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COLLOCATION METHOD USING QUARTIC B-SPLINE FOR NUMERICAL SOLUTION OF THE MODIFIED EQUAL WIDTH WAVE EQUATION

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ABSTRACT. A Numerical scheme based on collocation method using quartic B-spline functions is designed for the numerical solution of one-dimensional modified equal width wave (MEW) wave equation. Using Von-Neumann approach the scheme is shown to be unconditionally stable. Performance of the method is validated through test problems including single wave, interaction of two waves and use of Maxwellian initial condition. Using error norms L_2 and L_∞ and conservative properties of mass, momentum and energy, accuracy and efficiency of the suggested method is established through comparison with the existing numerical techniques.

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1. Introduction

We consider a model for nonlinear waves, the MEW equation

$$u_t + \epsilon u^2 u_x - \mu u_{xxt} = 0. \tag{1}$$

subject to the following physical boundary conditions

$$u(a,t) = \beta_1,\ u(b,t) = \beta_2, \tag{2}$$

along with collocation boundary conditions necessary for unique quartic B-spline solution

$$u_x(a,t) = u_x(b,t) = 0,\ u_{xx}(a,t) = u_{xx}(b,t) = 0, \tag{3}$$

and initial condition

$$u(x,0) = f(x),\ a \le x \le b. \tag{4}$$

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The parameter μ is a positive constant and ϵ is an arbitrary constant, $f(x)$ is a localized disturbance inside the interval $[a, b]$ and $u \to 0$ as $x \to \pm\infty$. Here the subscripts t and x denote differentiation with respect to t and x respectively. The MEW equation was introduced by [9] as a model for nonlinear dispersive waves and is related to the modified regularized long wave (MRLW) equation [1] and the modified Korteweg-de Vries (MKdV) equation [6]. Many authors have investigated numerical solution of the problem (1)-(4). These include finite difference method [4], He's variational iteration method [7], tanh and sine-cosine methods [13] and various forms of finite element methods including collocation and Galerkin methods, (see [3, 5, 11, 12, 14] and the references therein).

In Refs. [5, 11, 12, 14] the MEW equation is solved numerically by collocation methods based on quadratic, cubic and quintic B-splines. The present method solves Eqs. (1)-(4) by using quartic B-spline collocation method. In Section 2, a new numerical method is developed. The stability analysis of the method is established in section 3 and test problems are reported in section 4 to validate performance of the method.

2. Quartic B-spline solution

In order to develop the numerical method for approximating solution of boundary value problems like the one given in Eqs. (1)-(4), the interval $[a, b]$ is partitioned into $N + 1$ uniformly spaced points x_m such that $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$ and $h = \frac{b-a}{N}$. The quartic B-splines $B_m(x), m = -2, -1, \ldots, N+1$ at the knots x_m are defined as [10]:

$$B_m(x) = \frac{1}{h^4}\begin{cases} d_1 = (x - x_{m-2})^4, & [x_{m-2}, x_{m-1}], \\ d_2 = d_1 - 5(x - x_{m-1})^4, & [x_{m-1}, x_m], \\ d_3 = d_2 + 10(x - x_m)^4, & [x_m, x_{m+1}], \\ (x_{m+3} - x)^4 - 5(x_{m+2} - x)^4, & [x_{m+1}, x_{m+2}], \\ x_{m+3} - x)^4, & [x_{m+2}, x_{m+3}], \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

and the set of quartic B-splines $\{B_{-2}, B_{-1}, ..., B_{N+1}\}$ forms a basis over the interval $[a, b]$. The numerical solution $U(x, t)$ to $u(x, t)$ is given as:

$$U(x,t) = \sum_{m=-2}^{N+1} \delta_m(t) B_m(x), \tag{6}$$

where $\delta_m(t)$ are time dependent parameters to be determined at each time level. The nodal values U_m, U'_m and U''_m at the knots x_m are derived from Eqs. (5)-(6) in the following form

$$\begin{cases} U_m = \delta_{m-2} + 11\delta_{m-1} + 11\delta_m + \delta_{m+1}, \\ U'_m = \frac{4}{h}(-\delta_{m-2} - 3\delta_{m-1} + 3\delta_m + \delta_{m+1}), \\ U''_m = \frac{12}{h^2}(\delta_{m-2} - \delta_{m-1} - \delta_m + \delta_{m+1}). \end{cases} \tag{7}$$

Eq. (1) can be rewritten as

$$(u - \mu u_{xx})_t + \epsilon u^2 u_x = 0. \tag{8}$$

Apply forward difference formula and θ-weighted scheme ($0 \leq \theta \leq 1$) to the above equation, we get

$$\frac{(U^{n+1} - \mu U_{xx}^{n+1}) - (U^n - \mu U_{xx}^n)}{\Delta t} + \theta\epsilon(U^2 U_x)^{n+1} + (1-\theta)\epsilon(U^2 U_x)^n = 0, \tag{9}$$

where Δt is time step and the superscripts n and $n+1$ denote the adjecent time levels. Take $\theta = \frac{1}{2}$. Eq.(9) gives

$$\frac{(U^{n+1} - \mu U_{xx}^{n+1}) - (U^n - \mu U_{xx}^n)}{\Delta t} + \epsilon\frac{(U^2 U_x)^{n+1} + (U^2 U_x)^n}{2} = 0. \tag{10}$$

The nonlinear term in Eq. (10) is approximated using the Taylor series:

$$(U^2)^{n+1} U_x^{n+1} \approx (U^2)^n U_x^{n+1} + 2U^n U_x^n U^{n+1} - 2(U^2)^n U_x^n. \tag{11}$$

At the nth time step, we denote U_m, U'_m, U''_m at the knots x_m by the following expressions:

$$\begin{cases} L_{m1} = \delta_{m-2}^n + 11\delta_{m-1}^n + 11\delta_m^n + \delta_{m+1}^n, \\ L_{m2} = \frac{4}{h}(-\delta_{m-2}^n - 3\delta_{m-1}^n + 3\delta_m^n + \delta_{m+1}^n), \\ L_{m3} = \frac{12}{h^2}(\delta_{m-2}^n - \delta_{m-1}^n - \delta_m^n + \delta_{m+1}^n). \end{cases} \tag{12}$$

Using the knots $x_m, m = 0, 1, ..., N$ as collocation points, the following recurrence relation at point x_m is obtained using Eqs. (10)-(12):

$$a_{m1}\delta_{m-2}^{n+1} + a_{m2}\delta_{m-1}^{n+1} + a_{m3}\delta_m^{n+1} + a_{m4}\delta_{m+1}^{n+1} = h^2(2L_{m1} + \epsilon\Delta t L_{m1}^2 L_{m2} - 2\mu L_{m3}), \tag{13}$$

where

$$\begin{aligned} a_{m1} &= L_{m0} - 4\epsilon\Delta t h L_{m1}^2 - 24\mu, \\ a_{m2} &= 11L_{m0} - 12\epsilon\Delta t h L_{m1}^2 + 24\mu, \\ a_{m3} &= 11L_{m0} + 12\epsilon\Delta t h L_{m1}^2 + 24\mu, \\ a_{m4} &= L_{m0} + 4\epsilon\Delta t h L_{m1}^2 - 24\mu, \\ L_{m0} &= 2h^2(1 + \epsilon L_{m1} L_{m2}), \\ m &= 0, 1, ..., N. \end{aligned}$$

The Eq. (13) relates parameters at adjacent time levels and gives $N+1$ equations in $N+4$ unknowns $\delta_i, i = -2, -1, ..., N+1$. In order to get a unique solution; we eliminate the parameters $\{\delta_{-2}, \delta_{-1}, \delta_{N+1}\}$. Using Eq. (7) and the boundary conditions, the values of the parameters take the form

$$\begin{cases} \delta_{-2} = \frac{33}{4}\delta_0 + \frac{7}{4}\delta_1 - \frac{3}{8}\beta_1, \\ \delta_{-1} = -\frac{7}{4}\delta_0 - \frac{1}{4}\delta_1 + \frac{1}{8}\beta_1, \\ \delta_{N+1} = -\delta_{N-2} - 11\delta_{N-1} - 11\delta_N + \beta_2. \end{cases} \tag{14}$$

Elimination of the above parameters from Eq. (13) yields a 4-banded linear system of $N+1$ equations in $N+1$ unknown parameters. The linear system can be solved by a four-diagonal solver successively for $\delta_m^n, n = 1, 2, \ldots,$; once we calculate the initial parameters δ_m^0. Finally the approximate solution $U(x,t)$ will be obtained from Eq. (7).

Using the initial and boundary conditions, the values of the initial parameters δ_m^0 at the initial time are determined with the help of the following expressions:

$$\begin{aligned} U'(x_0,0) &= \frac{4}{h}\left(-\delta_{-2}^0 - 3\delta_{-1}^0 + 3\delta_1^0 + \delta_2^0\right) = 0, \\ U''(x_0,0) &= \frac{12}{h^2}\left(\delta_{-2}^0 - \delta_{-1}^0 - \delta_0^0 + \delta_1^0\right) = 0, \\ U(x_m,0) = u(x_m,0) &= \delta_{m-2}^0 + 11\delta_{m-1-1}^0 + 11\delta_m^0 + \delta_{m+1}^0 = f(x_m), \\ U'(x_N,0) &= \frac{4}{h}\left(-\delta_{N-2}^0 - 3\delta_{N-1}^0 + 3\delta_{N+1}^0 + \delta_{N+2}^0\right) = 0, \\ m &= 0,1,...,N. \end{aligned} \tag{15}$$

Eq. (15) consists of a linear $(N+4)\times(N+4)$ system which can also be solved by a four-diagonal solver.

3. Stability analysis

In this section we apply the Von-Neumann stability method [8] for the stability of scheme developed in the previous section. Since this method is applicable to linear schemes, the nonlinear term u^2u_x is linearized by taking u as a constant value k . The linearized form of proposed scheme takes the form

$$p_1\delta_{m-2}^{n+1} + p_2\delta_{m-1}^{n+1} + p_3\delta_m^{n+1} + p_4\delta_{m+1}^n = p_4\delta_{m-2}^n + p_3\delta_{m-1}^n + p_2\delta_m^n + p_1\delta_m^n, \tag{16}$$

where

$$\begin{aligned} p_1 &= 2h^2 - 24\mu - 4h\Delta t\epsilon k^2, \\ p_2 &= 22h^2 + 24\mu - 12h\Delta t\epsilon k^2, \\ p_3 &= 22h^2 + 24\mu + 12h\Delta t\epsilon k^2, \\ p_4 &= 2h^2 - 24\mu + 4h\Delta t\epsilon k^2,\ m = 0,1,...,N. \end{aligned}$$

Substitution of $\delta_m^n = exp(i\beta mh)\xi^n, i = \sqrt{-1}$ into Eq. (16) leads to

$$\begin{aligned} &\xi\{p_1\exp(-2i\beta h) + p_2\exp(-i\beta h) + p_3 + p_4\exp(i\beta h)\} \\ &= p_4\exp(-2i\beta h) + p_3\exp(-i\beta h) + p_2 + p_1\exp(i\beta h).m \end{aligned} \tag{17}$$

Simplifying Eq. (17), we get

$$\xi = \frac{A+iB}{C+iD}, \tag{18}$$

where

$$\begin{aligned} A &= 22h^2 - 12\gamma + 24\mu + \left(24h^2 + 8\gamma\right)\cos(h\beta) + \left(2h^2 + 4\gamma - 24\mu\right)\cos(2h\beta), \\ B &= \left(-20h^2 - 16\gamma - 48\mu\right)\sin(h\beta) + \left(-2h^2 - 4\gamma + 24\mu\right)\sin(2h\beta) \\ C &= 22h^2 + 12\gamma + 24\mu + \left(24h^2 - 8\gamma\right)\cos(h\beta) + \left(2h^2 - 4\gamma - 24\mu\right)\cos(2h\beta), \\ D &= \left(-20h^2 + 16\gamma - 48\mu\right)\sin(h\beta) + \left(-2h^2 + 4\gamma + 24\mu\right)\sin(2h\beta) \\ \gamma &= dth\epsilon h^2. \end{aligned}$$

After simplification, we obtain same expressions for A^2+B^2 and C^2+D^2 in the following form:

$$
\begin{aligned}
A^2+B^2 &= C^2+D^2 \\
&= 8\left(122h^4+40\gamma^2+240h^2\mu+288\mu^2\right) \\
&\quad +8\left(143h^4-12\gamma^2+24h^2\mu-144\mu^2\right)\cos(h\beta) \\
&\quad +8\left(22h^4-24\gamma^2-240h^2\mu-288\mu^2\right)\cos(2h\beta) \\
&\quad +8\left(h^4-4\gamma^2-24h^2\mu+144\mu^2\right)\cos(3h\beta),
\end{aligned} \tag{19}
$$

so that $|\xi|^2 = 1$ and the linearized numerical scheme for the MEW equation is unconditionally stable.

4. Test problems and discussion

In this section the numerical method outlined in the previous section is tested for a single solitary wave and interactions of two solitary waves. Moreover, the Maxwellian initial condition is also considered. The accuracy of the scheme is measured in terms of the root mean square and maximum norms given by

$$L_\infty = Max_i|u_i-U_i|, \quad L_2 = \sqrt{h\sum_{i=1}^{N}(u_i-U_i)^2},$$

where u and U are exact and approximate solutions respectively. The exact solitary wave solution of MEW equation is given in [3]:

$$u(x,t) = Asech(k(x-x_0-ct))) \tag{20}$$

where $c=\frac{\epsilon A^2}{6}$ and $k=\frac{1}{\sqrt{\mu}}$. A, c represent the amplitude and velocity of a single solitary wave initially centered at x_0. The initial condition for the above problem is given by

$$u(x,0) = Asech(k(x-x_0)), \tag{21}$$

and the boundary conditions are taken from Eq. (2) with $\beta_1=\beta_2=0$. We examine the conservation properties of the MEW equation related to mass, momentum and energy by calculating the following three invariants [14]:

$$C_1=\int_a^b u\,dx,\ C_2=\int_a^b (u^2+\mu u_x^2)dx,\ C_3=\int_a^b u^4 dx. \tag{22}$$

Integrals in Eq. (22) can be approximated by the trapezoidal rule.

4.1. A single solitary wave

Problem 1. *To compare our results with* [3, 4, 5, 11], *we choose the following parameters:*

$$a=0,\ b=80,\ \epsilon=3,\ A=0.25,\ h=0.1,\ \Delta t=0.2, 0.05,\ \mu=1,\ x_0=30.$$

In order to find error norms and the invariants C_1, C_2, C_3 at different times, the computations are carried out for times upto $t = 20$. We have compared present method with earlier published papers [3, 4, 5, 11] at $t = 20$ and the results are reported in Table 1. At $t = 20$ the error norms of the present method are $L_\infty = 0.010451\times10^{-4}$, 0.009269×10^{-4} and $L_2 = 0.015789\times10^{-4}$, 0.007867×10^{-4} for the time steps 0.2 and 0.05 respectively. It is clear from Table 1 that the errors in C_1, C_2, C_3 approach zero during the simulation, showing excellent conservation properties of the new method. Hence the performance of the new method is better than the above mentioned methods. Fig. 1 shows the graphs of the single solitary wave solutions at $t = 0,\ 20$. Initially the centre of solitary wave of amplitude 0.25 is located at $x = 30$. At time $t = 20$ its magnitude is 0.249922 centered at $x = 30.6$. The absolute difference in amplitudes over the time interval $[0,\ 20]$ is observed to be 7.8×10^{-5} while it is 2×10^{-5} in the case of velocities. It can be concluded that the solitary wave moves to the right with almost constant magnitude and velocity. The error graph at time $t = 20$ is reported in Fig. 2. It is clear from the graph that the maximum errors occur around the central position of the solitary wave.

The pointwise rates of convergence in space and time are calculated using the following expressions respectively, [2]:

$$\frac{\log_{10}\left(\|u-U_{h_i}\|/\|u-U_{h_{i+1}}\|\right)}{\log_{10}\left(h_i/h_{i+1}\right)} \quad \text{and} \quad \frac{\log_{10}\left(\|u-U_{\Delta t_i}\|/\|u-U_{\Delta t_{i+1}}\|\right)}{\log_{10}\left(\Delta t_i/\Delta t_{i+1}\right)},$$

where u represents exact solution and U_{h_i} and $U_{\Delta t_i}$ the numerical solutions with spatial step size h_i and time step size Δt_i respectively. Computations are carried out with different spatial and time step sizes in order to examine the point rates of convergence in space and time. In order to caculate the spatial rate of convergence the time step is kept fixed at $\Delta t = 0.05$ and the number of collocation points $N = 80,\ 160,\ 320,\ 640; 800$ is varied. The results are recorded in Table 2. It can observed from the table that the convergence rate decreases with the smaller spatial step size. For the computation of the time rate of convergence the number of collocation points is kept fixed at $N = 600$ and the time step size $\Delta t_i = 1,\ 0.5,\ 0.25,\ 0.125$ is varied. The results are tabulated in Table 3. It is concluded from the table that the time rate of convergence almost decreases for the smaller time step size.

The same problem is also considered for different values of the amplitude at time step of 0.01. In Table 4 the error norms and invariants are summarized for $A = 0.25,\ 0.5,\ 0.75,\ 1.0$. It is observed that the errors are smaller and the invariants remain constant during the simulation. The new method is compared with [4] and the comparison of error norms declares superiority of our scheme. Fig. 3 shows the graphs of the solutions for $A = 0.25,\ 0.5,\ 0.75,\ 1.0$ at $t = 20$.

Problem 2. *In order to compare our method with earlier work* [3, 12, 14] *we choose the parameters*

$$a = 0,\ b = 70,\ \epsilon = 3,\ A = 0.25,\ 0.5,\ 1.0,\ h = 0.1,\ \Delta t = 0.05,\ \mu = 1,\ x_0 = 30.$$

TABLE 1. Invariants and error norms for Problem 1 with single solitary wave

Δt	Time	$L_\infty \times 10^4$	$L_2 \times 10^4$	C_1	C_2	C_3
0.2	0	0.0	0.0	0.785398	0.166667	0.005208
	5	0.002706	0.004075	0.785398	0.166667	0.005208
	10	0.005377	0.008094	0.785398	0.166667	0.005208
	15	0.007944	0.012009	0.785398	0.166667	0.005208
	20	0.010451	0.015789	0.785398	0.166667	0.005208
	20[4]	2.576377	2.701647	0.785398	0.166474	0.005208
	20[5]	1.569539	2.021476	0.785286	0.166582	0.005206
	20[11]	1.744330	1.958879	0.784668	0.166434	0.005194
0.05	0	0.0	0.0	0.785398	0.166667	0.005208
	5	0.002357	0.002128	0.785398	0.166667	0.005208
	10	0.004788	0.004186	0.785398	0.166667	0.005208
	15	0.007141	0.006114	0.785398	0.166667	0.005208
	20	0.009269	0.007867	0.785398	0.166667	0.005208
	20[3]	0.465523	0.796940	0.785390	0.166761	0.005208
	20[4]	2.569972	2.692812	0.785398	0.166474	0.005208
	20[5]	2.498925	2.905166	0.784955	0.166477	0.005200

$\epsilon = 3,\ A = 0.25,\ h = 0.1,\ \mu = 1,\ x_0 = 30,\ 0 \leq x \leq 80$

TABLE 2. Space rate of convergence at $t = 20$, $A = 0.25$, $\Delta t = 0.05$, $x_0 = 30$, $0 \leq x \leq 80$

N	L_∞	Order
80	$1.1371E^{-3}$	4.7773
160	4.1468^{-5}	4.4718
320	$1.8687E^{-6}$	4.0216
640	$1.506E^{-7}$	4.2560

TABLE 3. Time rate of convergence at $t = 20$, $A = 0.25$, $N = 600$, $x_0 = 30$, $0 \leq x \leq 80$

Δt_i	L_∞	Order
1	$2.5234E^{-5}$	2.9770
0.5	3.2049^{-6}	2.4048
0.25	$6.0520E^{-7}$	0.7441
0.125	$3.6132E^{-7}$	0.9768

The simulation is performed upto time $t = 20$. Error norms and invariants are recorded for different values of t and tabulated in Table 5. It is observed that the accuracy of the scheme in terms of error norms increases for decreasing values of A. For $A = 1, 0.25$ and $t = 20$ the error norms of the present method are found as $L_\infty = 1.095929 \times 10^{-3},\ 9.27 \times 10^{-7}$ and $L_2 = 1.747622 \times 10^{-3},\ 7.878 \times 10^{-7}$.

The invariant quantities $C_1,\ C_2,\ C_3$ are almost constant during the simulation. In Table 5 we have also compared our results with lumped Galerkin method using quadratic B-spline functions [3], collocation and petro- Galerkin methods using quintic B-splines [12, 14] at $t = 20$. In this problem it is observed that the

TABLE 4. Invariants and error norms for Problem 1 with single solitary wave for various values of A

A	Time	$L_\infty \times 10^4$	$L_2 \times 10^4$	C_1	C_2	C_3
0.25	0	0.0	0.0	0.785398	0.166667	0.005208
	5	0.002431	0.002174	0.785398	0.166667	0.005208
	10	0.004952	0.004282	0.785398	0.166667	0.005208
	15	0.007409	0.006267	0.785398	0.166667	0.005208
	20	0.009651	0.0080877	0.785398	0.166667	0.005208
	20[4]	2.569562	2.692249	–	–	–
0.50	0	0.0	0.0	1.570796	0.666667	0.083333
	5	0.018825	0.015873	1.570796	0.666667	0.083333
	10	0.028470	0.026252	1.570796	0.666667	0.083333
	15	0.028486	0.032155	1.570796	0.666667	0.083333
	20	0.028502	0.036027	1.570796	0.666667	0.083333
	20[4]	14.57568	18.26059	–	–	–
0.75	0	0.0	0.0	2.356194	1.500000	0.421875
	5	0.035184	0.039624	2.356194	1.500000	0.421875
	10	0.033608	0.047029	2.356194	1.500000	0.421875
	15	0.035750	0.049007	2.356194	1.500000	0.421875
	20	0.036156	0.051698	2.356194	1.500000	0.421875
	20[4]	30.91793	43.95711	–	–	–
1.0	0	0.0	0.0	3.141593	2.666667	1.333333
	5	0.095600	0.133043	3.141593	2.666667	1.333333
	10	0.185455	0.261374	3.141593	2.666667	1.333333
	15	0.276137	0.403847	3.141593	2.666667	1.333333
	20	0.366993	0.549764	3.141593	2.666667	1.333333
	20[4]	56.82131	82.85314	–	–	–

$\epsilon = 3,\ h = 0.1,\ \Delta t = 0.01,\ \mu = 1,\ x_0 = 30,\ 0 \leq x \leq 80$

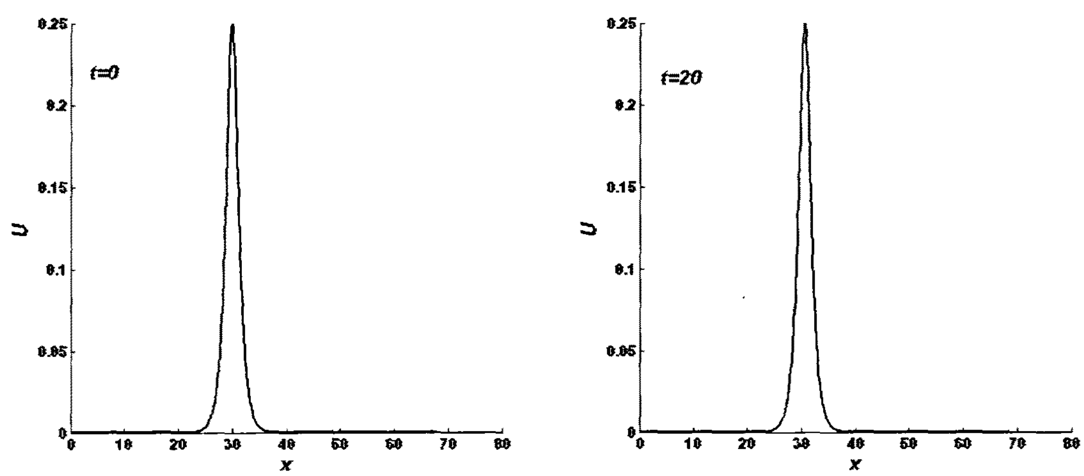

Figure 1. Single solitary wave solution at $t = 0,\ 20$

accuracy of different schemes also depend on amplitude A. Performance of the

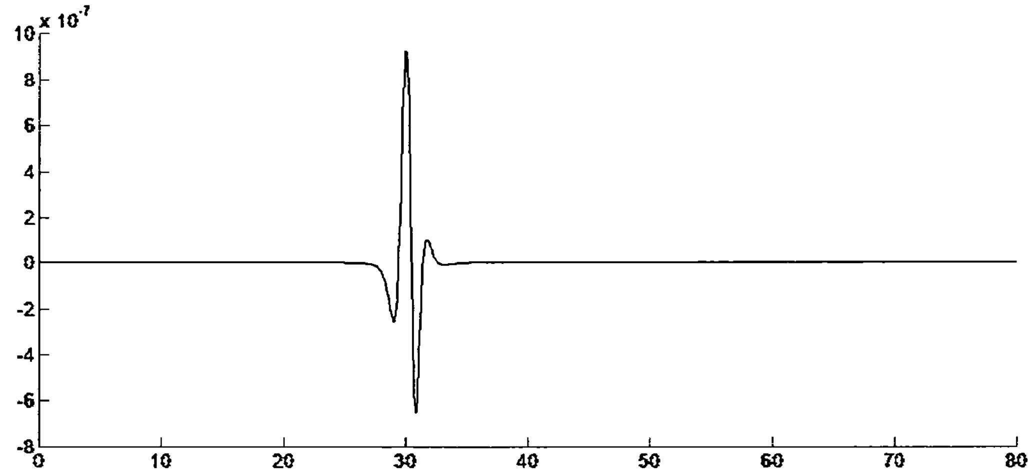

Figure 2. Error graph at $t = 20$

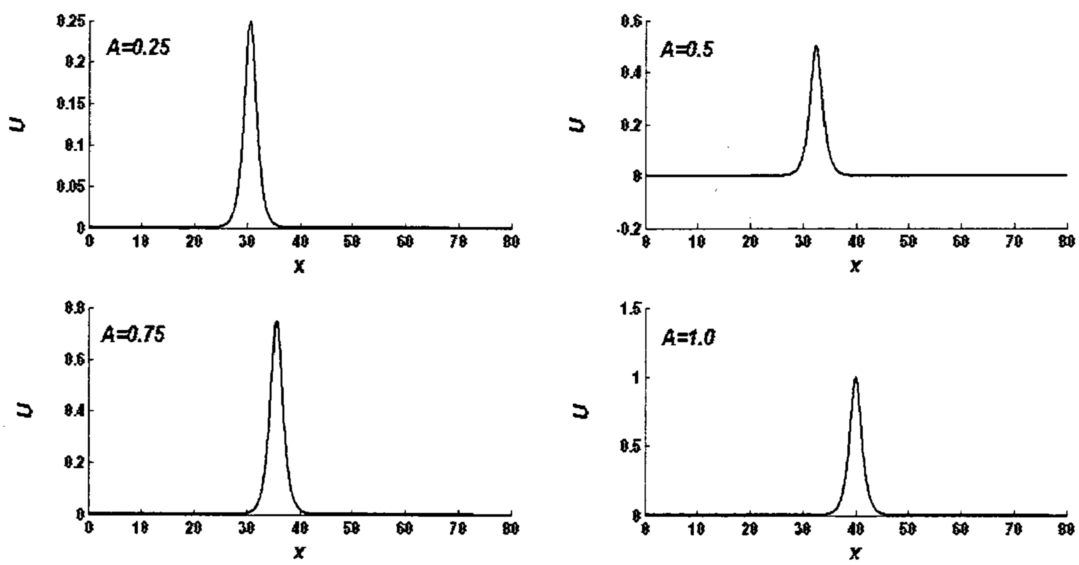

Figure 3. Solitary wave solution for various values of A at $t = 20$

present method is better than [3, 14] when A=0.25 but [12] has an edge in this case. Our scheme is comparable with [14] for $A = 0.50$ while [12] performs better. Furthermore for this problem error norms in Refs. [12, 14] are less than the present method when $A = 1$.

4.2. Interaction of two solitary waves

Problem 3. *For the sake of comparison with the results of Refs.* [3, 5, 12], *the following parameters are chosen*

$$\epsilon = 3,\ A_1 = 1.0,\ A_2 = 0.5,\ x_1 = 15,\ x_2 = 30,\ \mu = 1,\ h = 0.1,\ \Delta t = 0.2,\ 0 \le x \le 80.$$

TABLE 5. Invariants and error norms for Problem 2 with single solitary wave

A	Time	$L_\infty \times 10^3$	$L_2 \times 10^3$	C_1	C_2	C_3
0.25	0	0.0	0.0	0.785398	0.166667	0.005208
	5	0.000236	0.000213	0.785398	0.166667	0.005208
	10	0.000479	0.000419	0.785398	0.166667	0.005208
	15	0.000714	0.000611	0.785398	0.166667	0.005208
	20	0.000927	0.000787	0.785398	0.166667	0.005208
	20[5]	0.046009	0.080145	0.785393	0.166764	0.005208
	20[12]	0.00032	0.00027	0.785398	0.166667	0.005208
	20[14]	0.00203	0.00345	0.78539	0.16667	0.00521
0.50	0	0.0	0.0	1.570796	0.666667	0.083333
	5	0.002090	0.003158	1.570796	0.666667	0.083333
	10	0.004018	0.005902	1.570796	0.666667	0.083333
	15	0.005900	0.008422	1.570796	0.666667	0.083333
	20	0.007787	0.010999	1.570796	0.666667	0.083333
	20[12]	0.00640	0.00920	1.570796	0.666667	0.083333
	20[14]	0.00852	0.01172	1.57078	0.66666	0.08333
1.0	0	0.0	0.0	3.141593	2.666668	1.333333
	5	0.266834	0.410369	3.141593	2.666663	1.333329
	10	0.541117	0.845886	3.141593	2.666659	1.333324
	15	0.817482	1.294345	3.141593	2.666654	1.333320
	20	1.095929	1.747622	3.141593	2.666650	1.333316
	20[12]	0.65318	1.04778	3.141593	2.666667	1.333334
	20[14]	0.08360	0.14465	3.14165	2.66676	1.33343

$\epsilon = 3,\ h = 0.1,\ \Delta t = 0.05,\ \mu = 1,\ x_0 = 30,\ 0 \leq x \leq 70$

To study the interaction of two solitary waves we use the following initial condition:

$$u(x,0) = \sum_{1}^{2} A_i sech(k(x - x_i)), \tag{23}$$

where

$$A_i = \sqrt{\frac{6c_i}{\epsilon}},\ k = \frac{1}{\sqrt{\mu}}.$$

The parameters give two solitary waves having amplitudes of ratio 2 : 1 and their peak positions are located at $x = 15$ and 30. The analytical values of the invariants C_1, C_2, C_3 for the above parameters are given in [5, 12] as:

$$C_1 = \pi(A_1 + A_2) = 4.712389, C_2 = \frac{8}{3}\left(A_1^2 + A_2^2\right) = 3.333333, C_3 = \frac{8}{3}\left(A_1^4 + A_2^4\right) = 1.416667$$

The calculations are performed from $t = 0$ to $t = 80$ and values of the invariant quantities C_1, C_2, C_3 are tabulated in Table 6 for the present method and are compared with Refs. [4, 5]. It can be seen from the table that the invariants remain satisfactorily constant throughout the simulation. The upper bounds for absolute error in the invariants C_1, C_2, C_3 from $t = 0$ to $t = 55$ are less than 1.0×10^{-7}, 2.6×10^{-3} respectively. The same errors in Refs. [4, 5] are less than

TABLE 6. Invariants for Problem 3 with interaction of two waves

	present method			[4]			[5]		
Time	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3
0	4.712389	3.333337	1.416669	4.712388	3.329462	1.416669	4.712388	3.332357	1.416670
10	4.712389	3.332777	1.416108	4.712389	3.328927	1.416103	4.712022	3.324678	1.400768
20	4.712389	3.332191	1.415520	4.712387	3.328361	1.415523	4.711697	3.324210	1.401182
30	4.712389	3.330775	1.413861	4.712388	3.327818	1.413882	4.711242	3.346583	1.424847
40	4.712389	3.330942	1.414043	4.712385	3.327112	1.414050	4.711017	3.321250	1.398239
50	4.712389	3.330976	1.414314	4.712388	3.326632	1.414330	4.710804	3.320956	1.398729
55	4.712389	3.330701	1.414043	4.712386	3.326393	1.414062	4.710630	3.323628	1.399068
60	4.712389	3.330417	1.413763	4.712388	3.326228	1.413785	–	–	–
70	4.712389	3.329849	1.413199	4.712388	3.325891	1.413228	–	–	–
80	4.712389	3.329283	1.412635	4.712389	3.325434	1.412671	–	–	–

$\epsilon = 3,\ A_1 = 1,\ A_2 = 0.5,\ x_1 = 15,\ x_2 = 30,\ h = 0.1,\ \Delta t = 0.2,\ 0 \le x \le 80$

TABLE 7. Invariants quantities for Problem 3 with interaction of two waves

	present method			[12]		
Time	C_1	C_2	C_3	C_1	C_2	C_3
0	4.712389	3.333336	1.416669	4.7123884	3.3333358	1.4166697
5	4.712389	3.333336	1.416669	4.7123895	3.3333358	1.4166697
10	4.712389	3.333336	1.416669	4.7123896	3.3333358	1.4166697
15	4.712389	3.333335	1.416668	4.7123896	3.3333358	1.4166697
20	4.712389	3.333334	1.416667	4.7123896	3.3333358	1.4166696
25	4.712389	3.333332	1.416664	4.7123896	3.3333358	1.4166690
30	4.712389	3.333318	1.416647	4.7123896	3.3333359	1.4166648
35	4.712389	3.333295	1.416615	4.7123897	3.3333359	1.4166568
40	4.712389	3.333325	1.416655	4.7123896	3.3333358	1.4166669
45	4.712389	3.333332	1.416664	4.7123896	3.3333357	1.4166695
50	4.712389	3.333332	1.416665	4.7123896	3.3333357	1.4166698
55	4.712389	3.333332	1.416665	4.7123896	3.3333357	1.4166698

$\epsilon = 3,\ A_1 = 1,\ A_2 = 0.5,\ x_1 = 15,\ x_2 = 30,\ h = 0.1,\ \Delta t = 0.025,\ 0 \le x \le 80$

2.0×10^{-6}, 3.1×10^{-3}, 2.6×10^{-3}, 2.6×10^{-3} and 1.8×10^{-3}, 8.7×10^{-3}, 1.8×10^{-2}. Fig. 4 shows the state of interaction and then separation of solitary waves at times $t = 30$, 35, 40 and $t = 55$, 80 in sequel. Initially the larger wave of amplitude 1 is centered at $x = 15$ and the smaller one of amplitude 0.5 at $x = 30$. Since the velocity of larger wave is 0.5 and that of the smaller 0.125, the larger wave moves faster than the smaller and hence collides with it. At $t = 80$ the amplitude of the larger wave is 0.9993 centered at $x = 56.8$ and that of smaller 0.4988 with peak position located at $x = 37.7$. Hence during this interaction the amplitude is almost unchanged. The absolute difference in amplitude for larger wave is 7.0×10^{-4} and that of smaller 1.2×10^{-3}, consequently the velocities of the waves are almost maintained after the interaction. Thus the waves interact and then emerge from the collision by preserving their shapes and velocities. We have solved the same problem with $\Delta t = 0.025$ and the invariants are reported in Table 7 along with those of Ref. [12]. It is evident from the comparison of Tables 6-7 that the conservation properties of the present method are excellent when time step is reduced.

4.3. The Maxwellian initial condition

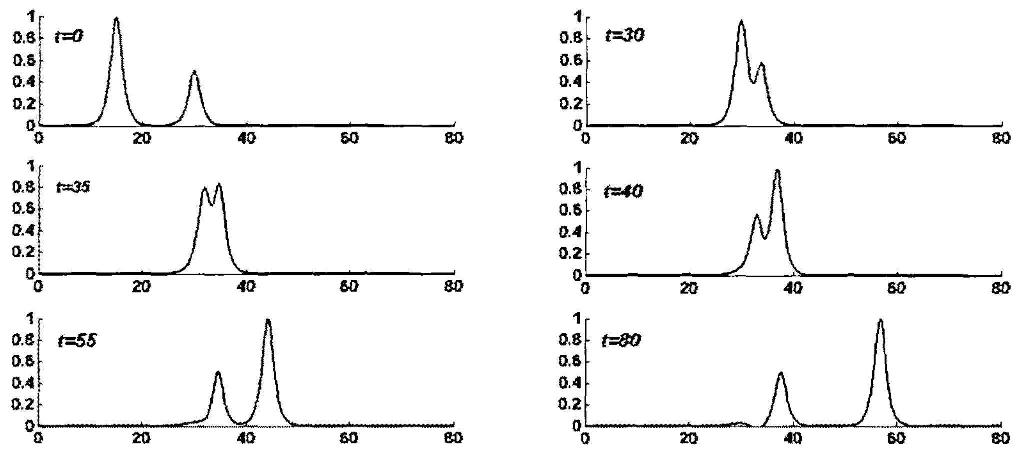

Figure 4. Interaction of two solitary waves at selected times

Problem 4. *The birth of solitary waves is considered using the Maxwellian initial condition*

$$u(x,0) = \exp\left(-x^2\right) \tag{24}$$

The following parameters are chosen

$$\epsilon = 3,\ \mu = 1,\ 0.5,\ 0.1,\ 0.05,\ 0.02,\ 0.005,\ h = 0.1,\ \Delta t = 0.01,\ -20 \leq x \leq 20.$$

In the case of Maxwellian condition the behavior of the solution depends on the values of μ. The Maxwellian does not break up into solutions for $\mu >> \mu_c$ where μ_c is some critical number, and exhibits rapidly oscilating wave packets. When $\mu \approx \mu_c$, a mixed type of solution is obtained consisting of a leading soliton with an oscillating tail [14]. The Maxwellian breaks up into a number of solitons according to the value of μ when $\mu << \mu_c$. Simulations are performed upto time $t = 12$. For $\mu = 1,\ 0.5$ the Maxwellian shows an oscillatory behavior and no clean waves are obtained as shown in Fig. 5. For $\mu = 0.1,\ 0.05,\ 0.02,\ 0.005$ the number of observed solitary waves is 1, 2, 3, 7 respectively as shown in Fig. 5. The graphs are in good agreement with earlier work [12, 14]. It is also clear from Fig. 5 that the peaks of solitary waves lie on the straight line. For various values of μ the conservative quantities are tabulated in Table 8 which remain almost constant during the simulation.

5. Conclusion

Quartic B-spline collocation method is employed to model the motion and interaction of solitary waves of the MEW equation. Four test problems are chosen from literature to validate performance of the suggested method. The maxwellian initial condition is also studied. The accuracy of the method is checked through L_2, L_∞ error norms and the conserved quantities C_1, C_2, C_3. It has been observed that the errors are sufficiently small and the invariants are almost kept constant during simulation. The obtained results are in agreement with some earlier results from literature. Linear stability analysis proved that the new

TABLE 8. Invariants for Maxwellian initial condition

μ	Time	C_1	C_2	C_3
1	0	1.772454	2.506634	0.886227
	3	1.772453	2.506637	0.886223
	6	1.772456	2.506629	0.886231
	9	1.772456	2.506627	0.886232
	12	1.772456	2.506627	0.886232
0.5	0	1.772454	1.879974	0.886227
	3	1.772452	1.879979	0.886219
	6	1.772454	1.879972	0.886227
	9	1.772454	1.879972	0.886227
	12	1.772454	1.879972	0.886226
0.1	0	1.772454	1.378646	0.886227
	3	1.772437	1.378721	0.886144
	6	1.772435	1.378718	0.886138
	9	1.772430	1.378715	0.886130
	12	1.772427	1.378712	0.886123
0.05	0	1.772454	1.315980	0.886227
	3	1.772355	1.316331	0.885265
	6	1.772273	1.315714	0.884471
	9	1.772224	1.315072	0.883746
	12	1.772167	1.314636	0.883001

$\epsilon = 3,\ h = 0.1,\ \Delta t = 0.01,\ -20 \leq x \leq 20$

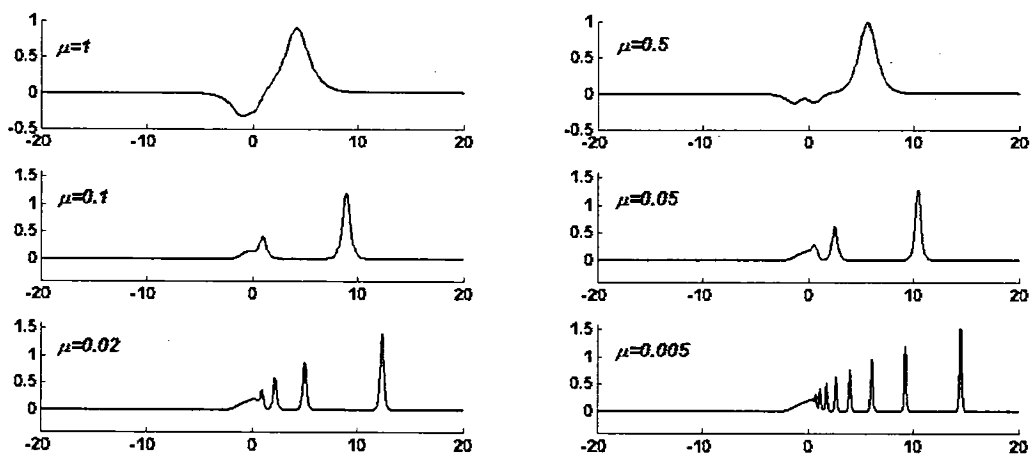

Figure 5. Maxwellian initial condition for different values of μ at $t = 12$

method is unconditionally stable theoretically and this has been supported by the test problems as well.

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