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APPLICATION OF PSEUDO Z_p INDEX THEORY TO PERIODIC SOLUTIONS WITH MINIMAL PERIOD FOR DISCRETE HAMILTONIAN SYSTEMS

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ABSTRACT. By making use of minimax theory and pseudo Z_p index theory, some results on the existence and multiplicity of periodic solutions with minimal period to nonconvex superquadratic discrete Hamiltonian systems are obtained.

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1. Introduction

In this paper, we shall define a new pseudo Z_p index theory and, as applications, give some existence and multiplicity results for a class of nonconvex superquadratic discrete Hamiltonian systems. Precisely, for any given integer $p > 1$, consider the problem

$$J\Delta x(n) = b(n)H'(Lx(n)) \tag{1}$$

where $x(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}, Lx(n) = \begin{pmatrix} x_1(n+1) \\ x_2(n) \end{pmatrix}, x_1, x_2 \in \mathbf{R}^N$, and $\Delta x_i(n) = x_i(n+1) - x_i(n)$, $i = 1, 2$, is the forward difference operator. J is the standard symplectic matrix $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ where I_N is the identity on $\mathbf{R}^N$ and N is a positive integer. $H(x) \in C^1(\mathbf{R}^{2N}, \mathbf{R})$ and $b(t) \in C(\mathbf{R}, \mathbf{R}^+)$ is periodic with minimal period T, here T is a given positive integer.

(1) can be regarded as a discrete analogue of

$$Jx'(t) = b(t)H'(x(t)). \tag{2}$$

As to (2), it is a classical Hamiltonian system and play an important pole in the study of modern physics, dynamics systems, astrophysics, nuclear physics and

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chemically reacting systems, etc.; see, for example [3, 4]. Classically, pT-periodic solutions for (2) are called subharmonic solutions. Concerning the existence of subharmonic solutions with minimal periods of (2) has a long and prestigious history, the first result should go back to [1] (see also [2] for a different proof). P.H.Rabinowitz gave a conjecture that the Hamiltonian systems had nonconstant solutions with prescribed minimal period under his given conditions in 1978[11]. After then, many out-standing researchers all-around the world such as Arnold, Morser, Ambrosetti, Rabinowitz, Eklend, Benci, Chang K.C, Long Y.M., ect., devoted themselves to the problem and made great progress on it, we read [2, 6, 7, 8, 9, 10, 12] for detail. (1) is the best approximations of (2) when one lets the step size not be equal to 1 but the variable's step size go to zero, so the solutions of (1) can give some desirable numerical features for (2).

(1) may arise from various fields such as electrical circuit analysis, matrix theory, control theory and discrete variational theory etc, see for example [13, 14]. At the same time, we also find that difference equations are closely related to differential equations in the sense that (i) a differential equation model is usually derived from a difference equation, and (ii) numerical solutions of a differential equation have to be obtained by discretizing the differential equation (thus resulting in difference equations). Therefore, it is of practical importance and mathematical significance to consider the existence and multiplicity of minimal periodic solutions of (1).

It is well-known that with the sharp development of difference equations, there are many excellent works have achieved in the past decade. Write $\nabla H(n, x(n)) = b(n)H'(x(n))$, concerning the subharmonic solutions for (1), Guo and Yu [18] obtained some existence and multiplicity results by Z_2 index theory and linking theorem when (1) are superquadratic Hamiltonian systems. In [19], when H is subquadratic at infinity, the authors gave some existence results of periodic solutions. Recently, [20] studied (1) with forced term, i.e.

$$-J\Delta x(n) + \nabla H(n, x(n)) = f(n)$$

where $f(n) = \begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix}$, by perturbation technique and dual least action principle, and gave some existence results of periodic solutions. However, as known to us, it seems that no similar results in the literature on the existence and the multiplicity of subharmonic solutions with minimal period of (1) have been obtained, because there are few known techniques for studying the multiplicity of minimal period problem of discrete systems. In view of this, the main purpose of this paper is to look for a new approach to study the multiplicity of periodic solutions with minimal periods for (1) by using geometrical theory and critical point theory.

Denote $x = (x_1^T, x_2^T)^T$, $H(x) = H(x_1, x_2)$. We need following assumptions:

Assumption 1. For all $x \in \mathbf{R}^{2N}$, $H(x) \geq a_0|x|^\beta$, where constants $a_0 > 0$ and $\beta > 2$.

Assumption 2. $\beta H(x) \leq (x, H'(x))$, $\forall x \in \mathbf{R}^{2N}$.

Assumption 3. $|H'(x)| \leq a_1|x|^{\beta-1}$, $\forall x \in \mathbf{R}^{2N}$, where constant $a_1 > 0$.
Assumption 4. If $x = x(n)$, $H'(x(n))$ both are periodic function with minimal period qT, q is rational, then q is necessarily an integer.

Remark 5. By assumption 2, there exist constants $b_1 > 0$, $b_2 > 0$ such that

$$H(x) \geq b_1|x|^\beta - b_2$$

holds for all $x \in \mathbf{R}^{2N}$. This means that $H(x)$ satisfies superquadratic condition at infinity.

For our convenience, let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ be the set of all natural numbers, integers, and real numbers, respectively. Throughout this paper, without special statement, $M = \max_{t\in[0,T]} b(t)$, $m = \min_{t\in[0,T]} b(t) > 0$, $|\cdot|$ denotes the usual norm in $\mathbf{R}^N$ with $N \in \mathbf{N}$, $(\cdot,\cdot)$ stands for the inner product, and $\cdot^\tau$ is the transpose of a matrix or vector. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a \leq b$. Set $s_p =$ the smallest prime factor of p.

2. Variational structure and some preparatory results

To apply minimax theory to study the existence of minimal period solutions of (1), we shall establish a suitable structure and give some preparatory results, which will be used in the proofs of our main results..

Let

$$S = \{\{x(n)\}|x(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} \in \mathbf{R}^{2N}, x_j(n) \in \mathbf{R}^N, j = 1, 2, n \in \mathbf{Z}\}$$

be the vector space. For any given positive integers p, T, define the subspace E_{pT} of S as $E_{pT} = \{x = \{x(n)\} \in S|x(n + pT) = x(n), n \in \mathbf{Z}\}$ equipped with the inner product

$$\langle x, y\rangle_{E_{pT}} = \sum_{j=1}^{pT} x(j)\cdot y(j), \quad \forall x \in E_{pT}, \quad y \in E_{pT},$$

and the norm

$$\|x\|_{E_{pT}} = \left(\sum_{n=1}^{pT} |x(n)|^2\right)^{\frac{1}{2}}, \forall x \in E_{pT},$$

where $(\cdot,\cdot)$ and $|\cdot|$ denote the usual inner product and norm of $\mathbf{R}^{2N}$ respectively. Then $(E_{pT}, \langle\cdot,\cdot\rangle)$ is a $2pTN$-dimensional Hilbert space and is linearly homeomorphic to $\mathbf{R}^{2pNT}$. For later use, we define another norm on $(E_{pT}, <\cdot,\cdot>_{E_{pT}})$ for $r > 1$ by

$$\|x\|_r = \left(\sum_{j=1}^{pT} |x(j)|^r\right)^{\frac{1}{r}}, \quad \forall x \in E_{pT}.$$

Write $\|x\|_{E_{pT}} = \|x\|$, it is obviously that $\|x\| = \|x\|_2$, then there exist constants $c_2 \geq c_1 > 0$ such that $c_1\|x\|_r \leq \|x\| \leq c_2\|x\|_r$, $\quad \forall x \in E_{pT}$.

For all $x \in E_{pT}$, consider the functional I

$$I(x) = \frac{1}{2}\sum_{n=1}^{pT}(J\Delta(Lx(n-1)), x(n)) - \sum_{n=1}^{pT} b(n)H(Lx(n)), \tag{3}$$

where $Lx(n) = \begin{pmatrix} x_1(n+1) \\ x_2(n) \end{pmatrix}$. It is easy to see that $I(x) \in C^1(E_{pT}, \mathbf{R})$ and

$$\frac{\partial I(x)}{\partial x_1(n+1)} = -\Delta x_2(n) + b(n)\cdot H_{x_1}(x_1(n+1), x_2(n)), \quad \forall n \in [1, pT],$$

$$\frac{\partial I(x)}{\partial x_2(n)} = \Delta x_1(n) + b(n)\cdot H_{x_2}(x_1(n+1), x_2(n)), \quad \forall n \in [1, pT].$$

Then it follows $I'(x) = 0$ holds if and only if when $n \in [1, pT]$,

$$\begin{cases} \Delta x_2(n) = b(n)\cdot H_{x_1}(x_1(n+1), x_2(n)) \\ \Delta x_1(n) = -b(n)\cdot H_{x_2}(x_1(n+1), x_2(n)) \end{cases}$$

that is, $J\Delta x(n) = b(n)H'(Lx(n))$, this is just (1). Hence, critical points x of $I(x)$ on E_{pT} are corresponding to pT-periodic solutions $x(n)$ of (1).

Next we consider the eigenvalue problem

$$J\Delta(Lx(n-1)) = \lambda x(n) \qquad x(n+pT) = x(n). \tag{4}$$

(4) can be reformed as

$$\begin{cases} x_1(n+1) = (1-\lambda^2)x_1(n) - \lambda x_2(n) \\ x_2(n+1) = \lambda x_1(n) + x_2(n) \\ x_1(n+pT) = x_1(n), x_2(n+pT) = x_2(n) \end{cases}$$

that is

$$\begin{pmatrix} (1-\lambda^2)I_N & -\lambda I_N \\ \lambda I_N & I_N \end{pmatrix}\begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} = \begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix}, x(n+pT) = x(n).$$

Denote $A(\lambda) = \begin{pmatrix} (1-\lambda^2)I_N & -\lambda I_N \\ \lambda I_N & I_N \end{pmatrix}$, then (4) can be expressed by

$$x(n+1) = A(\lambda)\cdot x(n) \qquad x(n+pT) = x(n). \tag{5}$$

Let $u(n) = v^n c$, where $c \in \mathbf{R}^{2N}$ and $c \neq 0$, be the corresponding eigenvector to the eigenvalue problem of (4), then $u(n)$ satisfies (5), hence $vc = A(\lambda)c, v^{pT} = 1$. By direct calculation, we have

$$v = e^{2k\pi i/pT}, \lambda = 2\sin\frac{k\pi}{pT} \quad k \in [-pT+1, pT-1]$$

because of $|A(\lambda) - vI_{2N}| = 0$ and $v^{pT} = 1$. For any $k \in [-pT+1, pT-1]$, write $v_k = e^{2k\pi i/pT}, \lambda_k = 2\sin\frac{k\pi}{pT}, \quad \rho \in \mathbf{R}^N$.

Next we have discussion on the eigenvector $u(n)$ according to the value of k.

case 1: $k = 0$

$\lambda_0 = 0$ is a $2N$ multiple eigenvalue and $u_0(n) = v_0^n \cdot c = c \in \mathbf{R}^{2N}$.

case 2: $k = \pm\frac{pT}{2}$ and pT is even (*i*) $k=\frac{pT}{2}$, $\lambda_{\frac{pT}{2}}=2$ is an N multiple eigenvalue, $v_{\frac{pT}{2}}=e^{k\pi i}=-1$, $c=(\rho^\tau,\rho^\tau)^\tau\in\mathbf{R}^{2N}$, $u_{\frac{pT}{2}}(n)=(-1)^n\cdot(\rho^\tau,\rho^\tau)^\tau$.

(*ii*) $k=-\frac{pT}{2}$, $\lambda_{-\frac{pT}{2}}=-2$ is an N multiple eigenvalue, $v_{-\frac{pT}{2}}=e^{-k\pi i}=-1$, $c=(\rho^\tau,-\rho^\tau)^\tau\in\mathbf{R}^{2N}$, $u_{-\frac{pT}{2}}(n)=(-1)^n\cdot(\rho^\tau,-\rho^\tau)^\tau$.

case 3: $k\in[-[\frac{pT-1}{2}],-1]\cup[1,[\frac{pT-1}{2}]]$, where $[\cdot]$ is the greatest integer function. v_k satisfies $(A(\lambda_k)-v_kI_{2N})c=0$, then λ_k is a $2N$ multiple eigenvalue and $c=-i\begin{pmatrix}\rho\\ ie^{-ik\pi/pT}\rho\end{pmatrix}$. Therefore $u_k(n)$ can be given by

$$u_k(n)\qquad=v_k^n\cdot c=e^{\frac{2k\pi i}{pT}n}\cdot\begin{pmatrix}-i\rho\\ e^{-ik\pi/pT}\rho\end{pmatrix}=\begin{pmatrix}e^{(\frac{2k\pi}{pT}n-\frac{\pi}{2})i}\rho\\ e^{(\frac{2k\pi}{pT}n-\frac{k\pi}{pT})i}\rho\end{pmatrix}.$$

From the above analysis, the space E_{pT} can be split as

$$E_{pT}=W\bigoplus Y \tag{6}$$

where $W=\operatorname{span}\{u_0\}$, $Y=\operatorname{span}\{u_k,k=-[\frac{pT}{2}],-[\frac{pT}{2}]+1,\cdots,-1,1,\cdots,[\frac{pT}{2}]\}$.

It follows that for any $x(n)\in Y$, we can write $x(n)$ as

$$x(n)=\sum_{k=-[\frac{pT}{2}],k\neq0}^{[\frac{pT}{2}]}\left[\begin{pmatrix}\sin(\frac{2k\pi}{pT}n)\cdot a_k\\ \cos(\frac{2k\pi}{pT}n-\frac{k\pi}{pT})\cdot a_k\end{pmatrix}+\begin{pmatrix}-\cos(\frac{2k\pi}{pT}n)\cdot b_k\\ \sin(\frac{2k\pi}{pT}n-\frac{k\pi}{pT})\cdot b_k\end{pmatrix}\right],$$

here $a_k,b_k\in\mathbf{R}^N$.

For the convenience of the later discussion, we express any $x(n)\in Y$ as

$$x(n)=\sum_{k=-[\frac{pT}{2}],k\neq0}^{[\frac{pT}{2}]}\begin{pmatrix}e^{(\frac{2k\pi}{pT}n-\frac{\pi}{2})i}\cdot\xi_k\\ e^{(\frac{2k\pi}{pT}n-\frac{k\pi}{pT})i}\cdot\xi_k\end{pmatrix},\quad \xi_k\in\mathbf{C}^N.$$

Lemma 6. *For any $x\in E_{pT}$, it has*

$$-\lambda_{max}\|x\|^2\le\sum_{n=1}^{pT}(J\Delta(Lx(n-1)),x(n))\le\lambda_{max}\|x\|^2,$$

and if $x\in Y$,

$$\lambda_{min}^2\|x\|^2\le\sum_{n=1}^{pT}|\Delta x(n)|^2\le\lambda_{max}^2\|x\|^2,$$

where $\lambda_{min}=\min\{\lambda_1,\cdots,\lambda_{pT-1}\}=2\sin\frac{\pi}{pT}$, $\lambda_{max}=\max\{\lambda_1,\cdots,\lambda_{pT-1}\}$.

Lemma 7. *For any $x(j)>0,y(j)>0,j\in[1,n],n\in\mathbf{Z}$,*

$$\sum_{j=1}^n x(j)y(j)\le\left(\sum_{j=1}^n x^r(j)\right)^{\frac{1}{r}}\cdot\left(\sum_{j=1}^n y^s(j)\right)^{\frac{1}{s}},$$

where $r>1,s>1,\frac{1}{r}+\frac{1}{s}=1$.

Lemma 8. *For any $x \in Y$*

$$\sum_{n=1}^{pT}(J\Delta(Lx(n-1)), x(n)) \leq (2\sin\frac{\pi}{pT})^{-1}\sum_{n=1}^{pT}|\Delta x(n)|^2.$$

Proof. For any $x \in Y$, by Lemmas 6 and 7, we have

$$\begin{aligned}\sum_{n=1}^{pT}(J\Delta(Lx(n-1)), x(n)) &\leq \left(\sum_{n=1}^{pT}|J\Delta(Lx(n-1))|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{n=1}^{pT}|x(n)|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=1}^{pT}|\Delta x(n)|^2\right)^{\frac{1}{2}} \cdot \left(2\sin\tfrac{\pi}{pT}\right)^{-1} \cdot \left(\sum_{n=1}^{pT}|\Delta x(n)|^2\right)^{\frac{1}{2}} \\ &= \left(2\sin\tfrac{\pi}{pT}\right)^{-1}\sum_{n=1}^{pT}|\Delta x(n)|^2.\end{aligned}$$

□

From (3) and (6), we can draw a conclusion that if $x = \bar{x} + y$, $\bar{x} \in W$, $y \in Y$, then $I(x) = I(\bar{x}+y) = I(y)$ holds for any $x \in E_{pT}$. Hence, we can study critical points of I(refer to (3)) only on the subspace Y.

3. Pseudo Z_p index theory

For the convenience of later discussion, we will recall the theory of Z_p index theory first, then state our pseudo Z_p index theory. Z_p index theory was established in 1976 by Ekeland and Lasry. Michalek developed it in [16]. Here we use the definition of Z_p index given by Liu [17] in 1993.

Throughout this section, we fix $p > 1$ as a positive integer and write it as

$$p = p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}, \tag{7}$$

where $p_1 < p_2 < \cdots < p_s$ are prime factors of p, $r_j > 0$ integers, $j = 1, 2, \cdots, s$.

Let X be a Banach space and μ be a linear isometric action of Z_p on X, where Z_p is a cyclic group with order p. A subset of A of X will be called μ-invariant if $\mu(A) \subset A$. A continuous map $f : A \to X$ is called μ-equivariant if $f(\mu x) = \mu f(x)$, for all $x \in A$.

Set

$$\Sigma = \{A \subset X | A \text{ is closed and } \mu\text{-invariant}\},$$
$$F_p = \{n \in \mathbf{N} | n = p_1^{t_1}p_2^{t_2}\cdots p_s^{t_s}, t_j \in \mathbf{Z}, t_j \geq 0, j = 1, 2, \cdots, s\}$$

and

$$\Upsilon = \{z \in \mathbf{C} | \arg z = \frac{2\pi j}{p}, \quad j = 0, 1, \cdots, p-1\}.$$

For any $n \in F_p$, we define the index $i_n : \Sigma \to \mathbf{N} \cup \{+\infty\}$, $\forall A \in \Sigma$ as follows:

$$i_n(A) = \min\{k \in \mathbf{N} | \text{ there exists a continuous map}$$
$$\Psi : A \to \Upsilon^k \setminus \{\theta\}. \text{ such that } \Psi(\mu x) = e^{in2\pi/p}\Psi(x)\}.$$

Set $i_n(\emptyset) = 0$ and $i_n(A) = +\infty$ if no such map Ψ exists, where $\emptyset$ stands for the empty set.

Definition 9. $i_n : \Sigma \to \mathbf{N} \cup \{+\infty\}$ defined above is a Z_p index if the following properties are true:

(1) $i_n(A) = 0 \Leftrightarrow A = \emptyset$;

(2) (monotonicity) if $\Psi : A_1 \to A_2$ is a continuous μ-equivariant map, then $i_n(A_1) \leq i_n(A_2)$, $\forall A_1, A_2 \in \Sigma$;

(3) (subadditivity) $i_n(A_1 \cup A_2) \leq i_n(A_1) + i_n(A_2)$, $\forall A_1, A_2 \in \Sigma$;

(4) (continuity) if $A \in \Sigma$ is a compact set, then there exists $\delta > 0$ such that $N_\delta(A) \in \Sigma$ and $i_n(N_\delta(A)) = i_n(A)$, where $N_\delta(A) = \{x \in X | \operatorname{dist}(x, A) \leq \delta\}$.

For any given μ-equivariant functional $g : X \to \mathbf{R}$, we can establish a pseudo Z_p index. Suppose $D = (a, b)$ is a finite interval and denote

$$\Phi_D^h(g) = \{\eta : X \to X | \mu - \text{ equivariant continuous homeomorphism satisfying}$$

$$\eta|_{g^{-1}(\mathbf{R} \setminus D)} = \mathrm{id}_{g^{-1}(\mathbf{R} \setminus D)}, g(\eta(x)) \leq g(x), \forall x \in X\}.$$

Definition 10. (Σ^*, i^*) is a pseudo index of Z_p index i related to $H^* = \Phi_D^\mu(g)$ when $i^* : \Sigma^* \to \mathbf{N} \cup \{+\infty\}$ testifies

(1) if $A \in \Sigma^*$, $B \in \Sigma$ and $\eta \in H^*$ then $\Sigma^* \subset \Sigma$, $\overline{A \setminus B} \in \Sigma^*$, $\overline{\eta(A)} \in \Sigma^*$;

(2)if $A \subset B$, then $i^*(A) \leq i^*(B)$, $\forall A, B \in \Sigma^*$;

(3)$i^*(\overline{A \setminus B}) \geq i^*(A) - i(B)$, $\forall A \in \Sigma^*, B \in \Sigma$;

(4)$i^*(\overline{\eta(A)}) \geq i^*(A)$, $\forall A \in \Sigma^*, \forall \eta \in H^*$.

Lemma 11[15]. *Let (Σ, i) be an index on X and $S \in \Sigma$ be a given close set, if*

$$i^*(A) = \inf_{\mu \in \Phi_D^\mu(g)} i(\mu(A) \cap S),$$

then (Σ, i^*) is a pseudo index for any $\Phi_D^\mu(g)$.

The most important aspect of an index theory lies in the applications of its dimensional property to estimate the critical point numbers of certain kinds of functionals. In the following we will discuss the property of the Z_p index i_n.

Let X_{2a} is the $\mu-$invariant $2a$-dimensional subspace of X, we identify X_{2a} with $\mathbf{C}^a$. A Z_p action μ on $\mathbf{C}^a$ is given by

$$\mu z = (e^{ik_1 2\pi/p} z_1, \cdots, e^{ik_a 2\pi/p} z_a), \tag{8}$$

for $z = (z_1, \cdots, z_a) \in \mathbf{C}^a$, where $k_j \neq 0$ integers, $j = 1, 2, \cdots, a$.

Let Ω be a bounded $\mu-$invariant domain in $\mathbf{C}^a$ and $\theta \in \Omega$, then the boundary of the domain $\partial\Omega$ is also a $\mu-$invariant set in Σ, that is, $\partial\Omega \in \Sigma$. We are going to estimate the index of $\partial\Omega$. For the given integer p defined in (7), we define $M = M' p_1^{l_1} \cdots p_s^{l_s}$ is the greatest common divisor of $\{|m_j|\}_{j=1}^a$ and $m = m' p_1^{t_1} \cdots p_s^{t_s}$ is the smallest common multiple of $\{|m_j|\}_{j=1}^a$, where M', m' are relatively prime to p. Write

$$n = \frac{m}{m'} = p_1^{t_1} \cdots p_s^{t_s}, \tag{9}$$

then $n \in F_p$.

Theorem 12. *If there exists some β, $1 \le \beta \le s$ such that*

$$a(t_\beta - l_\beta) < r_\beta - l_\beta, \tag{10}$$

then $i_n(\partial\Omega) = 2a$, where $n = \frac{m}{m'} = p_1^{t_1} \cdots p_s^{t_s}$ is defined above.

Corollary 13. *If $(m_j, p) = 1$, $j = 1, 2, \cdots, a$, then $i_1(\partial\Omega) = 2a$.*

Now we give an example to calculate the index of the following $\mu-$invariant set on the finite dimensional space Y by making use of theorem 12 and corollary 13.

Example. For given integers $p > 1$, $T > 0$, identify $\mathbf{R}^{2N}$ with $\mathbf{C}^N$ by $x = (x_1, x_2, \cdots, x_{2N}) \in \mathbf{R}^{2N}$ with

$$(x_1 + ix_{N+1}, x_2 + ix_{N+2}, \cdots, x_N + ix_{2N}) \in \mathbf{C}^N.$$

A Z_p action on Y is defined as $\mu x(n) = x(n+T)$. For any integer j, $|j| = 1, 2, \cdots, [\frac{pT-1}{2}]$, define a subspace

$$Y_j = \{x(n) | x(n) = \begin{pmatrix} e^{(\frac{2j\pi}{pT}n - \frac{\pi}{2})i} \cdot \xi_j \\ e^{(\frac{2j\pi}{pT}n - \frac{j\pi}{pT})i} \cdot \xi_j \end{pmatrix}, \quad \xi_j \in \mathbf{C}^N\}.$$

Consider $2kN$-dimensional subspace

$$Y_{2kN} = \bigoplus_{j=1}^{k} Y_j = \{x(n) = \sum_{j=-[\frac{k}{2}], j\neq 0}^{[\frac{k}{2}]} \begin{pmatrix} e^{(\frac{2j\pi}{pT}n - \frac{\pi}{2})i} \cdot \xi_j \\ e^{(\frac{2j\pi}{pT}n - \frac{j\pi}{pT})i} \cdot \xi_j \end{pmatrix}, \quad \xi_j \in \mathbf{C}^N\},$$

where $1 \le k \le pT - 1$. Then for any $x(n) \in Y_{2kN}$,

$$\begin{aligned} \mu x(n) \quad &= x(n+T) = \sum_{j=-[\frac{k}{2}], j\neq 0}^{[\frac{k}{2}]} \begin{pmatrix} e^{(\frac{2j\pi}{pT}(n+T) - \frac{\pi}{2})i} \cdot \xi_j \\ e^{(\frac{2j\pi}{pT}(n+T) - \frac{j\pi}{pT})i} \cdot \xi_j \end{pmatrix} \\ &= \sum_{j=-[\frac{k}{2}], j\neq 0}^{[\frac{k}{2}]} \begin{pmatrix} e^{(\frac{2j\pi}{pT}n + \frac{2j\pi}{p} - \frac{\pi}{2})i} \cdot \xi_j \\ e^{(\frac{2j\pi}{pT}n + \frac{2j\pi}{p} - \frac{j\pi}{pT})i} \cdot \xi_j \end{pmatrix} \\ &= \sum_{j=-[\frac{k}{2}], j\neq 0}^{[\frac{k}{2}]} \begin{pmatrix} e^{(\frac{2j\pi}{pT}n - \frac{\pi}{2})i} \cdot \xi_j \\ e^{(\frac{2j\pi}{pT}n - \frac{j\pi}{pT})i} \cdot \xi_j \end{pmatrix} \cdot e^{i\frac{2j\pi}{p}} \end{aligned}$$

Because of X_{2kN} being identified with $\mathbf{C}^{kN}$ by $\sum_{j=-[\frac{k}{2}], j\neq 0}^{[\frac{k}{2}]} \begin{pmatrix} e^{(\frac{2j\pi}{pT}n - \frac{\pi}{2})i} \cdot \xi_j \\ e^{(\frac{2j\pi}{pT}n - \frac{j\pi}{pT})i} \cdot \xi_j \end{pmatrix}$ with $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_k) \in \mathbf{C}^{kN}$, we get the representation of μ on $\mathbf{C}^{kN}$ as $\mu\zeta = (e^{i2\pi/p}\zeta_1, \cdots, e^{i2\pi k/p}\zeta_k)$. For $\rho > 0$, choose a set of Y_{2kN} as

$$\Gamma_{k,\rho} = \{x \in Y_{2kN} | \sum_{j=1}^{k} 2\sin\frac{j\pi}{pT}|\zeta_j|^2 = \rho^2\}, \tag{11}$$

then $\Gamma_{k,\rho}$ bounds an invariant domain containing θ. By theorem 12, we have

Corollary 14. *(a) If $1 \le k < s_p$, then $i_1(\Gamma_{k.\rho}) = 2kN$.*

(b) Set m is the smallest common multiple of $\{|j|\}_{j=1}^{k}$, i.e., $m = m' \cdot p_1^{t_1} \cdots p_s^{t_s}$, if there exists a t_β, $1 \le \beta \le s$, such that $kNt_\beta < r_\beta$, then $i_n(\Gamma_{k.\rho}) = 2kN$, where $n = p_1^{t_1} \cdots p_s^{t_s}$.

4. Main results and proofs

In Sect. 2, We turned the periodic solution problem of (1) to the corresponding critical point problem of the functional (3). Next we will achieve a series of new results on the existence and multiplicity of minimal period solutions of (1) by combining minimax method in critical theory with the pseudo Z_p index.

Recall any given positive integer p is expressed in (7). For integer q, $1 \le q \le p$, define

$$Q_q = \min\{l | l | p, q < l\}, \tag{12}$$

Clearly, when $q = 1$, $Q_q = s_p$. Define

$$\begin{aligned} K_q = &\text{ the smallest common multiple of } \{j | 1 \le j \le Q_q - 1\} \\ = &\, K' p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}, \end{aligned} \tag{13}$$

where $(K', p) = 1$ and $t_j (1 \le j \le s)$ are uniquely determined by q.

Our main results are following.

Theorem 15. *Let H satisfy assumption 1-4. Given integers p, q, $1 \le q < p$, Q_q and K_q are defined as (12) and (13) respectively satisfying*

$$\frac{a_0}{a_1} \ge \left(\frac{M \sin \frac{k\pi}{pT}}{\beta^2 m \sin \frac{Q_q \pi}{pT}} \right)^{\frac{1}{2}}, \tag{14}$$

if there exist some integers k $(1 \le k \le pT - 1)$ and t_γ $(1 \le \gamma \le s)$ such that

$$k t_\gamma N < r_\gamma \tag{15}$$

then (1) admits at least $2kN$ distinct solutions with minimal period pT/l, where $1 \le l \le q$ and $l | q$.

Without assumption 4, we have

Theorem 16. *Let H satisfy assumption 1-3, then for any given integer $p > 1$ satisfying*

$$\frac{a_0}{a_1} \ge \left(\frac{M \sin \frac{2\pi}{pT}}{\beta^2 m \sin \frac{\pi}{pT}} \right)^{\frac{1}{2}}$$

there exist at least $2N$ distinct solutions with minimal period pT of (1).

Let $q = 1$, then it follows $Q_1 = s_p$ by the definition of Q_q. So the condition (14) in theorem 15 can be turned into

$$\frac{a_0}{a_1} \geq \left(\frac{M \sin \frac{k\pi}{pT}}{\beta^2 m \sin \frac{s_p \pi}{pT}} \right)^{\frac{1}{2}}, \tag{16}$$

since $(K_q, p) = 1$, then $t_j \equiv 0$ for all $j = 1, 2, \cdots, s$, hence the condition (15) in theorem 15 is also true. It follows.

Corollary 17. *Let H satisfy assumption 1-4. For any given integer $p > 1$, if (16) holds for some integer $1 \leq k \leq [\frac{pT}{2}]$, then (1) admits at least $2kN$ distinct solutions with minimal period pT.*

In order to give the proof of theorem 15, we will state some propositions and lemmas at first.

For a positive integer f, define

$$B_f = \frac{\beta - 2}{2} m a_0 \left(\frac{2a_0 \beta \sin \frac{l\pi}{pT}}{a_1^2 M} \right)^{\frac{\beta}{\beta-2}} \cdot pT. \tag{17}$$

Proposition 18. *Under assumptions 1-3, we have*

(a) the functional I is bounded from below and $I(x) \geq B_1, \quad \forall x \in Y$;

(b) if $I(x) < B_2$, the $x(n)$ has minimal period pT; and further if we add assumption 4, there holds.

(c) if x is a critical point of I and

$$I(x) < B_f, \text{ for some positive integer } f \tag{18}$$

then $x(n)$ has minimal period pT/l, for certain l where $l < f$ and is a factor of p.

Remark 19. It is easy to see that $I(x) > 0$ for any $x \in Y$ by (a). Then we can draw a conclusion that there exists a sufficient small $\rho_0 > 0$ such that $\inf_{x \in S_{\rho_0} \cap Y} I(x) \geq c_0 > 0$ where $S_\rho = \{x \in Y | \|x\| < \rho\}$ and c_0 is a constant only related to ρ_0.

Remark 20. It follows from (c) of proposition 4.1, if x is a critical point of I and $I(x) < B_{s_p}$, then the solution of (1.1) related to x has minimal period pT.

Proof of Proposition 18 Suppose $x \in Y$ with period pT/l, where l is a positive integer and $l|p$.

Write

$$M_x = \max\{|Lx(n)|^{\beta-2}\}, \quad n \in [1, pT],$$

by assumptions 1-3, we have

$$\begin{aligned} \beta \sum_{n=1}^{pT} b(n) H(Lx(n)) = & \sum_{n=1}^{pT} b(n) \beta H(Lx(n)) \\ \leq & \sum_{n=1}^{pT} b(n) < Lx(n), \nabla H(Lx(n)) > \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{pT} < Lx(n), J\Delta x(n) > \\
&\leq (2\sin\frac{l\pi}{pT})^{-1} \|\Delta x(n)\| \cdot \|\Delta x(n)\| \\
&= (2\sin\frac{l\pi}{pT})^{-1} \|\Delta x(n)\|^2 \\
&= (2\sin\frac{l\pi}{pT})^{-1} \|b(n)\nabla H(Lx(n))\|^2 \\
&\leq (2\sin\frac{l\pi}{pT})^{-1} \cdot \sum_{n=1}^{pT} (b(n))^2 \cdot a_1^2 \cdot |Lx(n)|^{2\beta-2} \\
&\leq (2\sin\frac{l\pi}{pT})^{-1} \cdot a_1^2 \cdot M_x \cdot M \cdot \sum_{n=1}^{pT} [(b(n)) \cdot |Lx(n)|^{\beta}] \\
&\leq (2\sin\frac{l\pi}{pT})^{-1} \cdot \frac{a_1^2}{a_0} \cdot M_x \cdot M \cdot \sum_{n=1}^{pT} [(b(n)) \cdot |H(Lx(n))|].
\end{aligned}
$$

That is,

$$
\beta\sum_{n=1}^{pT} b(n)H(Lx(n)) \leq (2\sin\frac{l\pi}{pT})^{-1} \cdot \frac{a_1^2}{a_0} \cdot M_x \cdot M \cdot \sum_{n=1}^{pT} [(b(n)) \cdot |H(Lx(n))|],
$$

so we get $M_x \leq \frac{2a_0\beta\sin\frac{l\pi}{pT}}{a_1^2 M}$. From assumption 1, it follows that

$$
H(Lx(n)) \geq a_0 \cdot |Lx(n)|^{\beta} \geq a_0 \left(\frac{2a_0\beta\sin\frac{l\pi}{pT}}{a_1^2 M} \right)^{\frac{\beta}{\beta-2}},
$$

making use of assumption 2, there exists

$$
\beta\sum_{n=1}^{pT} b(n)H(Lx(n)) \leq \sum_{n=1}^{pT} (J\Delta(Lx(n-1)), x(n)),
$$

hence

$$
\begin{aligned}
I(x) = &\ \frac{1}{2}\sum_{n=1}^{pT} (J\Delta(Lx(n-1)), x(n)) - \sum_{n=1}^{pT} b(n)H(Lx(n)) \\
\geq &\ \frac{\beta}{2}\sum_{n=1}^{pT} b(n)H(Lx(n)) - \sum_{n=1}^{pT} b(n)H(Lx(n)) \\
\geq &\ \frac{\beta-2}{2} m a_0 \left(\frac{2a_0\beta\sin\frac{l\pi}{pT}}{a_1^2 M} \right)^{\frac{\beta}{\beta-2}} \cdot pT = B_l
\end{aligned}
$$

Because of $\beta > 2$, we have (a) and (b) are true.

(c) To prove (c), assume that $x(n)$ has minimal period pT/l for some integer $l \geq 1$. From above and (18), we have $l < f$. Since x is a critical point of I, then for some $\rho \in \mathbf{R}^{2N}$, $z(n) = x(n) + \rho \in H'(Lx(n))$ satisfies $J\Delta x(n) = b(n)H'(Lx(n))$ and has minimal period pT/l. By assumption 4, p/l is an integer, that is l is a factor of p. The proof of proposition 4.1 is complete. □

Lemma 21[15]. *The functional $g \in C^1(X, \mathbf{R})$ is $T(G)$-invariant and satisfies PS condition. Suppose (Σ^*, i^*) be a pseudo index related to the $T(G)$ index i of $\Phi_D^h(g)$ and*

$$c_j^* = \inf \sup_{i^*(A) \geq j, x \in A} f(x), \quad j = 1, 2, \cdots,$$

where $A \subset X$, $A \subset \Sigma^*$. There have
(1) *if $c_j^* \in D$, then c_j^* is a critical value of g;*
(2) *if there exists an integer $\gamma \in \mathbf{N}$ such that*

$$c = c_{\gamma+1} = \cdots = c_{\gamma+l} \in D, \quad l > 1,$$

then $i(K_c) \geq l$, here $K_c = \{x \in X; f'(x) = 0, f(x) = c\}$.

In order to employ lemma 21 to produce multiple critical points for the functional I, the following Z_p version of the deformation lemma (see [17]) is needed.

Lemma 22. *Let $\phi \in C^1(X, \mathbf{R})$ be a μ-invariant functional satisfying the PS condition. For any $c \in \mathbf{R}$ and any neighborhood U of $K_c = \{x \in X | \phi(x) = c, \phi'(x) = 0\}$, there exists a constant $\bar{\epsilon} > 0$ such that for any $0 < \epsilon < \bar{\epsilon}$, there is a continuous map $\eta : [0,1] \times X \to X$ such that*
$(a)\eta(0, x) = x, \quad \forall x \in X;$
$(b)\eta(t, x) = x, \quad \forall (t, x) \in [0, 1] \times \phi_{c-\epsilon};$
$(c)\eta(1, x) \in \phi_{c-\epsilon}, \quad \forall x \in \phi_{c+\epsilon} \setminus U;$
$(d)\eta$ *is a Z_p-equivariant homeomorphism.*

A full statement and proof of the Z_p deformation theorem can be found in [5] with obvious modification. Here we only to prove $I(x)$ satisfies Palais-Smale (PS) condition.

Lemma 23. *Under assumption 1-3, the functional $I(x)$(see (3)) defined on Y satisfies PS condition.*

Proof. Suppose $\{x^{(k)}\} \subset Y$, from proposition 18 we have $I(x^{(k)})$ is bounded from below, here we can suppose $I(x^{(k)})$ be bounded from above, that is, there exists a positive constant M_1 such that $I(x^{(k)}) \leq M_1, \quad \forall k \in \mathbf{N}$. From remark 5, lemma 6 and 8, we have

$$\begin{aligned}
I(x^{(k)}) &= \frac{1}{2}\sum_{n=1}^{pT}(J\Delta(Lx^{(k)}(n-1)), x^{(k)}(n)) - \sum_{n=1}^{pT} b(n)H(Lx^{(k)}(n)) \\
&\le \frac{1}{2}(2\sin\frac{\pi}{pT})^{-1}\sum_{n=1}^{pT}|\Delta x^{(k)}(n)|^2 - m\sum_{n=1}^{pT}\left(b_1|Lx^{(k)}(n)|^{\beta} - b_2\right) \\
&\le \frac{1}{2}(2\sin\frac{\pi}{pT})^{-1}\lambda_{max}^2\|x^{(k)}(n)\|^2 - mb_1\sum_{n=1}^{pT}|Lx^{(k)}(n)|^{\beta} + mb_2 pT \\
&= \frac{1}{2}(2\sin\frac{\pi}{pT})^{-1}\lambda_{max}^2\|x^{(k)}(n)\|^2 - mb_1\|x^{(k)}(n)\|_{\beta}^{\beta} + mb_2 pT
\end{aligned}$$

According to the equivalence of $\|x^{(k)}(n)\|$ and $\|x^{(k)}(n)\|_{\beta}$, there exists a constant $C > 0$ such that $\|x^{(k)}(n)\| \le C\|x^{(k)}(n)\|_{\beta}$. So

$$I(x^{(k)}) \le \frac{C^2}{2}(2\sin\frac{\pi}{pT})^{-1}\lambda_{max}^2\|x^{(k)}\|_{\beta}^2 - mb_1\|x^{(k)}(n)\|_{\beta}^{\beta} + +mb_2 pT \le M_1.$$

Since $\beta > 2$, there is a constant $M_2 > 0$ such that $\|x^{(k)}\| \le M_2, \quad \forall k \in$ **N**. Then $\{x^{(k)}\}$ is a bounded subsequence in the finite dimensional space Y. Obviously, $\{x^{(k)}\}$ possesses a convergent subsequence in Y. □

Lemma 24. *If integer $1 \le k \le pT - 1$, when $x \in \Gamma_{k,\rho}$, there holds*

$$I(x) \le pT \cdot \left(\sin\frac{k\pi}{pT}\right)^{\frac{\beta}{\beta-2}} \cdot \left(\frac{2}{\beta m A_0}\right)^{\frac{2}{\beta-2}} \cdot (1 - \frac{2}{\beta}) \tag{19}$$

Proof. When $x \in \Gamma_{k,\rho}$, we have

$$\begin{aligned}
I(x) &= \frac{1}{2}\sum_{n=1}^{pT}(J\Delta(Lx(n-1)), x(n)) - \sum_{n=1}^{pT} b(n)H(Lx(n)) \\
&\le \sin\frac{k\pi}{pT}\cdot\sum_{n=1}^{pT}|x(n)|^2 - \sum_{n=1}^{pT} b(n)H(Lx(n)) \\
&\le \sin\frac{k\pi}{pT}\cdot\sum_{n=1}^{pT}\left(|x(n)|^{\beta}\right)^{\frac{2}{\beta}}\cdot 1^{1-\frac{2}{\beta}} - m\sum_{n=1}^{pT} H|Lx(n)|^{\beta} \\
&\le \sin\frac{k\pi}{pT}\cdot\left(\sum_{n=1}^{pT}|x(n)|^{\beta}\right)^{\frac{2}{\beta}}\cdot\left(\sum_{n=1}^{pT}1\right)^{1-\frac{2}{\beta}} - a_0 m\sum_{n=1}^{pT}|Lx(n)|^{\beta} \\
&= (pT)^{\frac{\beta-2}{\beta}}\cdot\sin\frac{k\pi}{pT}\cdot\|x\|_{\beta}^2 - a_0 m\|x\|_{\beta}^{\beta}
\end{aligned}$$

We can write $\Theta(u) = (pT)^{\frac{\beta-2}{\beta}} \cdot \sin\frac{k\pi}{pT} \cdot u_\beta^2 - a_0 m u_\beta^\beta$, when we denote $\|x\|_\beta = u$, where $u \in (0, +\infty)$. Let $\Theta'(u) = 0$, we have

$$u_0 = \left(\frac{2\sin\frac{k\pi}{pT} \cdot (pT)^{\frac{\beta-2}{\beta}}}{\beta m a_0} \right)^{\frac{1}{\beta-2}}$$

Since $\beta > 2$, then $\Theta(u) \to -\infty$ when $u \to +\infty$, thus $\Theta(u)$ gets its maximum at $u = u_0$. Hence,

$$I(x) \le (pT)^{\frac{\beta-2}{\beta}} \cdot \sin\frac{k\pi}{pT} \cdot \left(\frac{2\sin\frac{k\pi}{pT} \cdot (pT)^{\frac{\beta-2}{\beta}}}{\beta m a_0} \right)^{\frac{2}{\beta-2}} - a_0 m \left(\frac{2\sin\frac{k\pi}{pT} \cdot (pT)^{\frac{\beta-2}{\beta}}}{\beta m a_0} \right)^{\frac{\beta}{\beta-2}}$$

$$= pT \cdot \left(\sin\frac{k\pi}{pT}\right)^{\frac{\beta}{\beta-2}} \cdot \left(\frac{2}{\beta m a_0}\right)^{\frac{2}{\beta-2}} \cdot \left(1 - \frac{2}{\beta}\right) \qquad \square$$

With the aid of above preparations, next we pay our attention to proving theorem 4.1.

Proof of Theorem 15: We are going to complete the proof by two steps. First, we look for critical points of the functional I by lemma 21 and prove the minimal period is pT/l, where l satisfies theorem 15. Second, we prove I has at least $2kM$ distinct critical points.

Step 1. As in Sect. 3, we have defined a Z_p index on the space E_{pT}. Denote $a = pT \cdot \left(\sin\frac{k\pi}{pT}\right)^{\frac{\beta}{\beta-2}} \cdot \left(\frac{2}{\beta m a_0}\right)^{\frac{2}{\beta-2}} \cdot \left(1 - \frac{2}{\beta}\right)$ and the interval $D = (0, a+1)$. For any $A \in \Sigma$, define $i_p^* = \inf_{\mu \in \Phi_D^\mu(I)} i\left(\mu(A) \cap (S_{\rho_0} \cap Y)\right)$, then (Σ, i_p^*) is a pseudo Z_p index by lemma 15. Write

$$c_j = \inf_{i_p^*(A) \ge j} \sup_{x \in A} I(x), \quad j = 1, 2, \cdots, 2kN.$$

From proposition 18 and lemma 24, there follows

$$0 < c_1 \le c_2 \le c_3 \le \cdots \le c_{2kN} \le a < a + 1. \tag{20}$$

By lemma 21, we have $\{c_j\}_{j=1}^{2kN}$ are critical values of I and when $c = c_{r+1} = \cdots = c_{r+l}$, where $l > 1$, $i_p^*(K_c) \ge l$, here $K_c = \{x \in E_{pT}, I'(x) = 0, I(x) = c\}$.

Next we show when $x \in K_{c_j}$, $j = 1, 2, \cdots, 2kN$, x has minimal period pT/l. Making use of proposition 18, lemma 24 and (19), we only to prove

$$a \le pT \cdot m a_0 \cdot \left(\frac{\beta-2}{2}\right) \cdot \left(\frac{2a_0\beta\sin\frac{Q_q\pi}{pT}}{a_1^2 M}\right)^{\frac{\beta}{\beta-2}},$$

where a was defined as above. By direct computation, we have when (14) in theorem 15 holds, $a \le pT \cdot m a_0 \cdot \left(\frac{\beta-2}{2}\right) \cdot \left(\frac{2a_0\beta\sin\frac{\pi}{pT}}{a_1^2 M}\right)^{\frac{\beta}{\beta-2}}$ is true.

From proposition 18, for any $x \in K_{c_j}$, $j = 1, 2, \cdots, 2kN$, x has minimal period pT/l, where l is an integer and $l < Q_q$, $l|p$. Using (12), $1 \le l \le q$.

Step 2: If $c_1 < c_2 < \cdots < c_{2kN}$, then I has at least $2kN$ distinct critical points.

If $c = c_j = \cdots = c_{j+l}$, $l \geq 1$, then by lemma 21, $i_p^*(K_c) \geq l+1 > 1$. Following we prove K_c contains infinity distinct critical points.

In fact, if $x \in K_c$, set $K_c = \{\mu^j x(n) | j = 0, 1, \cdots, pT/l - 1\}$, since x has minimal period pT/l, then if $j_1 \neq j_2$, we have $\mu^{j_1}x(n) \neq \mu^{j_2}x(n)$. Define $\Psi : K_c \to \Upsilon^k \backslash \{\theta\}$, then

$$\Psi(\mu^j x(n)) = e^{i\frac{2j\pi}{p}}, \quad j = 0, 1, \cdots, pT/l - 1.$$

Thus,

$$\Psi(\mu(\mu^j x(n))) = \Psi(\mu^{j+1}x(n)) = e^{i\frac{2j\pi}{p}} \cdot e^{i\frac{2\pi}{p}} = e^{i\frac{2\pi}{p}}\Psi(\mu^j x(n)).$$

Therefore,

$$i_n(K_c) = 1.$$

This is a contradictor. The proof of theorem 15 is completed. □

Theorem 16 can be proved by essentially the same arguments. We only need to prove under assumptions of theorem 16, $I(x) < B_2$ is true for any $x \in \Gamma_{1,\rho}$,

$$\Gamma_{1,\rho} = \{x \in S | x(n) = \begin{pmatrix} e^{(\frac{2\pi}{pT}n - \frac{\pi}{2})i} \cdot \xi \\ e^{(\frac{2\pi}{pT}n - \frac{\pi}{pT})i} \cdot \xi \end{pmatrix}, \xi \in \mathbf{C}^{2M}, 2\sin\frac{\pi}{pT}|\zeta|^2 = \rho^2\}.$$

Here we omit the proof.

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