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INTERVAL-VALUED FUZZY REGULAR LANGUAGE

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ABSTRACT. In this paper, a definition of interval-valued fuzzy regular language (IVFRL) is proposed and their related properties studied. A model of finite automaton (DFA and NDFA) with interval-valued fuzzy transitions is proposed. Acceptance of interval-valued fuzzy regular language by the finite automaton (DFA and NDFA) with interval-valued fuzzy transitions are examined. Moreover, a definition of finite automaton (DFA and NDFA) with interval-valued fuzzy (final) states is proposed. Acceptance of interval-valued fuzzy regular language by the finite automaton (DFA and NDFA) with interval-valued fuzzy (final) states are also discussed. It is observed that, the model finite automaton (DFA and NDFA) with interval-valued fuzzy (final) states is more suitable than the model finite automaton (DFA and NDFA) with interval-valued fuzzy transitions for recognizing the interval-valued fuzzy regular language. In the end, interval-valued fuzzy regular expressions are defined. We can use the proposed interval-valued fuzzy regular expressions in lexical analysis.

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1. Introduction

Fuzzy languages and grammar were formerly defined by Lee and Zadeh [3]. A fuzzy language '$\widetilde{L}$', in the set of finite alphabet Σ, is a class of strings $w \in \Sigma^*$ along with a grade of membership function $\mu_{\widetilde{L}}(w)$. This membership function $\mu_{\widetilde{L}}(w)$, $w \in \Sigma^*$, assigns to each string a grade of membership in [0, 1][5].

Using the concept of interval-valued fuzzy sets, an attempt has been made to generalize this membership function. The language so obtained is termed as an interval-valued fuzzy language. Here, the membership degree of each string is a closed subinterval in [0,1]. This membership function approximates the correct (but unknown) membership degree of each string in the given language. If each

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string of such language is regular, then the language is termed as interval-valued fuzzy regular language. The concept of interval-valued fuzzy language provides us with a flexible mathematical framework to cope with imprecise information. The finite automaton (DFA and NDFA) accepting these interval-valued fuzzy regular language, is termed as an interval-valued fuzzy automaton. An interval-valued fuzzy regular language can be generated by interval-valued fuzzy regular expression.

The main contributions of this paper are fourfold. Firstly, we have stated the basic definitions of interval-valued fuzzy set, fuzzy language and fuzzy regular language which has been already done. Secondly, we have proposed the definition of interval-valued fuzzy language and interval-valued fuzzy regular language and studied their related properties. Thirdly, we have discussed the acceptance of interval-valued fuzzy regular language through finite automaton (DFA and NDFA) with interval-valued fuzzy transitions and interval-valued fuzzy (final) states. Finally, we have proposed the definition of interval-valued fuzzy regular expressions.

2. Preliminaries

Definition 2.1. Let 'U' denote a universe of discourse. Let $I[0,1]$ denote the set of all closed subintervals of the interval $[0,1]$. An interval-valued fuzzy set (IVFS for short) 'A' is a mapping $\mu_A : U \rightarrow I[0,1]$.

For all $u \in U$, $\mu_A(u) = [\mu_A^l(u), \mu_A^u(u)]$, where $\mu_A^l(u) : U \rightarrow [0,1]$ and $\mu_A^u(u) : U \rightarrow [0,1]$ represents the lower and the upper membership values of each element $u \in U$ in 'A' such that

$$0 \leq \mu_A^l(u) \leq \mu_A^u(u) \leq 1, \ \forall u \in U.$$

Therefore, an interval-valued fuzzy set is characterized by an interval-valued membership function μ_A denoted as $A = \{(u, \mu_A(u)) \mid u \in U\}$.

Definition 2.2. Let Σ be a finite alphabet set and $f : \Sigma^* \rightarrow M$ a function, where M is a set of real numbers in $[0,1]$. Then the set $\widetilde{L} = \{(w, f(w)) \mid w \in \Sigma^*\}$ is called a fuzzy language [5] over Σ and f the membership function of $\widetilde{L}$.

Definition 2.3. Let '$\widetilde{L}$' be a fuzzy language over Σ the finite alphabet set and $f_{\widetilde{L}} : \Sigma^* \rightarrow M$, where M is a set of real numbers in $[0,1]$ the membership function of '$\widetilde{L}$'. Then, '$\widetilde{L}$' is a regular fuzzy language [5] if

(1) the set $\{m \in M \mid S_{\widetilde{L}}(m) \neq \emptyset\}$ is finite and
(2) for each $m \in M$, $S_{\widetilde{L}}(m)$ is regular, where $S_{\widetilde{L}}(m) = \{w \in \Sigma^* \mid f_{\widetilde{L}}(w) = m\}$ and $f_{\widetilde{L}}$ the membership function of '$\widetilde{L}$'.

3. Interval-valued fuzzy regular language

Definition 3.1. Let Σ be a finite alphabet set. Then we call the set $\widetilde{L} = \{(w, [f_{\widetilde{L}}^L(w), f_{\widetilde{L}}^U(w)]) \mid w \in \Sigma^*\}$ an interval-valued fuzzy language (IVFL for short), where $f_{\widetilde{L}}^L(w) : \Sigma^* \rightarrow [0,1], f_{\widetilde{L}}^U(w) : \Sigma^* \rightarrow [0,1]$ represents the lower and the upper membership functions of $\widetilde{L}$ respectively. For any $w \in \Sigma^*, 0 \le f^L_{\widetilde{L}}(w) \le f^U_{\widetilde{L}}(w) \le 1$.

In short, $\widetilde{L} = \{(w, f_{\widetilde{L}}(w)) \mid w \in \Sigma^*\}$, where $f_{\widetilde{L}}(w) = [f^L_{\widetilde{L}}(w), f^U_{\widetilde{L}}(w)]\ \forall w \in \Sigma^*$. In particular, $f_{\widetilde{L}}(w) : \Sigma^* \to I[0,1]$ denotes the membership function of $\widetilde{L}$. Here, $f_{\widetilde{L}}(w) \in I[0,1]$ and not a real number in $[0,1]$ which is assigned to each string of the language $\widetilde{L}$.

Example 3.1. If $\Sigma = \{a, b\}$ and $f_{\widetilde{L}}(w) : \Sigma^* \to I[0,1]$, then $\widetilde{L} = \{(a^*, [0.3, 0.5]), (a^*b, [0.4, 0.7]) \mid a^*, a^*b \in \Sigma^*\}$ represents an IVFL.

Let '$\widetilde{L}$' be an IVFL over Σ the finite alphabet set and $f_{\widetilde{L}}(w) : \Sigma^* \to I[0,1]$ the membership function of '$\widetilde{L}$'. Then, for each $[m,n] \in I[0,1]$, denote by $S_{\widetilde{L}}[m,n]$ the set

$$S_{\widetilde{L}}[m,n] = \{w \in \Sigma^* \mid f_{\widetilde{L}}(w) = [m,n]\}.$$

Note that $S_{\widetilde{L}}$ as a function is just $f^{-1}_{\widetilde{L}}$.

Definition 3.2. Let '$\widetilde{L}$' be an IVFL over Σ the finite alphabet set and $f_{\widetilde{L}}(w) : \Sigma^* \to I[0,1]$ the membership function of '$\widetilde{L}$'. Then we call '$\widetilde{L}$' an interval-valued fuzzy regular language (IVFRL for short) if;

(1) the set $\{[m,n] \in I[0,1] \mid S_{\widetilde{L}}[m,n] \neq \emptyset\}$ is finite and
(2) for each $[m,n] \in I[0,1]$ the string $S_{\widetilde{L}}[m,n]$ is regular.

Definition 3.3. Let $\widetilde{L}_1$ and $\widetilde{L}_2$ be two interval-valued fuzzy languages over Σ the finite alphabet set with the membership function $f_{\widetilde{L}_1}$ and $g_{\widetilde{L}_2}$ respectively. Then the basic operations such as union, intersection, complement, concatenation and star operations on $\widetilde{L}_1$ and $\widetilde{L}_2$ can be defined in the following way.

(1) Union: The union of $\widetilde{L}_1$ and $\widetilde{L}_2$ is defined by
$\widetilde{L} = \widetilde{L}_1 \cup \widetilde{L}_2 = \{(w, [max\{f^L_{\widetilde{L}_1}(w), g^L_{\widetilde{L}_2}(w)\}, max\{f^U_{\widetilde{L}_1}(w), g^U_{\widetilde{L}_2}(w)\}]) \mid w \in \Sigma^*\}$.
(2) Intersection: The intersection of $\widetilde{L}_1$ and $\widetilde{L}_2$ is defined by
$\widetilde{L} = \widetilde{L}_1 \cap \widetilde{L}_2 = \{(w, [min\{f^L_{\widetilde{L}_1}(w), g^L_{\widetilde{L}_2}(w)\}, min\{f^U_{\widetilde{L}_1}(w), g^U_{\widetilde{L}_2}(w)\}]) \mid w \in \Sigma^*\}$.
(3) Complement: The complement of $\widetilde{L}_1$ is defined by
$$\widetilde{L} = \widetilde{L}_1^c = \{(w, [1 - f^U_{\widetilde{L}_1}(w), 1 - f^L_{\widetilde{L}_1}(w)]) \mid w \in \Sigma^*\}.$$
(4) Concatenation: The concatenation of $\widetilde{L}_1$ and $\widetilde{L}_2$ is defined by
$\widetilde{L} = \widetilde{L}_1 \cdot \widetilde{L}_2 = \{(w, [max\{min(f^L_{\widetilde{L}_1}(x), g^L_{\widetilde{L}_2}(y))\}, max\{min(f^U_{\widetilde{L}_1}(x), g^U_{\widetilde{L}_2}(y))\}]) \mid w = xy, x, y \in \Sigma^*\}, w \in \Sigma^*$.
(5) Star: The star operation on $\widetilde{L}_1$ is defined by
$\widetilde{L} = \widetilde{L}_1^* = \{(w, [max\{min(f^L_{\widetilde{L}_1}(x_1), f^L_{\widetilde{L}_1}(x_2), ..., f^L_{\widetilde{L}_1}(x_n))\}, max\{min (f^U_{\widetilde{L}_1}(x_1), f^U_{\widetilde{L}_1}(x_2), ..., f^U_{\widetilde{L}_1}(x_n))\}]) \mid w = x_1x_2x_3...x_n, x_1, x_2, ..., x_n \in \Sigma^*,$

$n \geq 0\}$ assuming that $min\lambda = [0, 0]$, where λ being the empty string.
Hence, we have $\widetilde{L}^* = \bigcup_{i=0}^{\infty} \widetilde{L}^i = \widetilde{L}^0 \cup \widetilde{L}^1 \cup \widetilde{L}^2 \cup \ldots$. i.e., Kleene Closure is satisfied. The $+$ operation on $\widetilde{L}_1$ is defined by
$\widetilde{L} = \widetilde{L}_1^+ = \{(w, [max\{min(f_{\widetilde{L}_1}^L(x_1), f_{\widetilde{L}_1}^L(x_2), ..., f_{\widetilde{L}_1}^L(x_n))\}, max\{min(f_{\widetilde{L}_1}^U(x_1), f_{\widetilde{L}_1}^U(x_2), ..., f_{\widetilde{L}_1}^U(x_n))\}]) \mid w = x_1x_2x_3...x_n, x_1, x_2, ..., x_n \in \Sigma^*, n \geq 1\}$.
Hence, we have $\widetilde{L}^+ = \bigcup_{i=1}^{\infty} \widetilde{L}^i = \widetilde{L}^1 \cup \widetilde{L}^2 \cup \ldots$. i.e., Positive Closure is satisfied.

Since interval-valued fuzzy languages are the special class of interval-valued fuzzy sets, the equivalence and inclusion relations between two interval-valued fuzzy languages are the equivalence and inclusion relations between two interval-valued fuzzy sets.

Let $\widetilde{L}_1$ and $\widetilde{L}_2$ be two interval-valued fuzzy languages over Σ the finite alphabet set. Then
$\widetilde{L}_1 = \widetilde{L}_2$ iff $f_{\widetilde{L}_1}(w) = g_{\widetilde{L}_2}(w)$, $\forall w \in \Sigma^*$ and
$\widetilde{L}_1 \subseteq \widetilde{L}_2$ iff $f_{\widetilde{L}_1}(w) \subseteq g_{\widetilde{L}_2}(w)$, $\forall w \in \Sigma^*$.

Theorem 3.1. *Interval-valued fuzzy regular languages are closed under union, intersection, complement, concatenation and star operations.*

Proof. Let $\widetilde{L}_1$ and $\widetilde{L}_2$ be two IVFRL's over Σ the finite alphabet set. Let $f_{\widetilde{L}_1} : \Sigma^* \to M$ and $g_{\widetilde{L}_2} : \Sigma^* \to N$ be the membership function of $\widetilde{L}_1$ and $\widetilde{L}_2$ respectively, where M and N represents the set of all closed subintervals in [0, 1]. Obviously, $O \subseteq M \cup N$ (in the case of union, intersection, concatenation or star operation) and $O = \{[1 - f_{\widetilde{L}_1}^U, 1 - f_{\widetilde{L}_1}^L] \mid [f_{\widetilde{L}_1}^L, f_{\widetilde{L}_1}^U] \in M\}$ (in the case of complementation) is finite and corresponding strings are regular. Note that, O represents the membership function of new IVFRL obtained after an operation (union, intersection, complement, concatenation or star). It is described as follows.

(1) Union:

$$S_{\widetilde{L}}[m,n] = \begin{cases} S_{\widetilde{L}_1}[m,n] - \cup_{[m,n]' > [m,n]} S_{\widetilde{L}_2}[m,n]', & \text{if } [m,n] \in M - N, \\ S_{\widetilde{L}_2}[m,n] - \cup_{[m,n]' > [m,n]} S_{\widetilde{L}_1}[m,n]', & \text{if } [m,n] \in N - M, \\ ((S_{\widetilde{L}_1}[m,n] \cup S_{\widetilde{L}_2}[m,n]) - \cup_{[m,n]' > [m,n]} S_{\widetilde{L}_1}[m,n]') & \\ - \cup_{[m,n]'' > [m,n]} S_{\widetilde{L}_2}[m,n]'', & \text{if } [m,n] \in M \cap N. \end{cases}$$

(2) Intersection:

$$S_{\widetilde{L}}[m,n] = \begin{cases} S_{\widetilde{L}_1}[m,n] - \cup_{[m,n]' < [m,n]} S_{\widetilde{L}_2}[m,n]', & \text{if } [m,n] \in M - N, \\ S_{\widetilde{L}_2}[m,n] - \cup_{[m,n]' < [m,n]} S_{\widetilde{L}_1}[m,n]', & \text{if } [m,n] \in N - M, \\ ((S_{\widetilde{L}_1}[m,n] \cup S_{\widetilde{L}_2}[m,n]) - \cup_{[m,n]' < [m,n]} S_{\widetilde{L}_1}[m,n]') & \\ - \cup_{[m,n]'' < [m,n]} S_{\widetilde{L}_2}[m,n]'', & \text{if } [m,n] \in M \cap N. \end{cases}$$

(3) Complement:
$O = \{([1-n, 1-m]) \mid [m,n] \in M\}, \qquad S_{\widetilde{L}}([m,n]) = S_{\widetilde{L}_1}[1-n, 1-m].$

(4) Concatenation:
$$S_{\widetilde{L}}[m,n] = \bigcup_{\substack{min([m_1,n_1],[m_2,n_2])=[m,n],\\ [m_1,n_1]\in M\\ [m_2,n_2]\in N}} S_{\widetilde{L}_1}[m_1,n_1] S_{\widetilde{L}_2}[m_2,n_2] - \bigcup_{\substack{min([m_1,n_1]',[m_2,n_2]')>[m,n]\\ [m_1,n_1]'\in M\\ [m_2,n_2]'\in N}} S_{\widetilde{L}_1}[m_1,n_1]' S_{\widetilde{L}_2}[m_2,n_2]'.$$

(5) Star:
Assuming that $M = \{[m_1,n_1],[m_2,n_2],\ldots,[m_l,n_l]\}$ and
$[1,1] \geq [m_1,n_1] > [m_2,n_2] > \ldots > [m_l,n_l] \geq [0,0]$.
$S_{\widetilde{L}}[m_1,n_1] = (S_{\widetilde{L}_1}[m_1,n_1])^*$, if $[m_1,n_1] = [1,1], S_{\widetilde{L}}[1,1] = \lambda$ and
$S_{\widetilde{L}}[m_1,n_1] = (S_{\widetilde{L}_1}[m_1,n_1])^+ - \lambda$, if $[m_1,n_1] \neq [1,1]$.
$S_{\widetilde{L}}[m_i,n_i] = (\bigcup_{j\leq i} S_{\widetilde{L}_1}[m_j,n_j])^+ - (\bigcup_{k<i} S_{\widetilde{L}}[m_k,n_k]) - \lambda$, if $1 < i \leq l$.

Hence Kleene Closure is satisfied. □

4. Finite automata with interval-valued fuzzy transitions

Definition 4.1. A nondeterministic finite automata with interval-valued fuzzy transitions (NDFA-IVFT for short) $\widetilde{A}$ is a 5-tuple
$\widetilde{A} = (Q, \Sigma, \widetilde{\delta}, S, F)$, where Q = set of finite states, Σ = set of finite input alphabets, $\widetilde{\delta} : Q \times \Sigma \times Q \rightarrow I[0,1]$ is the degree function of state transitions, S is the initial state and $F \subseteq Q$ represents the set of final states.

For $x \in \Sigma^*$ and p, q $\in Q$, define

$$\widetilde{\delta}^*(p,x,q) = \begin{cases} [0,0], & \text{if } x = \lambda \text{ and } p \neq q, \\ [1,1], & \text{if } x = \lambda \text{ and } p = q, \\ \max_{r\in Q}\{\min(\widetilde{\delta}^*(p,x',r), \widetilde{\delta}(r,a,q)) \mid x = x'a \forall x' \in \Sigma^*, a \in \Sigma\}, \\ \text{otherwise.} \end{cases}$$

Then we say that, the string $x \in \Sigma^*$ is accepted by $\widetilde{A}$ with the degree $d_{\widetilde{A}}(x)$, where $d_{\widetilde{A}}(x) = \max\{\widetilde{\delta}^*(s,x,q) \mid q \in F\}$.
Also, we denote it as $\widetilde{L}(\widetilde{A})$ the set $\widetilde{L}(\widetilde{A}) = \{(x, d_{\widetilde{A}}(x)) \mid x \in \Sigma^*\}$.

Example 4.1.
In the above example,

$$\widetilde{L}(\widetilde{A}) = \{(x, [0.3, 0.7]) \mid x \in ab^*ab^*\} \cup \{(y, [0.3, 0.8]) \mid y \in ab^*\}$$

Definition 4.2. A deterministic finite automaton with interval-valued fuzzy transitions (DFA-IVFT for short) is a NDFA-IVFT with the condition that for each $p \in Q$ and $a \in \Sigma$, if $\widetilde{\delta}(p,a,q) > [0,0]$ and $\widetilde{\delta}(p,a,q') > [0,0]$ then $q = q'$.

Theorem 4.1. *'$\widetilde{L}$' is an interval-valued fuzzy regular language if and only if $\widetilde{L}$ is accepted by a NDFA-IVFT $\widetilde{A}$ with the exception of λ the empty string.*

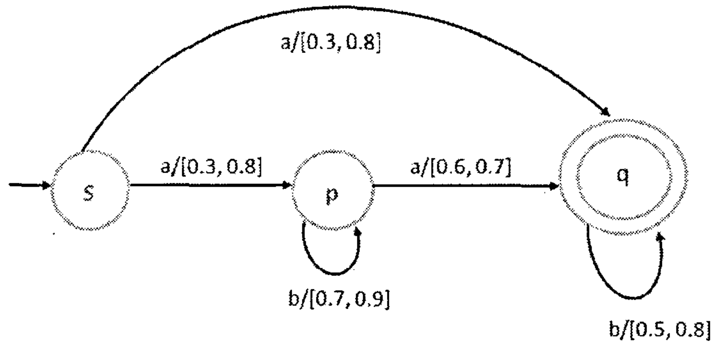

FIGURE 1. NDFA-IVFT

Proof. Let $\widetilde{L}$ be an interval-valued fuzzy regular language and $f_{\widetilde{L}} : \Sigma^* \to M$ be the membership function of $\widetilde{L}$, where M is the set of all closed subintervals in [0, 1]. Let $M = \{m_1, m_2, ..., m_n\}$ for some $n \geq 1$. Then $S_{\widetilde{L}}(m_i)$ is regular for each $m_i \in M, 1 \leq i \leq n$. Note that $S_{\widetilde{L}}(m_i) \cap S_{\widetilde{L}}(m_j) = \emptyset$ for $i \neq j$ since, $f_{\widetilde{L}}$ is a function.

Let $A_i = (Q_i, \Sigma, \delta_i, S_i, F_i)$ be a DFA (or a NDFA) such that $S_{\widetilde{L}}(m_i) = L(A_i)$, $1 \leq i \leq n$.
We construct $\widetilde{A}_i = (Q_i, \Sigma, \widetilde{\delta}_i, S_i, F_i)$, where

$$\widetilde{\delta}_i(p, a, q) = \begin{cases} m_i, & \text{if } (p, a, q) \in \delta_i, \\ [0,\ 0], & \text{otherwise} \end{cases}$$

We assume that $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Define, $\widetilde{A} = (Q, \Sigma, \widetilde{\delta}, S, F)$ such that $Q = Q_1 \cup Q_2 \cup ... \cup Q_n \cup \{S\}$, $S \notin Q_1 \cup Q_2 \cup ... \cup Q_n$, $F = F_1 \cup F_2 \cup ... \cup F_n$,

$$\widetilde{\delta}(p, a, q) = \begin{cases} \widetilde{\delta}_i(p, a, q), & \text{if } p, q \in Q_i \text{ for some } i \in 1, 2, ..., n, \\ \widetilde{\delta}_i(S_i, a, q), & \text{if } p = S \text{ and } q \in Q_i \text{ for some } i \in 1, 2, ..., n, \\ [0,\ 0], & \text{otherwise} \end{cases}$$

It is clear that, $\widetilde{A}$ accepts $\widetilde{L}$ with the exception of λ the empty string.

Conversly, Let $\widetilde{A} = (Q, \Sigma, \widetilde{\delta}, S, F)$ be a NDFA-IVFT. Define an interval-valued fuzzy language $\widetilde{L}$ with $f_{\widetilde{L}}(w) = d_{\widetilde{A}}(w)$ for each $w \in \Sigma^*$ (here $f_{\widetilde{L}}(\lambda) = [0, 0]$). Now we need to show that $\widetilde{L}$ is regular (i.e., IVFRL).

Let $M = \{[m, n] \mid \widetilde{\delta}(p, a, q) = [m, n]$ for some $p, q \in Q, a \in \Sigma\}$. Obviously, M is finite. Assume that $M = \{m_1, m_2, ..., m_n\}$ with $m_1 > m_2 > ... > m_n$, $n \geq 1$ for each i, $1 \leq i \leq n$, define a NDFA-IVFT $A_i = (Q, \Sigma, \delta_i, S, F)$, where $\delta_i = \{(p, a, q) \mid \widetilde{\delta}(p, a, q) \geq m_i\}$. Define the languages L_i, $1 \leq i \leq n$, as shown below in the increasing sequence of i:

$L_1 = L(A_1)$,
$L_2 = L(A_2) - L(A_1)$,
.
.

$L_i = L(A_i) - \bigcup_{j=1}^{i-1} L_j$.
Then $L_i = S_{\widetilde{L}}(m_i)$ and L_i is regular for each i, $1 \leq i \leq n$. Hence, $\widetilde{L}$ is an IVFRL. □

Theorem 4.2. *Let $\widetilde{L}$ be an interval-valued fuzzy regular language. Then $\widetilde{L}$ is accepted by a DFA-IVFT iff it satisfies the following condition: For $x, y \in \Sigma^+$, $u \in \Sigma^*$*

(1) $x = yu$ and $f_{\widetilde{L}}(y) > [0,0]$ implies that $f_{\widetilde{L}}(x) \leq f_{\widetilde{L}}(y)$.

Proof. Let $\widetilde{L}$ be accepted by a DFA-IVFT $\widetilde{A} = (Q, \Sigma, \widetilde{\delta}, S, F)$. We show that $\widetilde{L}$ satisfies the given condition (i.e.,(1)). Let $x = yu$ for $x, y \in \Sigma^+$, and $u \in \Sigma^*$. If $d_{\widetilde{A}}(x) = 0$ then, $f_{\widetilde{L}}(x) \leq f_{\widetilde{L}}(y)$ is true. Otherwise,
$f_{\widetilde{L}}(x) = d_{\widetilde{A}}(x) = min\{\widetilde{\delta}^*(s,y,q), \widetilde{\delta}^*(q,u,f)\} \leq \widetilde{\delta}^*(s,y,q) = d_{\widetilde{A}}(y) = f_{\widetilde{L}}(y)$, where $q, f \in F$.

For the reverse proof, let $\widetilde{L}$ be an IVFRL, with $f_{\widetilde{L}} : \Sigma^* \to M$ as membership function, where M is the set of all closed subintervals in [0, 1] satisfy the given condition (i.e.,(1)). Assume that $M = \{m_1, m_2, m_3, ..., m_n\}$ for some $n \geq 1$. It is clear from the theorem 4.1 that we can construct a DFA $A_i = (Q_i, \Sigma, \delta_i, S_i, F_i)$ such that $\widetilde{L}(A_i) = S_{\widetilde{L}}(m_i)$ for each i, $1 \leq i \leq n$.

Note that for $1 \leq i, j \leq n$ and $i \neq j$, $\widetilde{L}(A_i) \cap \widetilde{L}(A_j) = S_{\widetilde{L}}(m_i) \cap S_{\widetilde{L}}(m_j) = \emptyset$. Now we construct a DFA $A = (Q, \Sigma, \delta, S, F)$, where

$$Q = Q_1 \times Q_2 \times ... \times Q_n, S = (S_1, S_2, ..., S_n), \delta : Q \times \Sigma \to Q$$

is defined by $\delta((q_1, q_2, ..., q_n), a) = (\delta_1(q_1, a), \delta_2(q_2, a), ..., \delta_n(q_n, a))$ and $F = F_1' \cup F_2' \cup ... \cup F_n'$, where $F_i' = \{(q_1, q_2, ..., q_n) \in Q \mid q_i \in F_i$ and $q_j \in F_j$ for $i \neq j\}, 1 \leq i \leq n$. From this it is clear that $F_i' \cap F_j' = \emptyset$ for $i \neq j$ and $S_{\widetilde{L}}(m_i) = \{w \in \Sigma^* \mid \delta^*(s, w) \in F_i'\}$. Based on the above DFA A, we define a DFA-IVFT $\widetilde{A} = (Q, \Sigma, \widetilde{\delta}, S, F)$ such that

$$\widetilde{\delta}(p, a, q) = \begin{cases} m_i, & \text{if } \delta(p, a) = q \in F_i', \\ [1,\ 1], & \text{if } \delta(p, a) = q \notin F, \\ [0,\ 0], & \text{otherwise} \end{cases}$$

It remains to show that $d_{\widetilde{A}}(w) = f_{\widetilde{L}}(w)$, for each $w \in \Sigma^+$. But first we show that $\widetilde{A}$ has the following property:

For each $w \in \Sigma^+$ with $w = xa$, for $x \in \Sigma^*$ and $a \in \Sigma$,

(2) $d_{\widetilde{A}}(w) = m_i > [0,0]$ iff $\delta^*(s, x, p) \geq m_i$ and $\delta(p, a, q) = m_i$ for some $q \in F_i$, $1 \leq i \leq n$.

The if part holds simply. For the only if part, it holds trivially when $x = \lambda$. For $x \neq \lambda$, we assume that contrary, i.e., $\delta^*(s, x, p) = m_i$ and $\delta(p, a, q) = m_j > m_i$. Then there exists a decomposition of $x = ybz$, where $y, z \in \Sigma^*$ and $b \in \Sigma$, such that $\delta^*(s, y, r) \geq m_i, \delta(r, b, t) = m_i$ and $\delta(t, z, p) \geq m_i$. By the definition of $\widetilde{A}$, we know that $t \in F_i$ and $q \in F_j$. Thus, we have $f_{\widetilde{L}}(yb) = m_i$ and $f_{\widetilde{L}}(w) = m_j$. Since we assume that $m_j > m_i$, this is a contradiction to the given condition (i.e.,(1)).

So, (2) holds. Furthermore, the righthand side of (2) implies that $xa \in S_{\widetilde{L}}(m_i)$ i.e., $f_{\widetilde{L}}(w) = m_i$. Hence this completes the proof. □

5. Finite automaton with vague (final) states

In this section, the definition of finite automaton with interval-valued fuzzy (final) states (NDFA-IVFS and DFA-IVFS for short) is proposed. Unlike previous models (NDFA-IVFT and DFA-IVFT), in this model, NDFA-IVFS and DFA-IVFS are equivalent in accepting interval-valued fuzzy regular language. Interval-valued fuzzy regular language is accepted by NDFA-IVFS and DFA-IVFS without any restrictions and vice versa. So, this model is more suitable for the study of interval-valued fuzzy regular languages.

Definition 5.1. A nondeterministic finite automaton with interval-valued fuzzy (final) states (NDFA-IVFS for short) is a 5-tuple $\widetilde{A} = (Q, \Sigma, \widetilde{\delta}, S, F_{\widetilde{A}})$, where Q is the finite set of states, Σ the finite set of input alphabets, $\widetilde{\delta} : Q \times \Sigma \to 2^Q$ the transition function, S the starting state and $F_{\widetilde{A}} : Q \to I[0,1]$ the membership function of interval-valued fuzzy (final) state set. Define $d_{\widetilde{A}}(x) = max\{F_{\widetilde{A}}(q) \mid (s, x, q) \in \widetilde{\delta}^*\}$, where $\widetilde{\delta}^*$ is the reflexive and transitive closure of $\widetilde{\delta}$. Then we say that the string 'x' is accepted by $\widetilde{A}$ with the degree $d_{\widetilde{A}}(x)$. So the interval-valued fuzzy language accepted by $\widetilde{A}$ i.e., $\widetilde{L}(\widetilde{A})$ is given by the set $\{(x, d_{\widetilde{A}}(x)) \mid x \in \Sigma^*\}$.

Example 5.1.

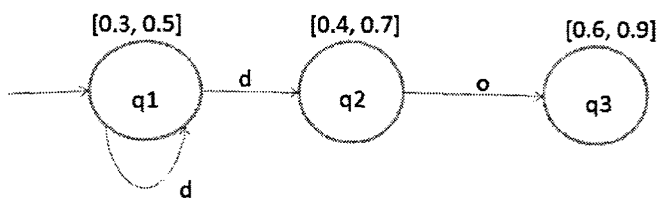

FIGURE 2. NDFA-IVFS

Let $\widetilde{A} = (Q, \Sigma, \widetilde{\delta}, S, F_{\widetilde{A}})$ be a NDFA-IVFS (Figure 2) with $Q = \{q_1, q_2, q_3\}$, $\Sigma = \{d, o\}$, $q_1 = S$ the interval-valued fuzzy starting state with membership value $F_{\widetilde{A}}(q_1) = [0.3, 0.5]$, $\widetilde{\delta} : Q \times \Sigma \to 2^Q$ the transition function given as $\widetilde{\delta}(q_1, d) = q_1, \widetilde{\delta}(q_1, d) = q_2$ and $\widetilde{\delta}(q_2, o) = q_3$, and $F_{\widetilde{A}} : Q \to M$, where M is the set of all closed subintervals in [0, 1] the membership function of interval-valued fuzzy (final) state set given as $F_{\widetilde{A}}(q_1) = [0.3, 0.5], F_{\widetilde{A}}(q_2) = [0.4, 0.7]$ and $F_{\widetilde{A}}(q_3) = [0.6, 0.9]$. Then, $d_{\widetilde{A}}(do) = [0.6, 0.9]$, $d_{\widetilde{A}}(d) = [0.4, 0.7]$ and $d_{\widetilde{A}}(dd^+) = [0.4, 0.7]$.

Definition 5.2. A deterministic finite automaton with interval-valued fuzzy (final) state (DFA-IVFS for short) $\widetilde{A} = (Q, \Sigma, \widetilde{\delta}, S, F_{\widetilde{A}})$ is a NDFA-IVFS with $\widetilde{\delta}$ being function $Q \times \Sigma \to Q$ instead of a relation. Thus for each $x \in \Sigma^*$,

$d_{\widetilde{A}}(x) = F_{\widetilde{A}}(q)$, where $q = \delta^*(s, x)$. Define, $d_{\widetilde{A}}(x) = [0, 0]$ if $\delta^*(s, x)$ is not defined.

Theorem 5.1. *Let $\widetilde{L}$ be an interval-valued fuzzy language. Then $\widetilde{L}$ is an interval-valued fuzzy regular language iff it is accepted by a DFA-IVFS.*

Proof. Let $\widetilde{L}$ be an interval-valued fuzzy language with the membership function $f_{\widetilde{L}} : \Sigma^* \to M$, where M is the set of all closed subintervals in [0, 1]. Assume that $\widetilde{L}$ is an interval-valued fuzzy regular language. Then M is finite and for each $m \in M$, the string $S_{\widetilde{L}}(m)$ is regular.

Assume that $M = \{m_1, m_2, ..., m_n\}$. We construct a DFA, $\widetilde{A}_i = (Q_i, \Sigma, \widetilde{\delta}_i, S_i, F_i)$ for each i, $1 \leq i \leq n$ such that $L(\widetilde{A}_i) = S_{\widetilde{L}}(m_i)$. Define a DFA-IVFS $\widetilde{A} = (Q, \Sigma, \widetilde{\delta}, S, F_{\widetilde{A}})$ to be the cross product of $\widetilde{A}_i$, $1 \leq i \leq n$ with

$$F_{\widetilde{A}}(q^{(1)}, q^{(2)}, ..., q^{(n)}) = \begin{cases} m_i, \text{ if } q^{(i)} \in F_i \text{ for some } i, 1 \leq i \leq n, q^{(j)} \notin F_j, \forall j \neq i, \\ [0,\ 0], \text{otherwise.} \end{cases}$$

Note that $(q^{(1)}, q^{(2)}, ..., q^{(n)})$ is reachable from $(S_1, S_2, ..., S_n)$ in $\widetilde{A}$, then it is not possible to get $q^{(i)} \in F_i$ and $q^{(j)} \in F_j$ for $i \neq j$, since $L(\widetilde{A}_i) \cap L(\widetilde{A}_j) = \emptyset$ for $i \neq j, 1 \leq i, j \leq n$. Hence, $\widetilde{A}$ accepts $\widetilde{L}$.

Conversely, Let $\widetilde{A} = (Q, \Sigma, \widetilde{\delta}, S, F_{\widetilde{A}})$ be a DFA-IVFS. Define $M = \{m \mid F_{\widetilde{A}}(q) = m$ for some $q \in Q\}$. So, M is finite. For each $m \in M$ define $A_m = (Q, \Sigma, \delta, S, F_m)$, where $F_m = \{q \mid F_{\widetilde{A}}(q) = m\}$. Let $\widetilde{L} = \widetilde{L}(\widetilde{A})$ i.e., $f_{\widetilde{L}}(w) = d_{\widetilde{A}}(w)$. Then clearly, for each $m \in M, S_{\widetilde{L}}(m) = L(A_m)$ is regular. Thus, $\widetilde{L}$ is an interval-valued fuzzy regular language. □

Theorem 5.2. *An interval-valued fuzzy language is accepted by a NDFA-IVFS iff it is accepted by a DFA-IVFS.*

Proof. Here we have to show if $\widetilde{A}$ is a NDFA-IVFS and $\widetilde{L} = \widetilde{L}(\widetilde{A})$ then $\widetilde{L} = \widetilde{L}(\widetilde{A}')$, where $\widetilde{A}'$ is a DFA-IVFS. Let $\widetilde{A} = (Q, \Sigma, \widetilde{\delta}, S, F_{\widetilde{A}})$ represents a NDFA-IVFS, we construct a DFA-IVFS $\widetilde{A}' = (Q', \Sigma, \widetilde{\delta}', S', F'_{\widetilde{A}})$ by using the method of standard subset construction and, for each $P \in Q'(P \subseteq Q)$, define $F'_{\widetilde{A}}(P) = max\{m \mid F_{\widetilde{A}}(q) = m, q \in P\}$, where $m \subseteq [0, 1]$ represents the membership value of the strings in the language. Hence $\widetilde{L} = \widetilde{L}(\widetilde{A}')$. □

6. Interval-valued fuzzy regular expressions (IVFREs)

Every string in an interval-valued fuzzy regular language has finite membership value in $I[0, 1]$. Set of all strings that are associated with these subinterval [0, 1] forms a regular language. Interval-valued fuzzy regular language can be described by a modified regular expressions having membership values in $I[0, 1]$. This modified interval-valued fuzzy regular expression can be used in lexical analysis of IVFRL.

For example: $(a + b)^*/[0.4, 0.8] + ab^*(b + a^*)/[0.6, 0.9]$ represents a modified interval-valued fuzzy regular expression.

We give a formal definition of interval-valued fuzzy regular expression as mentioned below.

Definition 6.1. Let Σ be a finite set of alphabet and $M = I[0,1]$ a finite set of closed subintervals in [0, 1].

(1) Let 'e' be a regular expression over Σ and $m \in M$. Then, we call $\widetilde{e} = (e)/m$ an interval-valued fuzzy regular expression (IVFRE for short), where m represents the membership value of 'e'.
(2) Let $\widetilde{e_1}$ and $\widetilde{e_2}$ be two IVFRE's, then the following will hold.
    (a) $\emptyset \in$ IVFRE with membership value [1, 1],
    (b) $\lambda \in$ IVFRE with membership value [1, 1],
    (c) $a \in$ IVFRE with membership value $m \in I[0,1]$ $\forall a \in \Sigma$,
    (d) forall $\widetilde{e_1}$ and $\widetilde{e_2} \in$ IVFRE's, $(\widetilde{e_1} + \widetilde{e_2}) \in$ IVFRE's, $(\widetilde{e_1} \cdot \widetilde{e_2}) \in$ IVFRE's and $(\widetilde{e_1})^* \in$ IVFRE's.
(3) By applying above mentioned steps((1) and (2)) finite number of times, an interval-valued fuzzy regular expression can be obtained.

Definition 6.2. Let $\widetilde{e}$ be an interval-valued fuzzy regular expression. Then, the language associated with it. i.e., interval-valued fuzzy regular language (IVFRL for short) is defined to be

$$\widetilde{L} = \widetilde{L}(\widetilde{e}) = \{(x, m) \mid x \in L(e)\}.$$

Here, L(e) represents language for regular expression 'e' and $m \in I[0,1]$ the membership value of the string x in $I[0,1]$. If $\widetilde{e} = (\widetilde{e_1} + \widetilde{e_2})$, $\widetilde{e} = (\widetilde{e_1}) \cdot (\widetilde{e_2})$ or $\widetilde{e} = (\widetilde{e_1})^*$. Then $\widetilde{L}(\widetilde{e}) = \widetilde{L}(\widetilde{e_1}) \cup \widetilde{L}(\widetilde{e_2})$, $\widetilde{L}(\widetilde{e}) = \widetilde{L}(\widetilde{e_1}) \cdot \widetilde{L}(\widetilde{e_2})$ or $\widetilde{L}(\widetilde{e}) = (\widetilde{L}(\widetilde{e_1}))^*$ respectively.

Definition 6.3. An interval-valued fuzzy regular expression over Σ is said to be normalized if it is of the form $e_1/m_1 + e_2/m_2 + ... + e_n/m_n$, where $e_1, e_2, ..., e_n$ represents regular expressions over Σ and $m_1, m_2, ..., m_n$ represents the closed subintervals in $I[0,1], n \geq 1$. Note that, if $m = [1,1]$ then e/m can simply be written as 'e'.

Example 6.1. The following are all valid IVFRE's.

(1) $(a + b \cdot c)^* \cdot (c + \phi)/[0.3, 0.5]$,
(2) $(0+1)^*00(0+1)^*$,
(3) $(b + ab^*a)^*ab^*b(a+b)^*/[0.5, 1] + a^*b/[0.4, 0.9]$.

The following is not a valid IVFRE.

$$(a^*b/[0.5, 0.8])/[0.2, 0.5] + ca/[0.4, 0.5].$$

Definition 6.4. An interval-valued fuzzy regular expression $\widetilde{e}$ is said to be strictly normalized if it is normalized. i.e., $\widetilde{e} = e_1/m_1 + e_2/m_2 + ... + e_n/m_n$ and for any $m_i \neq m_j, L(e_i) \cap L(e_j) = \emptyset$.

Example 6.2.

(1) $ab/[0.3, 0.5] + ac/[0.4, 0.7] + bc/[0.5, 0.9]$,

(2) $((11+110)^*0)/[0.5,1]+(\{0,1\}^*\{10\})/[0.4,0.8]$,

shows the IVFREs which are strictly normalized.

(1) $a^*b/[0.4,0.9]+(a+b)^*a(a+b)^*b/[0.3,0.7]$

shows an IVFRE which is not strictly normalized. It is clear from the above definitions that the families of languages as represented by IVFREs, normalized IVFREs, and strictly normalized IVFREs, respectively, are same as the family of interval-valued fuzzy regular languages.

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