real eigenvalues of $Q^{-1}B^TA^{-1}B$ in Theorem 2 and Theorem 3 are reasonable. For example, if Q is positive definite, then all the eigenvalues of $Q^{-1}B^TA^{-1}B$ are real and positive; if Q is negative definite, then all the eigenvalues of $Q^{-1}B^TA^{-1}B$ are real and negative.

Form (27), we can see that $\lim_{|h|\to\infty} \frac{4}{1+\sqrt{1+4\rho/|h|}} = 2$. It means that $0 < \omega < 2$ holds. Moreover, we have $\lim_{|h|\to 0} \frac{4}{1+\sqrt{1+4\rho/|h|}} = 0$. It means that the upper bound of ω decrease as $|h|$ decrease. Namely, the convergence region of this method become narrow as h decrease. It is pointed out that when $h = 1$, $\alpha = 0$ and all eigenvalues μ of $Q^{-1}B^TA^{-1}B$ are real and positive, Gloub and co-works [12] obtained the following results $0 < \omega < \frac{4}{1+\sqrt{1+4\rho}}$.

3. Numerical experiments

In this section, we illustrate the effectiveness of the generalized SOR-like methods by using two numerical examples. Our first example is a system of purely algebraic equations [13]. We consider the matrices $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times m}(n \geq m)$ in problem (1) as follows

$$a_{ij} = \begin{cases} i+1, & i=j, \\ 1, & |i-j|=1, \\ 0, & \text{otherwise}; \end{cases} \qquad b_{ij} = \begin{cases} j, & i=j+n-m, \\ 0, & \text{otherwise}. \end{cases}$$

The right-hand side vectors b and q are taken such that the exact solutions x and y are both vectors with all components being 1.

Now, we use the special choices of Q for solving problem (1) in the following: Choice (1): $Q = B^TB$. In this case, the eigenvalues of $(B^TB)^{-1}B^TA^{-1}B$ are non-negative. Choice (2): $Q = B^TA^{-1}B$. In this case, $Q^{-1}B^TA^{-1}B = I$. Choice (3): $Q = I \in R^{n\times n}$, In this case, the eigenvalues of $B^TA^{-1}B$ are non-negative. Choice (4): $Q = A \in R^{n\times n}$ and has the same structure with $A \in R^{m\times m}$. In this case, the eigenvalues of $A^{-1}B^TA^{-1}B$ are non-negative.

For simplicity, we only take n=50 and m=40 in above choices. The stopping criterion is $\frac{||r^{(k)}||_2}{||r^{(0)}||_2} \leq 10^{-6}$, where $r^{(k)} = \begin{pmatrix} b \\ q \end{pmatrix} - \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix}$, and the maximum number of iterations is 10^4. Here, we take $r = \omega$ and the initial guess is the vector with all zeros.

All pictures report the values of relaxation parameter ω versus the number of iterations. It is pointed out that the circle line, the triangle (up) line, the solid line and the dashdot line denote the curve of iteration steps of cases $h = 10^{-4}$, 10^{-3}, 10^{-2} and 10^{-1} in the left figures of all pictures, respectively. The dashed line, the point line, the dotted line and the diamond line denote the curve of iteration steps of cases $h = 1$, 10, 10^2 and 10^3 in the right figures of all pictures, respectively. When $\alpha = 0$, $0.25h$, $0.5h$ and $0.75h$, Figure 1-4 refer to the number of iterations for the different values of h in different choices (1)-(4), respectively.

It is obvious that the iteration steps decrease in the beginning, and then the iteration steps increase as ω increase in the convergence region of the GSOR-like method. Moreover, the convergence region of the GSOR-like method become wide as h increase. From the numerical results, we can see that the trend derived from all pictures is almost monotone. We report the number of iterations (denoted by IT) and the optimum relaxation parameter (denoted by ω_{opt}) in Tables 1-4, for different choices of Q and α. It is clear that the optimum relaxation parameter of the GSOR-like method increase as h increase in the different cases of α. When $h = 10^{-3}$, 10^{-2} and 10^{-1}, the number of iterations of GSOR-like method are less then the number of iterations of SOR-like method in Choice (1). When $h = 10^{-1}$, the number of iterations of GSOR-like method are less then the number of iterations of SOR-like method in Choice (4). When $h = 1$, the number of iterations of SOR-like method are less then the number of iterations of GSOR-like method in Choices (2) and (3).

Our second example is the augmented linear system in [17], we consider $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times m}(n \geq m)$ in problem (1) as follows

Table 1. IT and $\omega_{\mathbf{opt}}$ for Choice (1).

α	h	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3
0	ω_{opt}	0.06	0.2	0.55	1.21	1.82	1.97	1.98	1.99
	IT	509	132	33	47	336	3117	10^4	10^4
$0.25h$	ω_{opt}	0.06	0.19	0.52	1.10	1.72	1.96	1.97	1.98
	IT	512	140	37	36	202	1605	10^4	10^4
$0.5h$	ω_{opt}	0.06	0.19	0.49	0.99	1.51	1.81	1.93	1.98
	IT	510	139	40	25	99	321	1113	3719
$0.75h$	ω_{opt}	0.06	0.18	0.46	0.85	1.20	1.31	1.33	1.33
	IT	513	147	44	15	47	79	112	1151

Table 2. IT and $\omega_{\mathbf{opt}}$ for Choice (2).

α	h	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3
0	ω_{opt}	0.01	0.06	0.17	0.51	1.00	1.69	1.95	1.99
	IT	2883	508	138	38	2	71	637	6259
$0.25h$	ω_{opt}	0.01	0.06	0.17	0.49	0.96	1.57	1.92	1.99
	IT	2649	497	156	40	8	45	333	3145
$0.5h$	ω_{opt}	0.01	0.06	0.17	0.47	0.92	1.37	1.74	1.91
	IT	2705	494	145	42	11	25	91	289
$0.75h$	ω_{opt}	0.01	0.06	0.17	0.43	0.84	1.12	1.30	1.33
	IT	2737	483	151	49	15	13	21	26

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in R^{2p^2 \times 2p^2},$$
$$B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in R^{2p^2 \times p^2}$$

and $T = \frac{1}{h^2} \cdot \mathrm{tridiag}(-1, 2, -1) \in R^{p \times p}, F = \frac{1}{h} \cdot \mathrm{tridiag}(-1, 1, 0) \in R^{p \times p}$, where $\otimes$ denotes the Kronecker product symbol, $h = \frac{1}{p+1}$ is the discretization mesh size and $S = \mathrm{tridiag}(a, b, c)$ is a tridiagonal matrix with $S_{i,i} = b$, $S_{i-1,i} = a$ and $S_{i,i+1} = c$ for appropriate i. For this example, we set $m = 2p^2$ and $n = p^2$. Hence, the total number of variables is $m+n = 3p^2$. In order to show differences, we choose the same choices of Q and take the same stopping criterion in the first example for numerical experiments. Here, we only consider $p = 9$.

Table 3. IT and $\omega_{\mathbf{opt}}$ for Choice (3).

α	h	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3
0	ω_{opt}	0.003	0.01	0.03	0.10	0.32	0.84	1.59	1.94
	IT	10^4	3614	1110	303	224	767	4047	10^4
$0.25h$	ω_{opt}	0.003	0.01	0.03	0.10	0.31	0.78	1.46	1.89
	IT	10^4	3596	1111	301	216	666	2797	10^4
$0.5h$	ω_{opt}	0.003	0.01	0.03	0.10	0.30	0.71	1.27	1.69
	IT	10^4	3584	1112	304	207	588	1848	5903
$0.75h$	ω_{opt}	0.003	0.01	0.03	0.10	0.28	0.65	1.07	1.28
	IT	10^4	3611	1115	305	208	512	1187	2009

Table 4. IT and ω_{opt} for Choice (4).

α	h	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3
0	ω_{opt}	0.02	0.06	0.20	0.57	1.27	1.84	1.98	1.99
	IT	1739	525	135	33	130	909	8487	10^4
$0.25h$	ω_{opt}	0.02	0.06	0.20	0.54	1.16	1.75	1.96	1.98
	IT	1729	523	136	36	103	536	4370	10^4
$0.5h$	ω_{opt}	0.02	0.06	0.19	0.51	1.03	1.54	1.82	1.94
	IT	1721	527	143	39	80	248	826	2595
$0.75h$	ω_{opt}	0.02	0.06	0.19	0.48	0.90	1.22	1.31	1.33
	IT	1742	528	145	44	63	116	221	313

Table 5. IT and ω_{opt} for different choice as $\alpha = 0$.

Q	h	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3
B^TB	ω_{opt}	0.09	0.28	0.74	1.47	1.91	1.95	1.97	1.99
	IT	329	88	44	165	1271	10^4	10^4	10^4
$B^TA^{-1}B$	ω_{opt}	0.01	0.06	0.18	0.48	1.00	1.60	1.94	1.99
	IT	2530	508	154	37	2	52	391	3814
I	ω_{opt}	0.001	0.06	0.18	0.52	1.15	1.79	1.97	1.98
	IT	10^4	554	158	41	43	263	2448	10^4
A	ω_{opt}	0.13	0.39	0.94	1.66	1.95	1.96	1.97	1.99
	IT	219	58	46	194	1848	10^4	10^4	10^4

We show the number of iterations as ω increases in Figure 5 and report the number of iterations and the optimum relaxation parameter in Table 5, for different choices of Q and $\alpha = 0$. It is clear that the optimum relaxation parameter of the GSOR-like method increases as h increases. When $h = 10^{-4}$, 10^{-3}, 10^{-2} and 10^{-1}, the number of iterations of GSOR-like method are less then the number of iterations of SOR-like method in Choices (1) and (4). When $h = 1$, the number of iterations of SOR-like method are less then the number of iterations of GSOR-like method in Choices (2) and (3).

From a computational point of view, the preconditioned matrixes (1) and (4) are the best choices for the GSOR-like methods. The preconditioned matrix (2) is the best choice for the SOR-like methods. It is pointed out that if it is not easy to invert A, we can apply the inexact Uzawa method instead of A^{-1} for our algorithms. Especially, the authors proposed the nonlinear inexact Uzawa with mixed iteration method in [13]. Let $\hat{A}$ and $\hat{C}$ be two positive definite matrices, which are assumed to be the preconditioners of the matrices A and $C = B^TA^{-1}B$, respectively. In example 1, we use the above method instead of A^{-1} and choose the Jacobi preconditioner and the identity matrix to be the preconditioner for A and the approximative Schur complement $B^T\hat{A}^{-1}B$, respectively. In example 2, we take the multigrid preconditioner to be the preconditioner $\hat{A}$ and the identity matrix to be the preconditioner $\hat{C}$ for the Schur complement $C = B^TA^{-1}B$. The approximation $\Psi_A(\phi)$ is taken to be $\hat{A}^{-1}\phi$ for any ϕ in the nonlinear inexact Uzawa with mixed iteration method. And the approximation $\Psi_H(g_i)$ is generated by two conjugate gradient iterations for solving $H\psi = g_k = B^Tx_{k+1} - q$, where $H = B^T\hat{A}^{-1}B$. Here, we shall not have a detailed discussion.

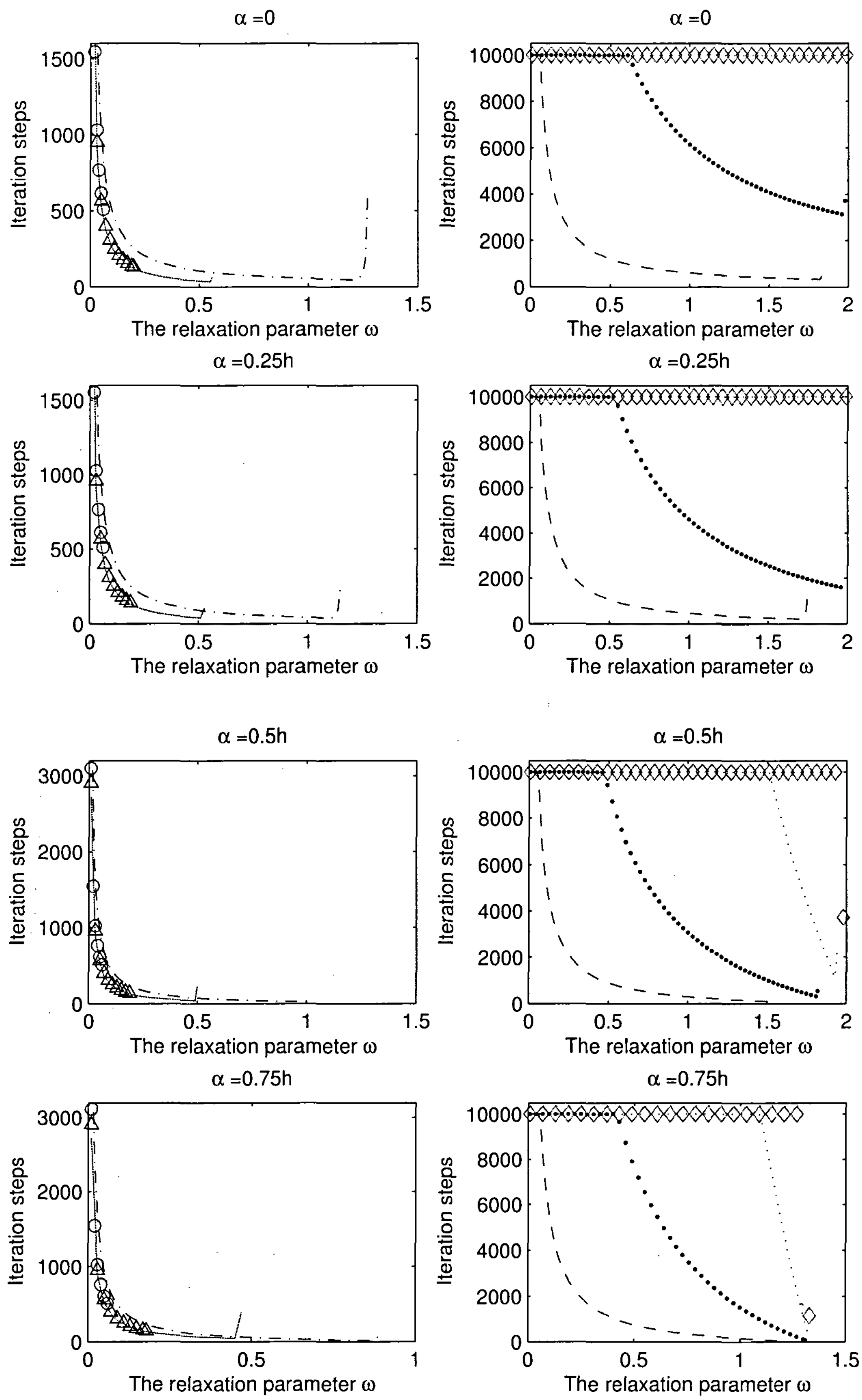

FIGURE 1. When $\alpha = 0$, $0.25h$, $0.5h$ and $0.75h$ respectively, the iteration steps as ω increase in Choice (1).

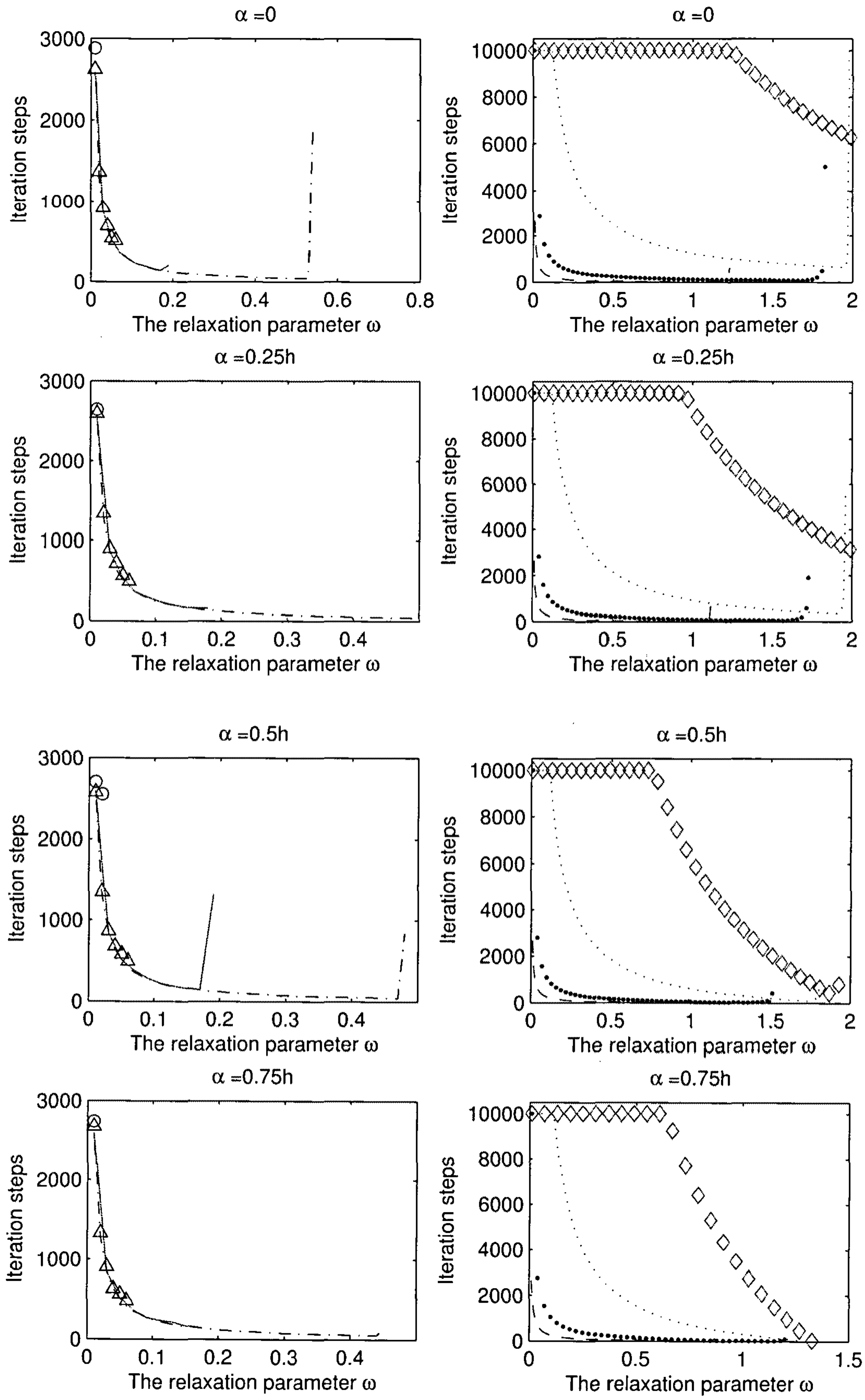

FIGURE 2. When $\alpha = 0$, $0.25h$, $0.5h$ and $0.75h$ respectively, the iteration steps as ω increase in Choice (2).

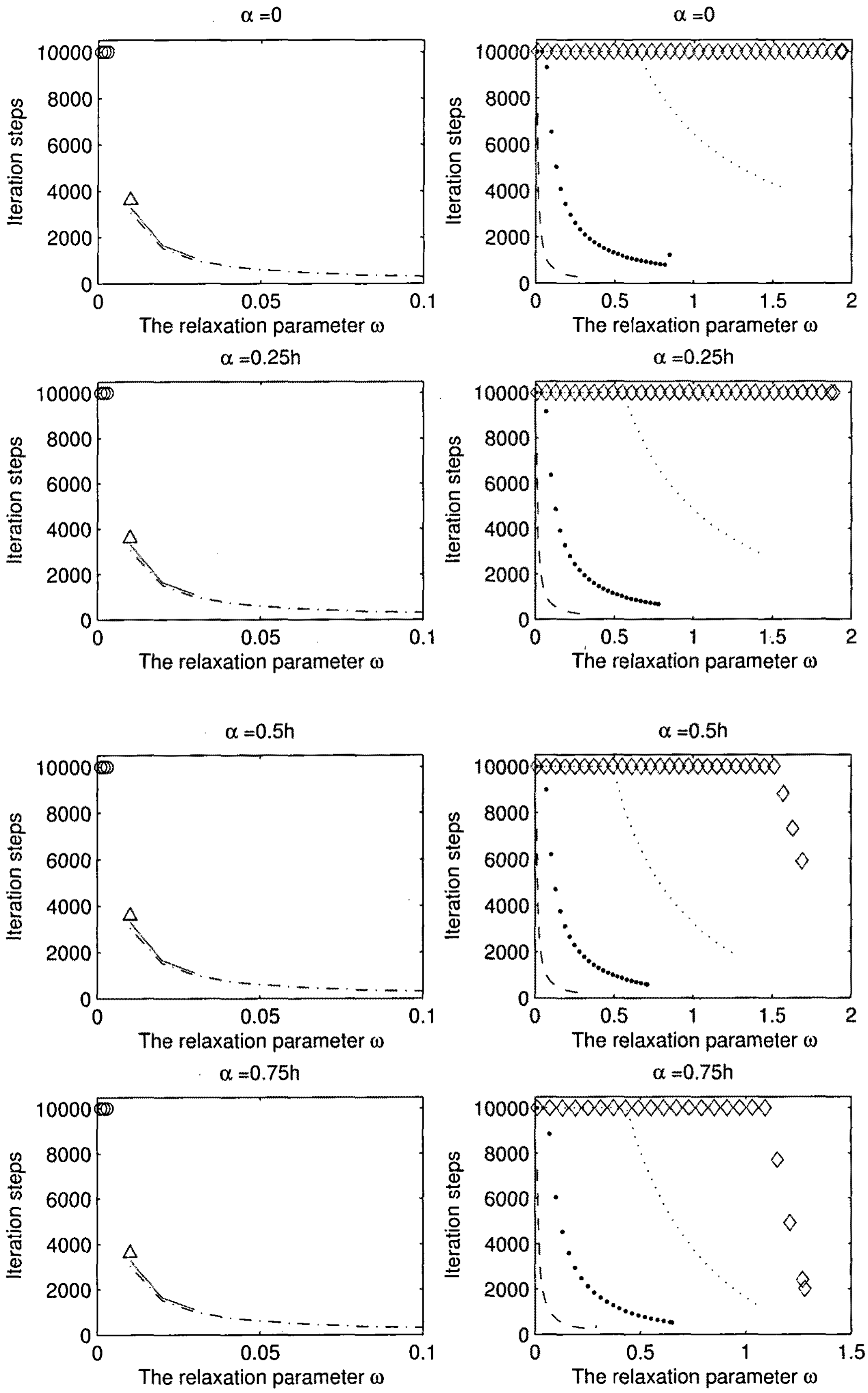

FIGURE 3. When $\alpha = 0$, $0.25h$, $0.5h$ and $0.75h$ respectively, the iteration steps as ω increase in Choice (3).

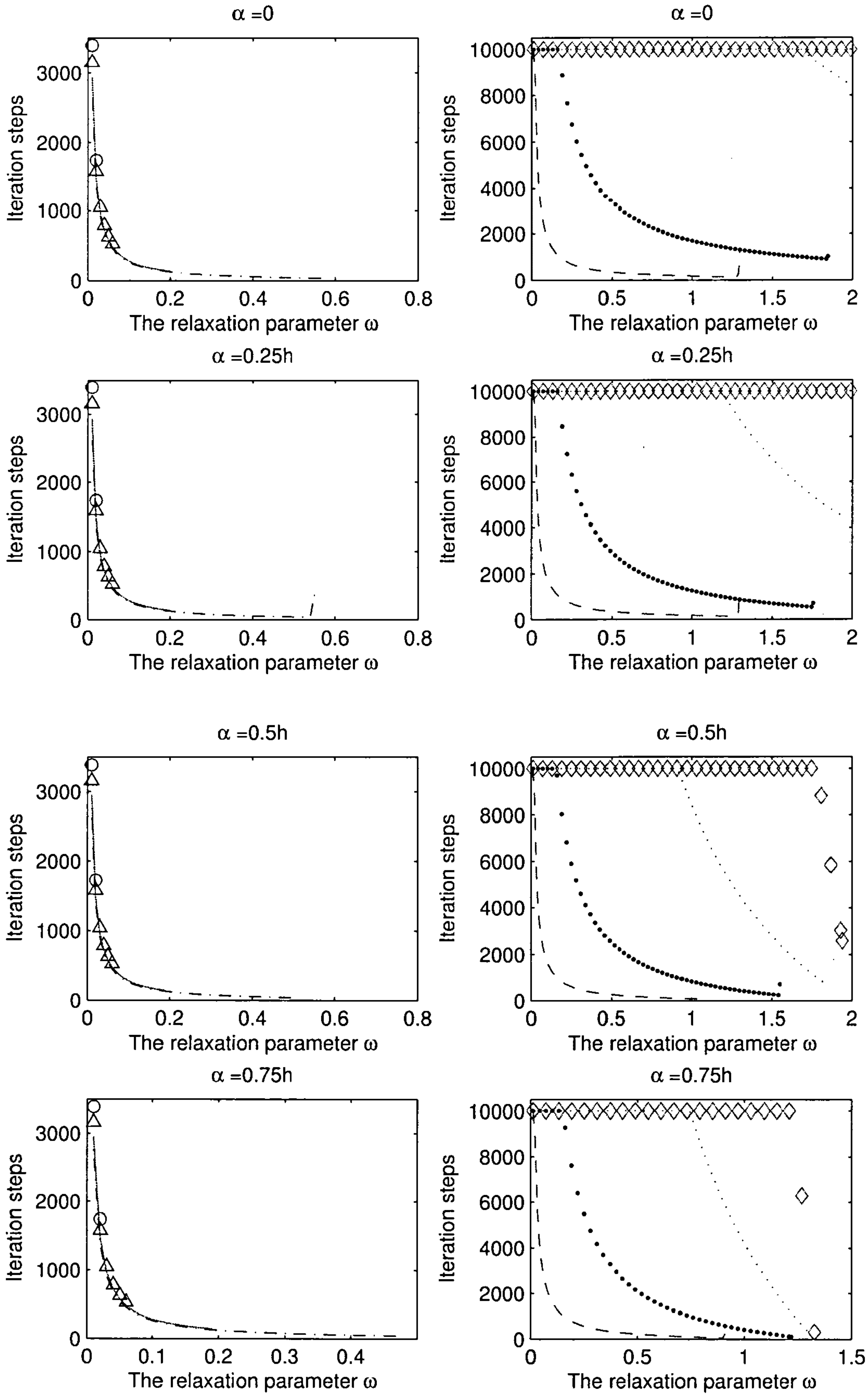

FIGURE 4. When $\alpha = 0$, $0.25h$, $0.5h$ and $0.75h$ respectively, the iteration steps as ω increase in Choice (4).

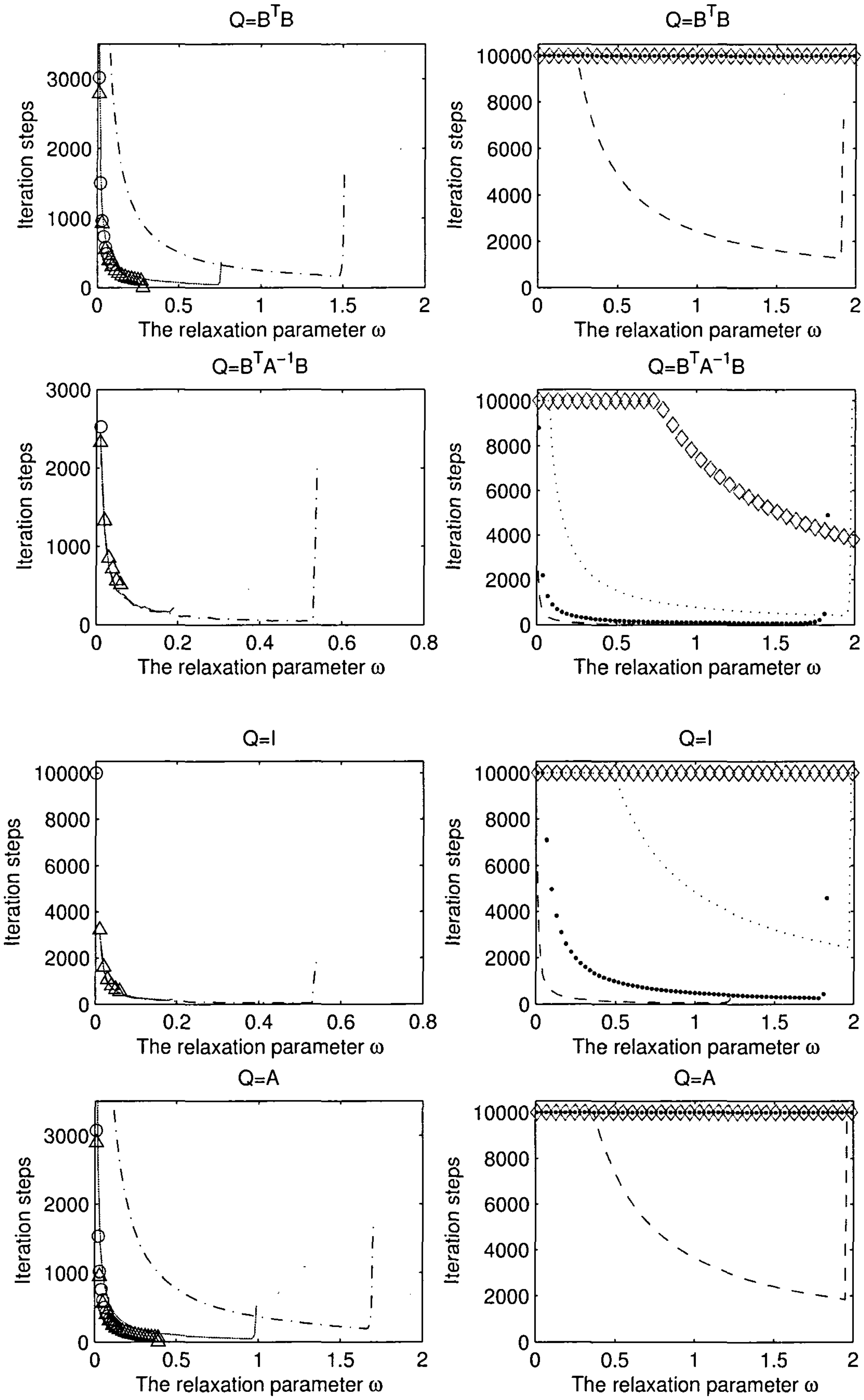

FIGURE 5. When $\alpha = 0$, the iteration steps as ω increase in Choices (1)-(4) respectively.

In a word, the GSOR-like method will converge faster than the SOR-like method by the choice of proper parameter values. Moveover, the convergence region of the GSOR-like method become wider as the parameter values increase. The determination of optimum value of the parameters need further studies.

REFERENCES

1. Z.Z Bai, B.N. Parlett and Z.Q Wang, *On generalized successive overrelaxation methods for augmented linear systems*, Numer. Math. **102**(2005), 1-38.
2. Z.Z. Bai and Z.Q. Wang, *On parameterized inexact Uzawa methods for generalized saddle point problems*, Linear Algebra Appl. **428**(2008), 2900-2932.
3. M. Benzi, M.J. Gander and G.H. Golub, *Optimization of the Hermitian and skew-Hermitian splitting iteration for saddle-point problems*, BIT, **43**(2003), 881-900.
4. J.H. Bramble, J.E. Pasciak and A.T. Vassilev, *Analysis of the inexact Uzawa algorithm for saddle point problems*, SIAM J. Numer. Anal. **34**(1997), 1072-1092.
5. Z.H. Cao, *Constraint Schur complement preconditioners for nonsymmetric saddle point problems*, Appl. Numer. Math.**59** (2009),151-169.
6. X. Chen,*On preconditioned Uzawa methods and SOR methods for saddle-point problems*, J. Comput. Appl. Math. **100** (1998), 207-224.
7. M.R. Cui, *Analysis of iterative algorithms of Uzawa type for saddle point problems*, Appl. Numer. Math.**50** (2004), 133-146.
8. H.C. Elman, *Preconditioned for saddle point problems arising in computational fluid dynamics*, Appl. Numer. Math. **43** (2002), 75-89.
9. H.C. Elman and G.H. Golub, *Inexact and preconditioned Uzawa algorithms for saddle point problems*, SIAM J. Numer. Anal. **31** (1994), 1645-1661.
10. X.L. Feng, Y.N. He and J.X. Meng, *Application of modified homotopy perturbation method for solving the augmented systems*, J. Comput. Appl. Math. **231** (2009), 288-301.
11. B. Fischer, A. Ramage, D.J. Silvester and A.J. Wathen, *Minimum residual methods for augmented systems*, BIT, **38** (1998), 527-543.
12. G.H. Golub, X. WU and J.Y. Yuan, *SOR-Like methods for augmented systems*, BIT, **41** (2001), 71-85.
13. Q.Y. Hu and J. Zou,*Nonlinear inexact Uzawa algorithms foe linear and nonlinear saddle-point problems*, SIAM J. Optim. **16** (2006), 798-825.
14. C.J. Li, Z. Li, X.H. Shao, Y.Y. Nie and D.J. Evans, *Optimum parameter for the SOR-like method for the augmented systems*, Int. J. Comput. Math. **78** (2004), 749-764.
15. X.H. Shao, H.L. Shen, C.J. Li and T. Zhang, *Generalized AOR method for augmented systems*, Journal on Numerical Methods and Computer Applications. **27** (2006), 241-248.
16. D.M.Young, *Iterative Solution for Large Linear Systems*, Academic Press, NewYork, 1971.
17. G.F. Zhang and Q.H. Lu, *On generalized symmetric SOR method for augmented systems*, J. Comput. Appl. Math. **219**(2008), 51-58.
18. Z.Q. Wang, *Optimization of the parameterized Uzawa preconditioners for saddle point matrices*, J. Comput. Appl. Math. **226** (2009), 136-154.

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