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EXISTENCE OF SOLUTIONS FOR BOUNDARY BLOW-UP QUASILINEAR ELLIPTIC SYSTEMS†

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ABSTRACT. In this paper, we are concerned with the quasilinear elliptic systems with boundary blow-up conditions in a smooth bounded domain. Using the method of lower and upper solutions, we prove the sufficient conditions for the existence of the positive solution. Our main results are new and extend the results in [Mingxin Wang, Lei Wei, Existence and boundary blow-up rates of solutions for boundary blow-up elliptic systems, Nonlinear Analysis(In Press)].

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1. Introduction

This paper is concerned with the study of positive boundary blow-up solutions to a quasilinear elliptic systems

$$\begin{cases} -\triangle_p u = u^{p-1}(a_1(x)u^{m_1} - b_1(x)u^{m-p+2} - l_1 v^n), & x \in \Omega, \\ -\triangle_q v = v^{q-1}(a_2(x)v^{t_1} - b_2(x)v^{t-q+2} - l_2 u^s), & x \in \Omega, \\ u = v = \infty, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbf{R}^N$ and $\triangle_p = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, $\triangle_q = \mathrm{div}(|\nabla v|^{q-2}\nabla v)$, $q > 1$. Functions $a_1, a_2 \in C^\eta(\overline{\Omega})$, the weight functions $b_1, b_2 \in C^\eta(\Omega)$ are positive and singular on the boundary $\partial\Omega$, $0 < \eta < 1$. Constants $m > m_1 \geq p-2$, $t > t_1 \geq q-2$, $n, s > 0$ and $l_1, l_2 > 0$. The boundary condition is assumed in the sense $u(x), v(x) \to +\infty$ when $d(x) \to 0^+$, where $d(x)$ stands for the distance function $\mathrm{dist}(x, \partial\Omega)$.

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Systems of the above form are mathematical models occuring in studies of the p-Laplace system, generalized reaction-diffusion theory, non-Newtonian fluid theory[1,26], non-Newtonian filtration[27] and the turbulent flow of a gas in porous medium. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids. When $p \neq 2$, the problem becomes more complicated since certain nice properties in herent to the case $p = 2$ seem to be lost or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ can be founded in [16,18].

There is a huge amount of works dealing with boundary blow-up problems with the semilinear case,

$$\begin{cases} -\triangle u = f(x,u), & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbf{R}^N$, see [2-5,8,10-13,21-25,29-30].

In [11], the authors considered positive solutions of the form

$$\begin{cases} -\triangle u = a_1 u - u^2 + b_1 uv, & x \in \Omega, \\ -\triangle v = a_2 v - v^2 + b_2 uv, & x \in \Omega, \\ u = v = \infty, & x \in \partial\Omega. \end{cases}$$

In [12], the authors studied the existence and uniqueness of positive solutions of the form

$$\begin{cases} \triangle u = u^p v^q, & x \in \Omega, \\ \triangle v = u^r v^s, & x \in \Omega, \\ u = v = \infty, & x \in \partial\Omega. \end{cases}$$

In [29], the authors studied the following problem

$$\begin{cases} \triangle u = u(a_1 u^{m_1} + b_1(x) u^m + \delta v^n), & x \in \Omega, \\ \triangle v = v(a_2 v^{p_1} + b_2(x) v^p + \delta u^q), & x \in \Omega, \\ u = v = \infty, & x \in \partial\Omega. \end{cases}$$

Recently, in [30], the authors considered the following problem

$$\begin{cases} -\triangle u = u(a_1(x) u^{m_1} - b_1(x) u^m - c_1 v^n), & x \in \Omega, \\ -\triangle v = v(a_2(x) v^{p_1} - b_2(x) v^p - c_2 u^q), & x \in \Omega, \\ u = v = \infty, & x \in \partial\Omega. \end{cases}$$

The quasilinear elliptic problem

$$\begin{cases} \triangle_p = f(u) & x \in \Omega, \\ u = \infty & x \in \partial\Omega, \end{cases}$$

for general nonlinearities $f(u)$ seems to have been first considered in [6]. The reader can also refer to papers [17,19,20].

In recent years, the "logistic" type equation has been studied, see [7-9].

In [9], the author considered the existence and uniqueness of the following equation

$$-\triangle_p u = a(x)|u|^{p-2}u - b(x)|u|^{q-1}u, \quad x \in R^N \ (N \geq 2).$$

In [31], the authors studied the problem

$$\begin{cases} \triangle_p u = m(x)f(u), & x \in \Omega, \\ u = \infty, & x \in \partial\Omega. \end{cases}$$

In [14], the author considered the uniqueness and boundary behavior for equations involving the p-Laplacian and singular weights

$$\begin{cases} \triangle_p u = a(x)u^q, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega. \end{cases}$$

In [15], the author extended the results in [12] and considered the positive boundary blow-up solutions to a quasilinear elliptic system of competitive type

$$\begin{cases} \triangle_p u = u^a v^b & x \in \Omega, \\ \triangle_p v = u^c v^e & x \in \Omega, \\ u = v = \infty & x \in \partial\Omega. \end{cases}$$

Our motivation comes from [30], where problem (1.1) was analyzed in the semilinear case $p = q = 2$. The main purpose of the present paper is to investigate the existence of positive solutions to the system (1.1). Our result do not depend on the exponents m_1 and t_1.

We will use the following assumptions on the growth rates of $b_1(x)$ and $b_2(x)$ near the boundary $\partial\Omega$

$$D_2 d(x)^{-\gamma_1} \leq b_1(x) \leq D_1 d(x)^{-\gamma_1}, \quad x \in \Omega, \tag{1.3}$$

$$K_2 d(x)^{-\gamma_2} \leq b_2(x) \leq K_1 d(x)^{-\gamma_2}, \quad x \in \Omega, \tag{1.4}$$

where $D_i, K_i > 0$, $0 \leq \gamma_1 < p$, $0 \leq \gamma_2 < q$ are constants.

Theorem 1. *Assume that $s(p-\gamma_1) \leq q(m-p+2)$, $n(q-\gamma_2) \leq p(t-q+2)$, and the assumptions (1.3) and (1.4) are satisfied. Then there exists $\lambda > 0$ such that when $0 < l_1, l_2 \leq \lambda$, the problem (1.1) has at least one positive solution.*

This work is organized as follows. In section 2, we consider some preliminaries on single equation which are instrumental in our proofs. In section 3, the main result for problem (1.1) will be proved.

2. Preliminaries

Before giving the definition of upper and lower solutions, we state some well-known results.

From the proof of Theorem 3.7 in [7], we have

Lemma 1. *Assume that $a(x) \in C(\overline{\Omega})$, $b(x) \in C^{\eta}(\Omega)$ $(0 < \eta < 1)$ and there exist constants $C_1, C_2 > 0$, $0 \le \gamma < p$ such that $C_1 d(x)^{-\gamma} \le b(x) \le C_2 d(x)^{-\gamma}$ in Ω. Then the following problem*

$$\begin{cases} -\triangle_p u = a(x)u^{p-1} - b(x)u^{r+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases}$$

have a positive solution u. Moreover, there exist positive constants M_1 and M_2, such that

$$M_2 d(x)^{-\frac{p-\gamma}{r-p+2}} \le u(x) \le M_1 d(x)^{-\frac{p-\gamma}{r-p+2}} \quad \text{in } \Omega.$$

From [14], we have

Lemma 2. *Let $q > p-1$ and $\gamma \in [0, p)$, the problem*

$$\begin{cases} \triangle_p u = d^{-\gamma} u^q & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

admits a unique positive solution denoted by $U_{q,\gamma}$, where $d(x) = dist(x, \partial\Omega)$. Moreover,

$$U_{q,\gamma} \sim ((p-1)\alpha^{p-1}(\alpha+1))^{\frac{1}{q-p+1}} d(x)^{-\alpha},$$

as $d(x) \to 0$, where $\alpha = \frac{p-\gamma}{q-p+1}$.

From [15], we have

Lemma 3. *Let $u \in C^{1,\eta}_{loc}(\Omega)$ for some $\eta \in (0,1)$ verify $\triangle_p u \ge C d(x)^{-\gamma} u^{r+1}$ in Ω with $u = \infty$ on $\partial\Omega$. Then $u \le C^{-\frac{1}{r-p+2}} U_{r,\gamma}$ in Ω, where $U_{r,\gamma}$ denotes the unique positive solution of the problem*

$$\begin{cases} \triangle_p u = d(x)^{-\gamma} u^{r+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega. \end{cases}$$

Similarly, if $\triangle_p u \le C d(x)^{-\gamma} u^{r+1}$ in Ω with $u = \infty$ on $\partial\Omega$. Then $u \ge C^{-\frac{1}{r-p+2}} U_{r,\gamma}$ in Ω.

Definition 1. Two pairs of functions $(\overline{u}, \overline{v})$, $(\underline{u}, \underline{v}) \in C^1(\Omega)$ are called upper and lower solutions of (1.1) if they satisfy

$$\begin{cases} -\triangle_p \overline{u} \geq \overline{u}^{p-1}(a_1(x)\overline{u}^{m_1} - b_1(x)\overline{u}^{m-p+2} - l_1\underline{v}^n), & x \in \Omega, \\ -\triangle_q \underline{v} \leq \underline{v}^{q-1}(a_2(x)\underline{v}^{t_1} - b_2(x)\underline{v}^{t-q+2} - l_2\overline{u}^s), & x \in \Omega, \end{cases}$$

$$\begin{cases} -\triangle_p \underline{u} \leq \underline{u}^{p-1}(a_1(x)\underline{u}^{m_1} - b_1(x)\underline{u}^{m-p+2} - l_1\overline{v}^n), & x \in \Omega, \\ -\triangle_q \overline{v} \geq \overline{v}^{q-1}(a_2(x)\overline{v}^{t_1} - b_2(x)\overline{v}^{t-q+2} - l_2\underline{u}^s), & x \in \Omega. \end{cases}$$

Proposition 1. *Assume that $(\overline{u}, \underline{v})$ and $(\underline{u}, \overline{v})$ are the positive upper and lower solutions of (1.1), and $\overline{u} = \underline{u} = \overline{v} = \underline{v} = \infty$ on $\partial\Omega$, $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$ in Ω. Then the problem (1.1) has at least a weak solution (u, v) with $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \leq v \leq \overline{v}$ in Ω and $u = v = \infty$ on $\partial\Omega$.*

Proof. Let $k > 0$, denote $\Omega_k = \{x \in \Omega : d(x) > \frac{1}{k}\}$, consider the problem

$$\begin{cases} -\triangle_p u = u^{p-1}(a_1(x)u^{m_1} - b_1(x)u^{m-p+2} - l_1 v^n), & x \in \Omega_k, \\ -\triangle_q v = v^{q-1}(a_2(x)v^{t_1} - b_2(x)v^{t-q+2} - l_2 u^s), & x \in \Omega_k, \\ u = \underline{u}, \quad v = \overline{v}, & x \in \partial\Omega_k. \end{cases} \tag{2.1}$$

By Theorem A.1 [15], the problem (2.1) has at least a solution (u_k, v_k) satisfying $\underline{u} \leq u_k \leq \overline{u}$, $\underline{v} \leq v_k \leq \overline{v}$ in Ω_k. This inequalities give bounds for the solutions (u_k, v_k), from the interior regularity estimate, we obtain bounds in $C^{1,\eta}_{loc}(\Omega)$. Then for a sequence $k \to \infty$, $u_k \to u$, we obtain that $v_k \to v$ in $C^{1,\eta}_{loc}(\Omega)$. So, (u, v) is a weak solution of the problem (1.1) verifying $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \leq v \leq \overline{v}$ in Ω. In particular, $u = v = \infty$ on $\partial\Omega$. □

Proposition 2. *Assume that constants $m > m_1 \geq p - 2$, $A \geq 0$, the function $b_1 \in C^\eta(\Omega)$ and satisfies (1.3). Then the following problem*

$$\begin{cases} -\triangle_p u = -Au^{p+m_1-1} - b_1(x)u^{m+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

has a positive solution u, and there exist constants $C_1, C_2 > 0$, such that

$$C_2 d(x)^{-\frac{p-\gamma}{m-p+2}} \leq u(x) \leq C_1 d(x)^{-\frac{p-\gamma}{m-p+2}}, \qquad x \in \Omega. \tag{2.3}$$

Proof. When $m_1 = 0$, the conclusion holds by Lemma 1. In the following we consider only the case $m_1 > 0$. Let U denote the positive solution of the problem

$$\begin{cases} \triangle_p u = b_1(x)u^{m+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega. \end{cases}$$

Set $\overline{u} = U$, $\underline{u} = \epsilon U$, where the constant $\epsilon > 0$ is to be determined. Let $\Omega_k = \{x \in \Omega : d(x) > \frac{1}{k}\}$. Consider the following problem

$$\begin{cases} -\triangle_p u = -Au^{p+m_1-1} - b_1(x)u^{m+1}, & x \in \Omega_k, \\ u = \underline{u}, & x \in \partial\Omega_k. \end{cases} \tag{2.4}$$

Clearly,

$$\triangle_p \overline{u} = b_1(x)\overline{u}^{m+1} \leq A\overline{u}^{p+m_1-1} + b_1(x)\overline{u}^{m+1} \quad \text{in } \Omega_k.$$

Taking into account $\gamma_1 \geq 0$ and $b_1(x) \geq D_2 d(x)^{-\gamma_1}$, we have $b_0 = \inf_\Omega b_1(x) > 0$. From the definition of U, $\alpha_0 = \inf_\Omega U(x) > 0$. If we take ϵ to be sufficient small such that $\epsilon^{m-p+2} \leq \frac{1}{2}$, it follows from $m > m_1$ that

$$\frac{U^{m+1}}{2} \geq \frac{A\epsilon^{m_1}U^{p+m_1-1}}{b_0}, \quad x \in \Omega_k,$$

$$U^{m+1} \geq \frac{A\epsilon^{m_1}U^{p+m_1-1}}{b_1(x)} + \epsilon^{m-p+2}U^{m+1}, \quad x \in \Omega_k.$$

It is easy to see that

$$b_1(x)U^{m+1} \geq A\epsilon^{m_1}U^{p+m_1-1} + \epsilon^{m-P+2}b_1(x)U^{m+1}, \quad x \in \Omega_k.$$

Therefore,

$$\triangle_p \underline{u} = \epsilon^{p-1}b_1(x)U^{m+1} \geq A(\epsilon U)^{p+m_1-1} + b_1(x)(\epsilon U)^{m+1}, \quad x \in \Omega_k.$$

By the upper and lower solutions argument, the problem (2.4) has at least one solution u_k satisfy $\underline{u} \leq u_k \leq \overline{u}$. From the interior regularity estimate [28], we obtain that for a sequence $k \to \infty$, $u_k \to u_0$ in $C^1_{loc}(\Omega)$. So u_0 is a positive solution of (2.2) and satisfies $\epsilon U \leq u_0 \leq U$. From Lemma 1 and Lemma 3, it follows that there exist positive constants C_1 and C_2 such that

$$C_2 d(x)^{-\frac{p-\gamma}{m-p+2}} \leq u_0(x) \leq C_1 d(x)^{-\frac{p-\gamma}{m-p+2}}, \quad x \in \Omega.$$

□

Now, we give a byproduct of Proposition 2 which will be useful to understand the boundary blow-up problem.

Proposition 3. *Assume $a_1 \in C(\overline{\Omega})$, $m > m_1 \geq p-2$, $b_1 \in C^\eta(\Omega)$, then the problem*

$$\begin{cases} -\triangle_p u = a_1(x)u^{p+m_1-1} - b_1(x)u^{m+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases} \tag{2.5}$$

has at least one positive solution u. Moreover, there exist two positive constants C_1 and C_2 such that (2.3) holds.

Proof. Denote $A_1 = \sup_{x\in\Omega}|a_1(x)|$, by Proposition 2, the following problem

$$\begin{cases} -\triangle_p u = -A_1 u^{p+m_1-1} - b_1(x)u^{m+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases} \tag{2.6}$$

has a positive solution, denoted by $\underline{u}$. Moreover, if U is the positive solution of the problem

$$\begin{cases} -\triangle_p u = -b_1(x)u^{m+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases}$$

and U satisfies (2.3), by the proof of Proposition 2, $\underline{u}$ satisfies $\underline{u} \le U$.

Now we consider the existence of positive solution of the problem

$$\begin{cases} -\triangle_p u = A_1 u^{p+m_1-1} - b_1(x)u^{m+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega. \end{cases} \tag{2.7}$$

It is clear that U is a lower solution of (2.7). In view of Lemma 1, the problem

$$\begin{cases} -\triangle_p u = A_1 u^{p-1} - b_1(x)u^{m+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega. \end{cases}$$

has a positive solution U_0 and U_0 satisfies (2.3), since $\inf_{x\in\Omega} b_1(x) > 0$ and $\inf_{x\in\Omega} U_0(x) > 0$, we can choose a large constant $k > 0$ such that

$$A_1 \le \frac{(k^{m-p+2}-1)}{k^{m_1}} b_1(x) U_0^{m-m_1-p+2}, \quad kU_0 \ge U, \ \forall\, x \in \Omega,$$

which implies that

$$A_1 k^{m_1} U_0^{m_1} - b_1(x) k^{m-p+2} U_0^{m-p+2} \le A_1 - b_1(x) U_0^{m-p+2}.$$

Through direct calculation, we arrive that $-\triangle_p(kU_0) \ge A_1(kU_0)^{p+m_1-1} - b_1(x)(kU_0)^{m+1}$. So, kU_0 is an upper solution of the problem (2.7). Similarly to Proposition 1, the problem (2.7) has at least one positive solution $\overline{u}$. Moreover, $\overline{u}$ satisfies $U \le \overline{u} \le kU_0$ and (2.3).

From the above argument, we obtain that $\overline{u}$ and $\underline{u}$ are an ordered upper and lower solutions of (2.5) and $\overline{u} = \underline{u} = \infty$ on $\partial\Omega$. Therefore, the problem (2.5) has at least one positive solution u and u satisfies $\underline{u} \le u \le \overline{u}$. Moreover, u satisfies (2.3). □

3. Proof of main result

Proof. We will divide the proof into three cases.

Case 1. $m_1 = t_1 = 0$. Since $b_1(x)$ and $b_2(x)$ satisfy (1.3) and (1.4) respectively, in view of Lemma 1, we know that the following problems

$$\begin{cases} -\triangle_p u = a_1(x)u^{p-1} - b_1(x)u^{m+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\triangle_q v = a_2(x)v^{q-1} - b_2(x)v^{t+1}, & x \in \Omega, \\ v = \infty, & x \in \partial\Omega, \end{cases}$$

have positive solutions, denoted by U and V respectively. Moreover, U and V satisfy

$$C_2 d(x)^{-\frac{p-\gamma_1}{m-p+2}} \le U(x) \le C_1 d(x)^{-\frac{p-\gamma_1}{m-p+2}}, \qquad x \in \Omega,$$

$$C_2 d(x)^{-\frac{q-\gamma_2}{t-q+2}} \le V(x) \le C_1 d(x)^{-\frac{q-\gamma_2}{t-q+2}}, \qquad x \in \Omega,$$

for some positive constants C_1 and C_2.

Note that $s(p-\gamma_1) \le q(m-p+2)$, $n(q-\gamma_2) \le p(t-q+2)$, we have

$$\lambda_1 := \min\left\{ \frac{C_1^{-s} C_2^{t-q+2} K_2}{2} \inf_\Omega d(x)^{\frac{s(p-\gamma_1)}{m-p+2}-q}, \ \frac{C_1^{-n} C_2^{m-p+2} D_2}{2} \inf_\Omega d(x)^{\frac{n(q-\gamma_2)}{t-q+2}-p} \right\} > 0.$$

In order to show that the problem (1.1) has at least one positive solutions when $l_1, l_2 \le \lambda_1$. We employ the method of upper and lower solutions.

Take $\epsilon > 0$, to be such that $\max\{\epsilon^{m-p+2}, \epsilon^{t-q+2}\} \le \frac{1}{2}$. Let $(\overline{u}, \underline{v}) = (U, \epsilon V)$ and $(\underline{u}, \overline{v}) = (\epsilon U, V)$. It is not hard to see that

$$\begin{aligned} -\triangle_p \overline{u} &= a_1(x)U^{p-1} - b_1(x)U^{m+1} \\ &\ge U^{p-1}(a_1(x) - b_1(x)U^{m-p+2} - l_1(\epsilon V)^n) \\ &= \overline{u}^{p-1}(a_1(x) - b_1(x)\overline{u}^{m-p+2} - l_1\underline{v}^n), \quad x \in \Omega. \end{aligned}$$

$$\begin{aligned} -\triangle_q \overline{v} &= a_2(x)V^{q-1} - b_2(x)V^{t+1} \\ &\ge V^{q-1}(a_2(x) - b_2(x)V^{t-q+2} - l_2(\epsilon U)^s) \\ &= \overline{v}^{q-1}(a_2(x) - b_2(x)\overline{v}^{t-q+2} - l_2\underline{u}^s), \quad x \in \Omega. \end{aligned}$$

When $l_1 < \lambda_1$, we have

$$l_1 < \frac{C_1^{-n} C_2^{m-p+2} D_2}{2} d(x)^{\frac{n(q-\gamma_2)}{t-q+2}-p}.$$

so,

$$\frac{D_2 C_2^{m-p+2}}{2} d(x)^{-p} \ge l_1 C_1^n d(x)^{-\frac{n(q-\gamma_2)}{t-q+2}}, \qquad x \in \Omega,$$

since

$$\frac{b_1(x)U^{m-p+2}}{2} \ge \frac{D_2 C_2^{m-p+2} d(x)^{-p}}{2}, \quad V^n \le C_1^n d(x)^{-\frac{n(q-\gamma_2)}{t-q+2}}, \quad x \in \Omega.$$

We obtain $\frac{b_1(x)U^{m-p+2}}{2} \ge l_1 V^n, \quad x \in \Omega$. Then

$$b_1(x)U^{m-p+2} \ge b_1(x)\epsilon^{m-p+2}U^{m-p+2} + l_1 V^n = b_1(x)\underline{u}^{m-p+2} + l_1\overline{v}^n,$$

we conclude that

$$\begin{aligned} -\triangle_p \underline{u} &= \epsilon^{p-1}(a_1(x)U^{p-1} - b_1(x)U^{m+1}) \\ &= \epsilon^{p-1}U^{p-1}(a_1(x) - b_1(x)U^{m-p+2}) \\ &\leq \underline{u}^{p-1}(a_1(x) - b_1(x)\underline{u}^{m-p+2} - l_1\overline{v}^n), \quad x \in \Omega. \end{aligned}$$

When $l_2 < \lambda_1$, we have

$$l_2 < \frac{C_1^{-s}C_2^{t-q+2}K_2}{2} d(x)^{\frac{s(p-\gamma_1)}{m-p+2}-q},$$

so,

$$\frac{K_2C_2^{t-q+2}}{2} d(x)^{-q} \geq l_2C_1^s d(x)^{-\frac{s(p-\gamma_1)}{m-p+2}}, \qquad x \in \Omega.$$

Since

$$\frac{b_2(x)V^{t-q+2}}{2} \geq \frac{K_2C_2^{t-q+2}d(x)^{-q}}{2}, \quad U^s \leq C_1^s d(x)^{-\frac{s(p-\gamma_1)}{m-p+2}}, \quad x \in \Omega.$$

We obtain

$$\frac{b_2(x)V^{t-q+2}}{2} \geq l_2U^s, \quad x \in \Omega.$$

Then

$$b_2(x)V^{t-q+2} \geq b_2(x)\epsilon^{t-q+2}V^{t-q+2} + l_2U^s, \quad x \in \Omega,$$

we conclude that

$$\begin{aligned} -\triangle_q \underline{v} &= \epsilon^{q-1}(a_2(x)V^{q-1} - b_2(x)V^{t+1}) \\ &= \epsilon^{q-1}V^{q-1}(a_2(x) - b_2(x)V^{t-q+2}) \\ &\leq \underline{v}^{q-1}(a_2(x) - b_2(x)\underline{v}^{t-q+2} - l_2\overline{u}^s), \quad x \in \Omega. \end{aligned}$$

By Proposition 1, the problem (1.1) has at least one positive solution (u, v) and satisfies $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \leq v \leq \overline{v}$.

Case 2. $m_1 > 0$, $t_1 > 0$. Since $b_1(x)$ and $b_2(x)$ satisfy (1.3) and (1.4) respectively, in view of Proposition 2, we know that the following problems

$$\begin{cases} -\triangle_p u = -A_1u^{p+m_1-1} - b_1(x)u^{m+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\triangle_q v = -A_2v^{q+t_1-1} - b_2(x)v^{t+1}, & x \in \Omega, \\ v = \infty, & x \in \partial\Omega, \end{cases}$$

have positive solutions, denoted by U and V respectively, where $A_1 = \sup_{x\in\Omega}|a_1(x)|$, $A_2 = \sup_{x\in\Omega}|a_2(x)|$. Let $(\overline{u}, \underline{v}) = (MU, \epsilon V)$, $(\underline{u}, \overline{v}) = (\epsilon U, MV)$, where ϵ and M are positive constants, $\max\{\epsilon^{m-p+2}, \epsilon^{t-q+2}\} \leq$

$\frac{1}{2}$. Since $m > m_1$, $\inf_{x\in\Omega} b_1(x) > 0$ and $\inf_{x\in\Omega} U(x) > 0$, we can take M large enough satisfying $M^{m-p+2} > 2$ such that

$$A_1 + a_1(x)M^{m_1} \leq \frac{M^{m-p+2}}{2} b_1(x)U^{m-m_1-p+2}.$$

Then we have

$$A_1 + a_1(x)M^{m_1} \leq b_1(x)M^{m-p+2}U^{m-m_1-p+2} - b_1(x)U^{m-m_1-p+2},$$
$$a_1(x)M^{m_1}U^{m_1} - b_1(x)M^{m-p+2}U^{m-p+2} \leq -A_1U^{m_1} - b_1(x)U^{m-p+2},$$

we conclude that

$$\begin{aligned} -\triangle_p \overline{u} &= M^{p-1}(-A_1U^{p+m_1-1} - b_1(x)U^{m+1}) \\ &\geq \overline{u}^{p-1}(a_1(x)M^{m_1}U^{m_1} - b_1(x)M^{m-p+2}U^{m-p+2}) \\ &\geq \overline{u}^{p-1}(a_1(x)\overline{u}^{m_1} - b_1(x)\overline{u}^{m-p+2} - l_1\underline{v}^n), \quad x \in \Omega. \end{aligned}$$

Similarly, when M is large enough, $M^{t-q+2} \geq 2$, the following holds

$$-\triangle_q \overline{v} \geq \overline{v}^{q-1}(a_2(x)\overline{v}^{t_1} - b_2(x)\overline{v}^{t-q+2} - l_2\underline{u}^s), \quad x \in \Omega.$$

Denote

$$\lambda_2 := \min\left\{\frac{C_1^{-s}C_2^{t-q+2}K_2}{2M^s}\inf_{\Omega} d(x)^{\frac{s(p-\gamma_1)}{m-p+2}-q}, \ \frac{C_1^{-n}C_2^{m-p+2}D_2}{2M^n}\inf_{\Omega} d(x)^{\frac{n(q-\gamma_2)}{t-q+2}-p}\right\} > 0.$$

We shall prove that the problem (1.1) has at least one positive solution when $l_1, l_2 \leq \lambda_2$.

When $l_1 < \lambda_2$, we have

$$l_1 \leq \frac{b_1(x)U^{m-p+2}V^{-n}}{2M^n}, \quad \text{i.e.} \quad l_1M^nV^n \leq \frac{b_2(x)U^{m-p+2}}{2},$$

by $\epsilon^{t-q+2} \leq \frac{1}{2}$, we have

$$l_1M^nV^n \leq b_1(x)U^{m-p+2} - b_1(x)\epsilon^{m-p+2}U^{m-p+2}.$$

Furthermore, in view of $-A_1U^{m_1} \leq a_1(x)U^{m_1}\epsilon^{m_1}$, it follows that

$$-A_1U^{m_1} - b_1(x)U^{m-p+2} \leq a_1(x)U^{m_1}\epsilon^{m_1} - b_1(x)\epsilon^{m-p+2}U^{m-p+2} - l_1M^nV^n.$$

It is not hard to get

$$\begin{aligned} -\triangle_p \underline{u} &= \epsilon^{p-1}(-A_1U^{p+m_1-1} - b_1(x)U^{m+1}) \\ &= \underline{u}^{p-1}(-A_1U^{m_1} - b_1(x)U^{m-p+2}) \\ &\leq \underline{u}^{p-1}(a_1(x)\underline{u}^{m_1} - b_1(x)\underline{u}^{m-p+2} - l_1\overline{v}^n), \quad x \in \Omega. \end{aligned}$$

Similarly, we obtain

$$-\triangle_q \underline{v} \leq \underline{v}^{q-1}(a_2(x)\underline{v}^{t_1} - b_2(x)\underline{v}^{t-q+2} - l_2\overline{u}^s), \quad x \in \Omega.$$

Case 3. $m_1 > 0$ and $t_1 = 0$, or $m_1 = 0$ and $t_1 > 0$.

We only give the proof for the case $m_1 > 0$ and $t_1 = 0$. We consider the following problems

$$\begin{cases} -\triangle_p u = -A_1 u^{p+m_1-1} - b_1(x)u^{m+1}, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\triangle_q v = a_2(x)v^{q-1} - b_2(x)v^{t+1}, & x \in \Omega, \\ v = \infty, & x \in \partial\Omega, \end{cases}$$

have positive solutions, denoted by U and V respectively, where $A_1 = \sup_{x\in\Omega}|a_1(x)|$. Let $(\overline{u}, \underline{v}) = (MU, \epsilon V)$, $(\underline{u}, \overline{v}) = (\epsilon U, V)$, where ϵ and M are suitable positive constants. It is clear that

$$-\triangle_q \overline{v} = a_2(x)V^{q-1} - b_2(x)V^{t+1} \geq \overline{v}^{q-1}(a_2(x) - b_2(x)\overline{v}^{t-q+2} - l_2\underline{u}^s), \quad x \in \Omega.$$

Similarly to case 2, if M is large enough, we can obtain that

$$-\triangle_p \overline{u} \geq \overline{u}^{p-1}(a_1(x) - b_1(x)\overline{u}^{m-p+2} - l_1\underline{v}^n), \quad x \in \Omega.$$

Denote

$$\lambda_3 := \min\left\{\frac{C_1^{-s}C_2^{t-q+2}K_2}{2M^s}\inf_\Omega d(x)^{\frac{s(p-\gamma_1)}{m-p+2}-q}, \ \frac{C_1^{-n}C_2^{m-p+2}D_2}{2}\inf_\Omega d(x)^{\frac{n(q-\gamma_2)}{t-q+2}-p}\right\} > 0,$$

with the same method in Case 1 and Case 2, we prove that when $l_1, l_2 \leq \lambda_3$, the following hold

$$\begin{cases} -\triangle_p \underline{u} \leq \underline{u}^{p-1}(a_1(x)\underline{u}^{m_1} - b_1(x)\underline{u}^{m-p+2} - l_1\overline{v}^n), & x \in \Omega, \\ -\triangle_q \underline{v} \leq \underline{v}^{q-1}(a_2(x) - b_2(x)\underline{v}^{t-q+2} - l_2\overline{u}^s), & x \in \Omega. \end{cases}$$

Thanks to Proposition 1, the problem (1.1) has at least one positive solution. □

Remark. According to Proposition 2 and Lemma 1, the positive solution (u, v) of the problem (1.1) obtained in Theorem 1 satisfies

$$\begin{cases} C_2 d(x)^{-\frac{p-\gamma_1}{m-p+2}} \leq u(x) \leq C_1 d(x)^{-\frac{p-\gamma_1}{m-p+2}}, & x \in \Omega, \\ C_2 d(x)^{-\frac{q-\gamma_2}{t-q+2}} \leq v(x) \leq C_1 d(x)^{-\frac{q-\gamma_2}{t-q+2}}, & x \in \Omega, \end{cases}$$

for some positive constants C_1 and C_2.

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