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Classification of the Analytic Hierarchy Process Approaches by Application Circumstances

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ABSTRACT

This paper studies six different AHP (Analytic Hierarchy Process) approaches and suggests that the features of the approaches are classified by application circumstances in order to contribute to the applicability and quality usage of the AHP. Our study investigates the hierarchical principles and characteristics of the AHP, and historical debates on the AHP evaluation in which the six approaches have been involved. One of six approaches is an ANP (Analytic Network Process) application that is directly connected to AHP usage. The application differences among the six approaches are validated with a plain example. Then, the four circumstances of AHP applications are classified by two dimensions: the first dimension is whether or not the importance (weights) of criteria is independent of restrictively setting alternatives, and the second dimension is whether or not preference (priorities) of alternatives is independent of adding alternative(s) to or removing alternative(s) from the considering set of alternatives. Then featuring way of weighting criteria is classified. We suggest the distinguishing manners and describe the implications of the AHP application. Finally, we discuss rank reversal and multiplicative AHP.

Keywords: Multiple Criteria Analysis, Analytic Hierarchy Process Approaches, Application Classification, Priority Dimensions

1. Introduction

Since full description of the AHP (Analytic Hierarchy Process) by Saaty [16], its use-

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fulness has been verified by its rich applications in almost all industries [8, 32, 37]. The context of the AHP application has been extended by the ANP (Analytic Network Process) as the generalized form of the AHP [20]. The AHP application usually employs three stages: (1) the hierarchical compositions of criteria and alternatives, (2) the simple pairwise comparisons in order to calculate local priorities, and (3) the synthesis of global priorities. The synthesis stage states that the priority of the ultimate objective is hierarchically propagated to criteria according to the amount of contribution to the criteria (weighing criteria). The priorities of the criteria are distributed to alternatives (prioritizing alternatives) and finally to be synthesized with respect to each alternative in order to choose.

The phase of synthesis triggered the debates about the AHP measurement and its rank reversal since the early 1980s. The alternate approaches to Saaty's AHP (hereafter referred to as the *Conventional AHP*) have appeared from the debate. This paper deals with the differences among 6 AHP approaches including Conventional AHP and the ANP originally developed by Saaty [16, 20] (the ANP discussed in this paper is confined to only certain dependency between criteria and alternatives). The other four approaches have come out, directly or indirectly, from the AHP debates since the early 1980s [2, 11, 21, 25].

Based on the belief there is no one basic rule that can explain or cover all kinds of circumstances in the areas of social sciences or human psychology, we are motivated in our research to understand the circumstances of AHP applications and to classify the circumstances to better understanding them. In order to find the characteristics that can classify the circumstances, we investigate what aspects of the AHP invoked such debates as the above and explain how the aspects are related with the different view of approaches to the Conventional AHP and with the circumstances of AHP applications. In this paper, it is not our intention to contribute further to the debate. Instead, it is our intention to contribute to extending and enriching the applicability of the AHP as well as understanding the contextual features of AHP applications.

For the sake of this purpose, we set up our objectives. First, we outline the features of the Conventional AHP [16] and five alternative approaches; Ideal mode AHP [6, 8], Referenced AHP [25], B-G AHP [2, 3], Dominant AHP [11], and ANP [20]. Liking pin AHP [27] is regarded, in this paper, as the modification of B-G AHP. Second, we classify the six approaches using two dimensions with respect to each of which the approaches are commonly characterized. Finally we propose guidelines to apply the AHP with characteristics of application circumstances.

There have been the AHP studies using multiplicative function, rather than additive function, as an aggregation method that is free from rank reversal problem [1, 31]. However, the multiplicative AHP has been criticized with counterexamples [22, 33] and still has not been accepted in general. Therefore we focus on the six models of additive function and later we discuss the multiplicative AHP technique.

The next section describes two principles concerned with AHP synthesis. We, then, briefly review the AHP debates. In Section 3, we reveal two agendas that are involved in AHP sensitivity and rank reversal: (1) how to elicit the importance of criteria, and (2) how to normalize the preference of alternatives. In this section we illustrate the six different AHP approaches and show how each approach deals with the two agendas. In Section 4, we classify the six approaches into a 2×2 matrix with two dimensions and describe application guidelines using circumstances from a practical view. In Section 5, we discuss the model with multiplication form and application of the AHP, and we finally conclude this paper with final remarks.

2. The AHP Synthesis

2.1 The Independence and Ratio Scale in the AHP

The AHP is characterized by its principles and axioms. Three principles are the hierarchical composition of complexity, measurement with a ratio scale, and synthesis. The 4 axioms are reciprocal, homogeneity, independence, and expectation axioms [16, 17].

We are, in this paper, concerned with the synthesis of the criteria weighs and the alternative priorities. In accordance with this subject, we must answer the two questions: *"What is the independence axiom?"* and *"Why is the ratio scale in the AHP?"* These two questions are also helpful in understanding the basis to spin the classification axes (described in Section 4) for the AHP approaches that have come out of the AHP debates. Both questions are related to the hierarchical composition of complexity.

For the question of ratio scale, the principle of hierarchical composition means hierarchically breaking down a complex problem into smaller and smaller pieces until it is possible to clearly address the smaller pieces. The priorities (or weights) of the elements are also broken down into the lower level elements consistently. Here is the key point why the AHP must use ratio scales. The priorities of the elements at any level of the hierarchy are determined by multiplying the priorities of the elements within a group in that level by the priorities of the parent element. Thus, the sum of priorities of elements is equal to the priority of the parent in the hierarchy. Because the product of two interval-level measures is mathematically meaningless, ratio scales are required for this multiplication [16].

The independency axiom, out of the 4 axioms, is also associated with the principle of hierarchical composition. The judgment about or the priorities of the elements in a hierarchy do not depend on lower level elements. Finally the weights of the criteria are independent of consideration of the alternatives [9].

2.2 A Brief Review of the AHP Debates

Watson and Freeling [34] raised a question about the meaning and interpretation of "relative importance" which, they clamed, was meaningless without any guidelines (for example, a measurement scale) to respondents with respect to the relative importance. In response to Watson and Freeling, Saaty *et al.* [24] replied, in their example, that the AHP required the user to refer to the magnitude (sum or average) of the alternatives with respect to a criterion, which was referred to as the *Referenced AHP* by Shoner and Wedley [25]. Watson and Freeling [35] doubted whether or not the restriction that the Referenced AHP imposed on the comparison of the importance of criteria was ever implemented in practice. Saaty *et al.* [24] replied that the AHP approach was a unit free approach. Shoner and Wedley [25] argued that unit free was not appropriate for the answer and pointed out that those writings on the AHP have subsequently ignored the restriction of the Referenced AHP.

Belton and Gear [2] showed that the Conventional AHP could suffer the problem of rank reversal by adding a new alternative that provided no additional information on the relative ratings of existing alternatives. Later, rank reversal is defined as a changing of rank among previously existing alternatives when irrelevant alternatives are added to or removed form the decision. Here, the irrelevant alternative is defined as an alternative dominated by one or more previously existing alternatives [8]. Saaty and Vargas [23] tried to impose legitimacy to rank reversals. They observed that indeed rank reversal did happen in the real world and that it depended on the relationship between this new alternative and the old ones under each criterion. The value system of human beings varies by scarcity [19]. Belton and Gear [3] highlighted the inherent ambiguity in the term "weight." They argued that weights for criteria were established by paired comparisons of the relative importance of values represented by the largest valued alternatives under each attribute. We call this approach the *B-G AHP*.

Shoner and Wedly [25] showed that the Referenced AHP and B-G AHP satisfy the necessary condition that leads to correct estimation without unwarranted rank reversals. Forman [7] criticized by asserting that the Referenced AHP approach was too restrictive to apply the AHP. The example by Schoner and Wedley [25] was just for the sake of illustration and not appropriate for an AHP application because the restriction limits AHP to applications involving only absolute measurements on the tangible with a single criterion. Schoner *et al.* [26] showed that the Referenced AHP approach could work for multiple criteria with nonlinear objectives.

Dyer [4, 5] argued that the AHP yielded arbitrary rankings and that such a problem was the result of the fact that the weight of each criterion was independent of the evaluation of the available alternatives over the criterion. Saaty [18] replied that the criteria weights in the AHP can be independent of alternatives and scales with absolute measurement or relative measurement. Relative measurement can involve a kind of dependency among the alternatives. This dependency justifies the legitimacy of rank reversals [10]. Based on the approach of Belton and Gear [2], the Conventional AHP is extended to the name of *Ideal mode AHP*. However we distinguish, in this paper, what follows the Ideal mode AHP from the B-G AHP.

Differently certified alternatives can affect primitively on getting the result of pairwise comparisons so that alternative method is required [36]. Kinoshita and Nakanish [11] developed another AHP approach in arguing that it seemed more practical to judge the criteria weights depending on a preferred or well-known alternative rather than considering all alternatives. They called the alternative used to judge the criteria weights a dominant alternative and their approach, the *Dominant AHP*. Their approach could prevent rank reversals from adding or removing alternative(s). They extended the Dominant AHP to the *CCM* (Concurrent Convergent Method) which was developed to handle any discrepancy between a criteria weights vector elicited from a dominant alternative and corresponding estimated vectors derived from other dominant alternatives. Hence, we deal with the Dominant AHP in this paper (See [13] for more details about CCM and [28] for comparison between CCM and ANP).

3. Illustrations of the AHP Approaches

3.1 An Example

In order to easily illustrate and clarify the differences among the 6 approaches, let us set up a simple decision problem with two criteria (C1 and C2) and three alternatives (A1, A2, and A3) in Table 1. We assume that the two criteria with respect to which the three alternatives are measured are represented by a monetary value. To find how each approach deals with rank reversals, each approach is considered, at first, with two alternatives and then with adding another later. In general, the irrelevant alternative is defined as an alternative that is dominated by one or more previously existing alternatives [8]. We can regard A3 as an "irrelevant alternative" in terms of total value because if A1 is greater than A2 out of the choice set {A1, A2}, then introducing a third alternative A3, thus expanding the choice set to {A1, A2, A3}, does not make A2 greater than A1, and vice versa. This paper regards the monetary value as benefit.

Since the two criteria are represented by a monetary value, the total sum for each alternative is simply obtained by adding the two values of the two criteria. This example shows inequalities, A1 > A3 > A2, in Table 1.

Table 1. An example with two criteria and three alternatives

	C1	C2	Total	Normalized for A1 and A2	Normalized for all 3 alternatives
A1	36	16	52	0.54	0.37
A2	12	32	44	0.46	0.31
A3	42	4	46		0.32
Sum	90	52	142	1.00	1.00

Accordingly, let us use following notations in this paper.

w_1 and w_2: the priorities [weights] of C1 and C2, respectively, and subject to

$$w_1 + w_2 = 1$$

x_{i1} and x_{i2}: the monetary values of alternative Ai for C1 and C2, respectively

v_{i1} and v_{i2}: the derived priorities corresponding to x_{i1} and x_{i2}, respectively

z_i : the aggregate value of the alternative Ai by $v_{i1}\ w_1 + v_{i2}\ w_2$

3.2 Conventional AHP and Two agendas

Applying the Conventional AHP to the example, two weights of criteria, C1 and C2 are assumed to be derived from a pairwise comparison matrix and normalized as $w_1 + w_2 = 1$. The priorities of the alternatives for a criterion j using pairwise comparisons are normalized as $v_{1j} + v_{2j} = 1$ for j = 1 and 2, as shown in Equation (E1). Finally, the aggregate value of alternative Ai is represented as the composite form of $z_i = v_{i1} w_1 + v_{i2} w_2$ and $z_1 + z_2 = 1$ automatically. Adding the third alternative A3, we can say $\sum_{i=1}^{3} z_i = 1$ because the Conventional AHP still holds with $\sum_{i=1}^{3} v_{ij} = 1$ for each criterion j.

Since decision making is regarded as an estimation process for the best or better alternative(s) which is directly influenced by the criteria weights, it is possible to investigate the sensitivity of alternatives evaluation by the criteria weights. In this example, the aggregate value of each alternative can be represented by the function of w_1 because $w_2 = 1 - w_1$.

The derived priorities of the two alternatives (A1 and A2) with respect to each criterion can be calculated from Table 1. The priorities are as follows:

$$\begin{pmatrix} 1 & 36/12 \\ 12/36 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} \text{ for C1, and } \begin{pmatrix} 1 & 16/32 \\ 32/16 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \text{ for C2}$$

Then, the aggregate function is shown in the following equation (E1) for the two alternatives, A1 and A2.

$$z = \begin{pmatrix} 3/4 & 1/3 \\ 1/4 & 2/3 \end{pmatrix} \begin{pmatrix} w_1 \\ 1 - w_1 \end{pmatrix} = \begin{pmatrix} 0.42 w_1 + 0.33 \\ -0.42 w_1 + 0.67 \end{pmatrix} \qquad \text{(E1)}$$

If the third alternative (A3) is added, the aggregate functions for A1 and A2 are modified respectively by the normalizing vectors ($\sum_{i=1}^{3} v_{ij} = 1$ for j = 1 and 2) which are the same as the eigenvectors derived from the pairwise comparison matrix among the three alternatives. The final aggregation function is given in (E2).

$$z = \begin{pmatrix} 6/15 & 4/13 \\ 2/15 & 8/13 \\ 7/15 & 1/13 \end{pmatrix} \begin{pmatrix} w_1 \\ 1-w_1 \end{pmatrix} = \begin{pmatrix} 0.09w_1 + 0.31 \\ -0.48w_1 + 0.61 \\ 0.39w_1 + 0.08 \end{pmatrix} \tag{E2}$$

The results of two equations, (E1) and (E2), are shown in Figure 1. The range of $w_1 > a$ that keeps such a relationship of A1 > A2 like Table 1 has changed to $b < w_1 < c$ for A1 as the best alternative among three alternatives because the aggregate functions for A1 and A2 have shifted by adding A3. The shift of functions is caused by normalizing the sum of the priorities to 1. Assuming that the weights of the two criteria are equal, i.e., $w_1 = 0.5$ and $w_2 = 0.5$, this example shows aggregate values $z_1 = 0.54$ and $z_2 = 0.46$ with A1 and A2. However, with the three alternatives, $z_1 = 0.35$, $z_2 = 0.37$ and $z_3 = 0.27$ which are different from the result of Table 1. This case shows the typical rank reversal between A1 and A2 in applying the Conventional AHP (A1 > A2 → A2 > A1 > A3).

Accordingly, two agendas (discussion factors) in this paper are focused on here:

(1) *how to elicit the importance of criteria,*

(2) *how to normalize the preferences of alternatives.*

The first factor is involved in the sensitivity of Conventional AHP and both of the two factors are involved in the rank reversal. According to the AHP debates, the first one is concerned with whether or not the criteria weights can be elicited regardless of alternatives and the second is whether or not normalizing the sum of alternative priorities equals to 1. The first agenda is related to the independence axiom and the second agenda is to the ratio scale prioritization.

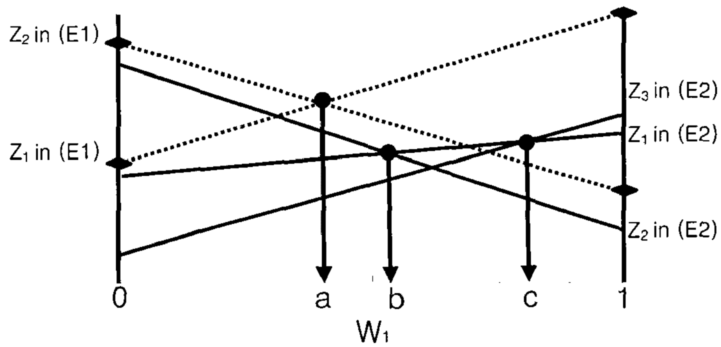

Figure 1. The Sensitivity of the Conventional AHP

3.3 Other AHP approaches

3.3.1 The Ideal mode AHP

This approach is an extension of the Conventional AHP; however, the idea was adopted from those ideas of Belton and Gear [2]. This approach sets up an ideal alternative that has the most preferred value, for every criterion, out of those given by alternatives on hand. In the example of two criteria, the ideal alternative is established with values, $\left(\max_i\{x_{i1}\},\ \max_i\{x_{i2}\}\right)$. This ideal alternative serves as a reference to compare and has a priority of 1.0 for each criterion, while each real alternative has its priority relative to the ideal alternative. Although an irrelevant alternative is added to (or removed from) the current set of alternatives, the priorities allocated to the existing alternatives under each criterion do not change.

The only operational difference between the Ideal mode AHP and the Conventional AHP occurs when a synthesis is performed [8]. Both approaches conduct criteria evaluation independent of alternatives. However, the Ideal mode AHP does not change the once established priorities with either adding or removing alternative(s) because the alternatives have been compared to the ideal alternative.

For A1 = (36, 16) and A2 = (12, 32) in Table 1, the values of ideal alternative are (36, 32). The values of A1 and A2 are compared to those of the ideal alternative, respectively, and have resulted in the first and the second row vectors of the matrix in (E3).

$$z = \begin{pmatrix} 1 & 1/2 \\ 1/3 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ 1-w_1 \end{pmatrix} = \begin{pmatrix} 0.50w_1 + 0.50 \\ -0.67w_1 + 1.00 \end{pmatrix} \tag{E3}$$

Even if the third alternative A3 is added, the previous weights vector and the previous ideal alternative are invariant. Thus, the previous aggregate functions of w_1 for A1 and A2 are invariant as in (E4). This system is not affected by adding or removing alternative(s).

$$z = \begin{pmatrix} 1 & 1/2 \\ 1/3 & 1 \\ 7/6 & 1/8 \end{pmatrix} \begin{pmatrix} w_1 \\ 1-w_1 \end{pmatrix} = \begin{pmatrix} 0.50w_1 + 0.50 \\ -0.67w_1 + 1.00 \\ 1.04w_1 + 0.13 \end{pmatrix} \tag{E4}$$

3.3.2 The Referenced AHP

The Referenced AHP originated in the debate between Watson and Freeling [34, 35] and Saaty *et al.* [24]. The Referenced AHP [25] requires the decision maker to refer to the magnitude of the alternatives. In the Referenced AHP, the relative importance of a criterion must be proportional to the sum [or average] (adjusted by its scaling factor, if necessary) of alternative measures on that criterion. In this example, the weights of two criteria (w_1 and w_2) satisfy the following equation which is the condition that distinguishes the Referenced AHP from the Conventional AHP.

$$(w_1, w_2) = k\left(\sum_{i=1} x_{i1}, \sum_{i=1} x_{i2}\right), \text{ where } k \text{ is normalizing coefficient.}$$

For two alternatives A1 and A2, the weights of C1 and C2 are equally important by (1/96)(36 + 12, 16 + 32) = (0.5, 0.5). The aggregate values for A1 and A2 are:

$$z = \begin{pmatrix} 3/4 & 1/3 \\ 1/4 & 2/3 \end{pmatrix} \begin{pmatrix} 0.50 \\ 0.50 \end{pmatrix} = \begin{pmatrix} 0.54 \\ 0.46 \end{pmatrix} \tag{E5}$$

With the addition of A3, the weights of C1 and C2 are given by (1/142)(36 + 12 + 42, 16 + 32 + 4) = (0.63, 0.37). Finally, the inequality, A1 > A3 > A2 is drawn in (E6).

$$z = \begin{pmatrix} 6/15 & 4/13 \\ 2/15 & 8/13 \\ 7/15 & 1/13 \end{pmatrix} \begin{pmatrix} 0.63 \\ 0.37 \end{pmatrix} = \begin{pmatrix} 0.37 \\ 0.31 \\ 0.32 \end{pmatrix} \tag{E6}$$

This approach preserves the rank order (A1 > A2) by altering the weights of the criteria. The normalized results of the alternatives priorities are the same as the real values in Table 1.

3.3.3 The B-G approach

Belton and Gear [2] tired to reveal the AHP weakness in ordering ranks that was caused by relatively distributing priorities to the alternatives and their aggregation in the AHP. They considered the definition of weight as the value on a unit scale on which the criterion was measured. To be consistent with this view, one should normalize the eigenvectors so that the maximum entry is 1 rather than the entries sum to 1. Finally, the criteria weights are established by comparisons of the maximum values of alternatives with respect to the criteria. Applying this procedure to the example of this paper, the weights of the two criteria (w_1 and w_2) are derived as follows.

$$(w_1, w_2) = k\left(\max_i\{x_{i1}\}, \max_i\{x_{i2}\}\right), \text{ where } k \text{ is the normalizing coefficient.}$$

With respect the two alternatives, A1 and A2, the weights of C1 and C2 are determined as $k(\max\{36, 12\}, \max\{16, 32\}) =$ (1/68)(36, 32) = (0.53, 0.47). In (E7), the synthesized results are 0.77 and 0.65 and they are normalized as 0.54 and 0.46 (normalization is represented by "$\Rightarrow$" in this paper).

$$z = \begin{pmatrix} 1 & 1/2 \\ 1/3 & 1 \end{pmatrix} \begin{pmatrix} 0.53 \\ 0.47 \end{pmatrix} = \begin{pmatrix} 0.77 \\ 0.65 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.54 \\ 0.46 \end{pmatrix} \tag{E7}$$

Adding A3 to A1 and A2, the weights of C1 and C2 should be adjusted because the reference unit for criterion C1 changes from 36 to 42 (See Table 1), conforming to [3]. The normalized weights appear as $k(\max\{36, 12, 42\}, \max\{16, 32, 4\}) =$ (1/74)(42, 32) = (0.57, 0.43). The normalized result of the alternatives priorities (0.37, 0.31, 0.32) in (E8) are the same as the Reference AHP produced. Finally, we have the relationship, A1 > A3 > A2. This approach preserves previous rank order (A1 > A2).

$$z = \begin{pmatrix} 6/7 & 1/2 \\ 2/7 & 1 \\ 1 & 1/8 \end{pmatrix} \begin{pmatrix} 0.57 \\ 0.43 \end{pmatrix} = \begin{pmatrix} 0.70 \\ 0.60 \\ 0.62 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.37 \\ 0.31 \\ 0.32 \end{pmatrix} \tag{E8}$$

3.3.4 The Dominant AHP

This approach was developed by Kinoshita and Nakanishi [11]. The idea of this approach is distinct from those ideas of the other approaches in the aspect that the reference unit for the criteria weights is elicited from an alternative, called the dominant alternative. It seems more realistic when a decision maker knows well about a certain alternative to prioritize criteria. A dominant alternative is the reference to compare with so that it is given 1 for every criterion. To be consistent with this point of view, the criteria weights are based on the measures of the dominant alternative.

If Ai is the dominant alternative in this example, the values in Ai provide the weights, w_1 (Ai) and w_2 (Ai), which are regarded as w_1 and w_2. That is,

$(w_1, w_2) = k(w_1(Ai), w_2(Ai)) = k(x_{i1}, x_{i2})$, where k is the normalizing coefficient.

Since this approach derives the weights of criteria with regard to which alternative has been realized as the dominant alternative, a set of new weights has to be generated if the dominant alternative changes.

For the example described in this paper, let A1 be the dominant alternative. Then the weights of C1 and C2 are derived as $k(x_{11}, x_{12}) = k$ (36, 16). Since the criteria values are assumed to be monetary values in this example, k = 1/(36+16) and the weights become (0.69, 0.31). In a given alternative set, the weight of the criteria would be varied by altering the dominant alternative. If A2 becomes the dominant alternative, then the criteria weights are (1/44)(12, 32) = (0.27, 0.73). However, whatever is selected as the dominant alternative, the dominant AHP guarantees the same normalized priorities of aggregation among the multiple alternatives. The results of applying this approach to the case of the two alternatives, A1 and A2, are shown in (E9). The normalized values are same as (0.54, 0.46) for the two alternatives and the values are consistent with those of the actual values in Table 1.

$$\left.\begin{array}{l} Applying\ A1,\ z = \begin{pmatrix} 1 & 1 \\ 1/3 & 2 \end{pmatrix} \begin{pmatrix} 0.69 \\ 0.31 \end{pmatrix} = \begin{pmatrix} 1.00 \\ 0.85 \end{pmatrix} \\ Applying\ A2,\ z = \begin{pmatrix} 3 & 1/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0.27 \\ 0.73 \end{pmatrix} = \begin{pmatrix} 1.18 \\ 1.00 \end{pmatrix} \end{array}\right. \Rightarrow \begin{pmatrix} 0.54 \\ 0.46 \end{pmatrix} \tag{E9}$$

Even though A3 is added, this approach, unlike the Referenced AHP and the B-G AHP, keeps the weights of C1 and C2 unless the previous dominant alternative alters. However, if A3 becomes the new dominant alternative, the weights vector is $k(x_{31}, x_{32})$ = (1/46)(42, 4) = (0.91, 0.09).

This approach draws the same normalized priorities of alternatives as those of the Referenced AHP and the B-G AHP. (E10) shows the case when A1 is the dominant alternative. All the alternatives draw the same normalized vectors and show the inequality A1 > A3 > A2. This is compatible with the previous rank order, A1 > A2.

$$Applying\ A1,\ \begin{pmatrix} 1 & 1 \\ 1/3 & 2 \\ 7/6 & 1/4 \end{pmatrix} \begin{pmatrix} 0.69 \\ 0.31 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.85 \\ 0.89 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.37 \\ 0.31 \\ 0.32 \end{pmatrix} \quad \text{(E10)}$$

If A2 or A3 is chosen as the dominant alternative in (E10), respectively, but normalized priorities vector of alternatives is same as that of A1.

In accordance with the case of multiple dominant alternatives, the Concurrent Convergence Method (CCM) that extends the dominant AHP [12] is activated when there is any discrepancy between normalized criteria weight vectors elicited from multiple dominant alternatives. Vectors with such discrepancies are guaranteed to converge on one vector using the CCM algorithm (See [13] for more details).

3.3.5 The ANP (Analytic Network Process) Approach

Saaty's ANP [16, 20] is the generalized form of the AHP. The AHP holds the idea of a linear propagation of properties of a given objective to the alternatives at the bottom of the hierarchy. The dominance of a property is assumed to flow from criteria to alternatives. If this assumption is violated, that is, any influence from alternatives to the criteria is necessary to be incorporated. The ANP is triggered to contain the structure of mutual influences. However, this approach requires an elaborate network design and high density of data to fit to the network. The ANP is represented graphically as a network instead of a hierarchy and algebraically as a column stochastic matrix (supermatrix, S) incorporating, among criteria and alternatives, all influences whose sum is normalized to a unit in every column. The S of this paper is formulated as follows.

$$S = \begin{pmatrix} \varphi & W \\ V & \varphi \end{pmatrix} \quad \text{(E11)}$$

where the sub-matrix W is the column eigenvectors of criteria with respect to each alternative and the sub-matrix V is the column eigenvectors of alternatives with respect to each criterion. φ is a null matrix. The eigenvector of the initial supermatrix converges to a stationary matrix (so called limit supermatrix) by the infinite power method, that is, S^{∞}.

Constructing the initial supermatrix (S) in Table 2, we input the normalized alternatives ratios for each criterion from Equation (E2) (the same as those given by Conventional AHP) and criteria ratios on behalf of the three alternatives, respectively, (the same as those given by the Dominant alternative approach in Equation (E10)). The results in Table 3 are given by $\lim_{k\to\infty} S^{2k+1}$ (see [20] for more details).

Table 2. Initial Supermatrix

	$C1$	$C2$	$A1$	$A2$	$A3$
$C1$	0	0	0.69	0.27	0.91
$C2$	0	0	0.31	0.73	0.09
$A1$	6 / 15	4 / 13	0	0	0
$A2$	2 / 15	8 / 13	0	0	0
$A3$	7 / 15	1 / 13	0	0	0

Table 3. Limiting Supermatrix

	$C1$	$C2$	$A1$	$A2$	$A3$
$C1$	0	0	0.63	0.63	0.63
$C2$	0	0	0.37	0.37	0.37
$A1$	0.37	0.37	0	0	0
$A2$	0.31	0.31	0	0	0
$A3$	0.32	0.32	0	0	0

The weights of the criteria and the aggregate value of the three alternatives have resulted as completely the same as those of the real value or the Referenced AHP above. The resulting weight vector is the overall weight vector of the three different weight vectors.

4. Classification of the AHP Application Circumstances

This paper proposes to classify the various AHP approaches that have emerged from historical debates, in addition to the Conventional AHP and ANP. The classification model, as such, should have the implication that guides the circumstances of AHP applications. The classification is given by 2 dimensions, one of which is associated with prioritizing criteria and the other is with prioritizing alternatives.

4.1 Criteria Prioritization

Watson and Freeling [34] argued that if the AHP question (which is better and how much?) would be asked to a decision maker without further explanation, it should become meaningless. In their behalf, it is obvious that the prioritizing criteria have no choice but to depend on alternative measurement because the explanation about comparing units with which the importance of criteria are evaluated is realized by alternative measurement. Belton and Gear [2] and Dyer [4] argued that rank reversal in the AHP is because of the fact that the weight of each criterion is independent of the evaluations of the available alternatives over this criterion. Their argument is concerning the manner in which these weights are elicited.

To the contrary, Saaty *et al.* [24] described
"However in practice, when the alternatives are not known in advance, one may simplify the analysis initially by attempting to elicit priorities on the attributes without knowledge of the particular alternatives. ⋯,
The AHP approach is not affected by a change of scale on the attribute. ⋯
Thus, it is unit free approach."

This view is consistent with the independence axiom. Therefore, prioritizing criteria in the Conventional AHP is independent of the alternative value or scale. Consistent with the Conventional AHP, the Ideal mode AHP establishes the weights based on the decision maker's judgment unit. The criteria weights, if once established by the Ideal mode AHP, are maintained consistently no matter which new alternative is added or removed (See (E3) and (E4)).

Depending on a set of alternatives implies that the criteria weights are required to particularly correspond to alternatives' measures which are shown in the four approaches: Referenced AHP, B-G AHP, Dominant AHP, and ANP. On the behalf of the alternative-dependent approaches, the criteria weights are not invariant because the relative importance of criteria is constrained to the measures realized in the alternatives and to the measures generally changed with alternatives altering. The B-G AHP constrains the relative importance of criteria to the ratios among the maximum values of given alternatives with respect to each criterion. If adding or removing an alternative causes any change to the maximum values with respect to any criterion, then such change should be accompanied with a change in the relative importance of the criteria. Accordingly, the weights of the criteria in B-G AHP depend on alternative measures. The example of this paper shows the case where adding the third alternative, A3, causes a modification of the maximum value and then a change of their ratio (See (E7) and (E8)). In the Referenced AHP, the criteria weights are elicited on the basis of the average (or sum) of the alternative measures with respect to each criterion. It is trivial to show that the average (or sum) of the alternatives' measures changes with adding or removing alternative(s) (See (E5) and (E6)). The Dominant AHP provides the judgment standards for the criteria weights depending on the dominant alternative. Altering the dominant alternative is accompanied with the change of judgment bases and, thus, the criteria weights (See (E9) and (E10)). The ANP generates the criteria weights through the supermatrix operation. The supermatrix in the example of this paper, as in (E11), shows that the overall weight vector is changeable depending on weight vectors that constitute the sub-matrix, W.

4.2 Alternatives Prioritization

The Conventional AHP uses the relative ratio scale because ratio scales are appropriate for hierarchical decomposition. The sum of priorities of the children elements, at any level of the hierarchy, becomes equal to the priority of their parent by multiplying the priority of the parent and priorities of children whose priorities are summed to one at first. This hierarchical propagation of priorities is consistent with the meaning of hierarchical decomposition. The eigenvector solution for a pairwise comparison matrix is unique by summing elements to one. Accordingly, the Conventional AHP normalizes the alternatives priorities, at the bottom level of the hierarchy, derived from the pairwise comparison matrix by dividing their sum.

Such normalization, as summing to one, is also concerned with rank reversals in the Conventional AHP. Rank reversal or adjustment in the AHP does not occur because of eigenvector calculations, nine point scale used in the AHP, inconsistencies in judgments, nor because "exact" copies are included in an evaluation. Rank reversal occurs because of scarcity (an abundance or dilution effect) [7, 18, 23, 24]. Even more, it can take place not only with the legitimacy of scarcity but also with any technique that decomposes and synthesizes, regardless of whether it uses the AHP. Perez [15] showed this phenomenon with the example of multidistrict proportional elections and concluded that it is necessary to identify the kind of situations in which the AHP is suitable.

Forman and Gass [8] introduce a closed or an open system. To be a closed system or an open system is concerning the aspect that scarcity is legitimate in the evaluation of alternatives and that the sum of priorities distributed to the alternatives increase or decrease if new alternatives are added or existing alternatives are removed from consideration. The sum of the alternatives' priorities is usually subject to 1 in the closed system, but it is not the case in the open system. The AHP originally started from a closed system.

The alternatives' priorities of the Referenced AHP and the ANP also should sum to 1 because they are exactly the same as the Conventional AHP in eliciting alternatives priorities. However, the Ideal mode AHP does not change the alternatives priorities that have been once established when some alternatives are added to or removed from current consideration. Let us see the following description by Saaty [21]:

> "In accordance with the ideal mode, by deriving priorities from paired comparisons, rank is always preserved if one idealizes only the first time, and then compares each alternative with the ideal alternative, allowing the value to exceed one. On the other hand, idealizing repeatedly only preserves rank from irrelevant alternatives."

The B-G AHP does not change the alternatives priorities while the current set of the maximum values of the alternatives with respect to each criterion is kept with adding or removing alternative(s). Neither does the Dominant AHP providing that a dominant alternative has not caused an alteration.

4.3 Classification of Circumstances

We construct a 2×2 classification matrix by 2 dimensions: alternative-dependent or independent for the relative importance of criteria and closed or open prioritization for the preferences of the alternatives.

The first dimension of vertical axis is concerned with how to elicit judgments for the relative importance of criteria. This dimension represents the circumstance under which the independence axiom of the AHP that implies judging the criteria weights in a hierarchy do not depend on alternatives is kept or violated. In general, the judgments require a reference unit under which a decision maker compares the amount of contributions among criteria to a given objective. The distinguishing point in the first dimension is whether or not such judgments units depend on a set of alternatives at hand when the decision maker evaluates relative importance of criteria.

Meanwhile, the second dimension (closed or open system) is how to normalize the preference of alternatives in terms of each criterion, which finally characterizes the aggregation of such preferences for the alternatives. In other words, allocating priorities to alternatives are given by relative comparison or by absolute standard among alternatives. Finally, the total amount of priorities for alternatives is fixed or not (closed or open). The fixed number is usually 1 in the AHP, that is, in the example above, $\sum_i v_{ij} = 1$ for each criterion j.

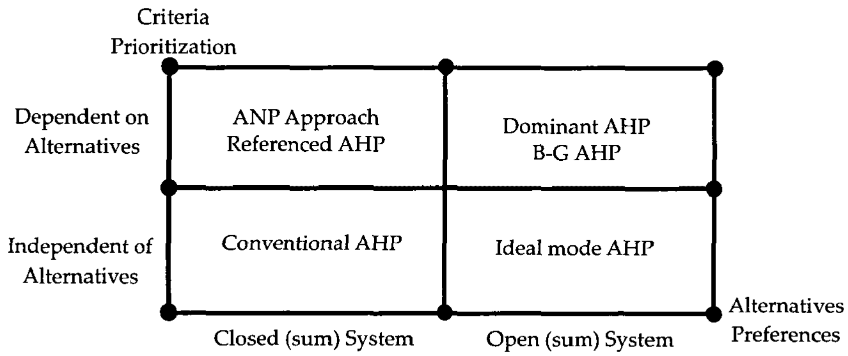

Figure 2. Classification of AHP Applications

4.3.1 Criteria Prioritization: Independent of or Dependent on Alternatives

For the characteristic of criteria, it is alternatives-independent if we elicit judgmental bases independent of alternatives measurements. Hence, the Conventional AHP and Ideal mode AHP are alternatives-independent and the other approaches are alternative-dependent.

Let us assume that there is a decision maker to choose a most appropriate automobile out of three or four candidates. However, the decision maker has enough experience and knowledge about not only the alternatives at hand but also all kinds of automobiles which are latent or implicit alternatives. When the decision maker evaluates the importance of the criteria and judges the relative importance not based on the limited alternatives but on all alternatives that he/she has experienced or virtually expects, it is not necessary for him/her to change the criteria weights with adding or deleting alternatives because the weights are elicited by considering all the alternatives.

According to the above interpretation, it is possible for a human being to elicit one's own judgment unit, consciously or unconsciously, considering all the possible alternatives that one can expect with one's experience or knowledge. This interpretation is available when a decision maker or an evaluator certainly has pre-experience or pre-knowledge about the judgment beyond the limited alternatives. Accordingly, the criteria weights are independent of limited alternatives and a decision maker can judge the criteria weights without specific alternative(s).

Such a way of interpretation is meaningful in applying the alternative-dependent AHPs. If the criteria weights by the pre-experience or pre-knowledge are quite different from those by a set of given alternatives, the alterative-dependent AHPs are activated to incorporate the adjustment of the criteria weights. If a decision maker re-evaluates the criteria weights in a given hierarchy until he/she is satisfied with the AHP results, then the reevaluation process also can be a part of the alternative-dependent AHPs.

4.3.2 Alternatives Preferences: Closed or Open System

AHP application situations can be divided into two kinds with respect to which preference of alternative is affected, or not, by adding or removing alternatives. The situation is named closed or open system, in this paper, which also can be decided by regarding the alternatives' priorities as resources or budgets, or by checking the number of alternatives to be evaluated.

Open system includes such circumstance as a decision maker wants to prevent reversals in rank order of the original set of alternatives from occurring when a new dominated alternative is added. If the amount of priorities for alternatives has been once allocated, it is not necessarily adjusted by adding or removing alternative(s) because priorities can be added or removed as required. As a result, the previously assigned priorities of alternatives are invariant and rank reversals among previously existing alternatives cannot occur. Otherwise, the allocation of a fixed amount of resources or budget is an appropriate example in which scarcity is meaningful.

Open system can cover the circumstance where there are many alternatives, usually more than 10 [21], because one can use the rating approach in which alternatives are evaluated one at a time after the base alternative such as an ideal alternative in the Ideal mode AHP or a dominant alternative in the Dominant AHP is established.

4.4 Suggestion of Choice Rule

It is not uncommon that the decision maker can judge or must judge criteria without specific alternatives. In this circumstance, the interpretation of criteria independence can be conducted so that the decision maker does not depend on a specific set of alternatives but considers all the explicit and implicit alternatives he/she has experienced or can imagine. Hence, the associated choice for criteria will be given as the answer for the question Q1: *"Is it possible to evaluate criteria without alternative(s)?"* or *"Are the criteria weights stable even if the set of alternatives changes?"* If the answer is "yes," then the prioritizing criteria is independent of explicit or changing alternatives. Otherwise, dependence on explicit or changing alternatives is adopted.

Concerning features of criteria with alternatives to classify application circumstances, we can add the question Q1+: *Which is the most appropriate reference to feature criteria among three modes-Sum [Average], Ideal, or Each alternative(s) consideration?* This question is more obvious in case of the alternatives-dependent criteria evaluation because of explicit alternatives. Getting sum [average] of explicit alternatives is allowed for the Reference AHP and dependence on Ideal case out of explicit alternatives is for the B-G AHP. The ANP approach and the Dominant AHP depend on each alternative for prioritizing criteria. Otherwise the decision maker uses his/her own units or Ideal alternative out of implicit [latent] alternatives.

To be open or to be closed is guided by the simple question Q2: *"Is it possible to regard the alternatives priorities as closed (limited) sum [resources or budget] for the given criterion?"* or *"Can adding or removing alternatives(s) affect the previous alternatives priorities?"* If the answer is "yes," then closed system of alternatives priorities is appropriate. Otherwise, open system is adopted. Additionally, according to Saaty [21], if the number of alternatives is greater than or equal to 10, then open sum of priorities approaches are considered (Q2+). Otherwise, reorganizing alternatives hierarchically is necessary to use pairwise comparisons.

For more details, we compare the AHP approaches as follows and summarize guidelines to choose in Table 4.

4.4.1 Ideal mode AHP, B-G AHP, and Dominant AHP

They are all classified into Open system but differentiated by prioritizing criteria. Even though the idea of the Ideal mode AHP is adopted from Belton and Gear's [2, 3], the basic operations of the two approaches are quite different when considering a new alternative that has any value exceeding the current maximum values with respect to criteria. Allowing the value to exceed one implies that the Ideal mode AHP does not adjust the criteria weights once derived because the Ideal mode AHP is based on the implicit alternatives rather than given or limited alternatives. However, for the same cases, the B-G approach requires adjusting the criteria weights, instead of assigning the value to exceed one, which shows scaling the criteria weights depending on the given alternatives. Comparing (E4) and (E8), it is trivial to show such a difference. However, if the same data set is used for the two equations, the results of syntheses normalizing to 1 are the same.

The Dominant AHP is differentiated from the Ideal mode AHP and the B-G AHP in light of eliciting comparing units with which the relative importance of criteria is derived. The Dominant AHP provides a judgment unit from an alternative but the other AHP approaches should investigate all alternatives at hand. The relative importance, depending on a dominant alternative, is independent of adding or removing alternatives like depending on the Ideal alternative. In the Dominant AHP, the only case that causes any change to the criteria weights is to alter the dominant alternative. Meanwhile, the B-G AHP permits any change to the criteria weights unless a chosen ideal alternative is preserved.

4.4.2 ANP approach and Referenced AHP

It is not rare to have complex decision problems where the judgment of the criteria, such as ambience or food quality of restaurant, is psychological and conceptual [27]. For such kinds of criteria, it is not easy for a decision maker to take the total sum or the average of alternatives. In this circumstance, the ANP approach seems to be logically better than the Referenced AHP because the ANP compares criteria as such with respect to each alternative. However, the ANP approach requires more input data than the Referenced AHP does.

4.4.3 ANP approach and Dominant AHP

In the example of this paper, if there are the criteria weights prior to the derived priorities for the criteria with respect to each alternative, the previous criteria weights are absorbed into the limiting supermatrix of the ANP without any effect on its result (See Appendix for mathematical proof). In case of such feedback from alternatives to criteria, it is necessary for the ANP to calculate the criteria weights for each alternative. As shown in the ANP example, the criteria weights with respect to each alternative can be regarded as the results by the Dominant AHP questioning. However, the Dominant AHP could be effective where the confidence or certainty levels in measuring alternatives are not equal. The most confident or reliable alternative will be the dominant alternative. In addition, prioritizing alternatives distinguish the two methods into the closed or the open system.

4.4.4 *Guidelines for Choice*

As represented in Table 4, ranking irregularity as adding or deleting alternative(s) is acceptable only to the Conventional AHP. The other AHP approaches preserve previous ranks of alternatives. However, the way of keeping ranks is different. The ideal mode uses the Ideal alternative which has nothing to do with change of alternatives. The B-G AHP and the Dominant AHP preserve previous ranks but their criteria weights change when the Ideal or Dominant alternative alters. In the Dominant AHP, a dominant alternative receives the numerical value 1 representing the standard priority to compare for every criterion. Each of the other alternatives receives a priority by comparing its preference to that of the dominant alternative with respect to each criterion. Accordingly, the numerical values to be assigned to the other alternatives for each criterion are positive but free upper limit, which is unlike the B-G AHP.

Table 4. Guidelines for choice

Approach	Q1: Criteria weights-Alternatives	Q1+: Featuring criteria weights	Q2: Alternative priorities measurement	Q2+: Numerous alternatives?	Ranks of aggregation as changing alternatives
Conventional	Independence	Implicit unit	Closed	Reorganizing	Irregularity acceptable
Ideal mode	Independence	Implicit Ideal	Open	Yes	Preserved
ANP approach	Dependence	Each alternative	Closed	Reorganizing	Incorporating all bilateral relations
Referenced	Dependence	Explicit unit (sum or average)	Closed	Reorganizing	Need to change criteria weights
B-G	Dependence	Explicit Ideal	Open (Maximum = 1)	Yes	- Preserved - Different criteria weights per Ideal
Dominant	Dependence	Each alternative	Open (Standard = 1)	Yes	- Preserved - Different criteria weights per Dominant

The ANP and the Referenced AHP preserve previous ranks by weights change. The Referenced AHP requires that the sum or average of the alternatives measures with respect to each criterion should be recalculated and compared when adding or removing alternative(s) is required. The ANP also can prevent rank reversals by incorporating the weight vectors with respect to alternatives as is the case for the Dominant AHP (See E(10) and Table 2). The ANP logically constrains rank adjustments, if needed, by adjusting the criteria weights. Adjusting the criteria weights is conducted in the supermatrix by adding or removing weight vector(s) based on the correspondingly added or removed alternative(s).

5. Discussion and Concluding Remarks

5.1 Rank Reversals and Multiplicative AHP

While the approaches in the open system do not allow for scarcity and do not permit rank reversals, the closed prioritization admits the scarcity legitimacy as such. However the approaches in the closed system can prevent rank reversals by keeping a required condition (See [24] for the condition in tail). In addition, there have been the AHP studies that have treated ranking irregularities by different approach using multiplicative function rather than additive function.

Barzilai and Lootsma [1] proposed to use a multiplicative variant of the AHP in order to model power relationships in group decision making. Triantaphyllou [31] used a similar approach for single decision-maker problems. In these studies, the ranking irregularity which occurs when one compares the rank derived when all the alternatives are considered and the rank derived when subset of alternatives are considered occurs often. It is also testified that ranking irregularities are not possible when a multiplicative variant of the AHP is used.

However, Vargas [33] showed the relative weights of the objects obtained with the multiplicative principle are different from the real outcome given in an example. This indicates that multiplicative synthesis gives the wrong result in relative measurement. Saaty [22] also showed that multiplicative composition is faulty because, for example, an alternative with the same value under two criteria receives a smaller value for the more important criteria. In addition, enforcing ranking preservation all the time is a mistake when paired comparisons are involved.

In contrast, Stam and Silva [29] explored difference between the additive AHP and multiplicative AHP by theory and simulation experiment and suggest the relative attractiveness of the multiplicative AHP and its situation. In addition to the comparativeness between two modes of AHP [30], Ishizaka *et al.* [14] argued that the variation of multiplicative AHP such as usage of the logarithmic scale is in agreement with consumer choice theory. There is special case such as imperfection of human behavior that the AHP should deal with [38].

Always preserving ranks with adding or deleting alternative(s) is a part of AHP application and many of AHP approaches meet such requirement, which has been shown in this paper. Concerning ranking irregularity, the fact that in performing paired comparisons there is dependence of the measurement of one element on the quality of what it is compared with need not be overlooked. Examining real-life cases, ranking irregularities are likely to occur. We consider the additive AHP as a valid MCDM technique still widely used in practice, however further research should concentrate on properties that the AHP variations argue.

5.2 Concluding Remarks

The Conventional AHP starts with hierarchical composition, pairwise comparison and its eigenvector method. However, reviewing the AHP debates with alternative approaches, and recognizing the principle of hierarchy, we have illustrated that the AHP decision making is sensitive to two factors (how to elicit the criteria weights and how to normalize the alternative priorities), which are related to the independency axiom and the priorities of ratio scale measures. These two factors are separated into two dimensions: (1) the criteria prioritization-dependent on or independent of limited alternatives, (2) the alternatives prioritization-closed or open system. 6 different views for AHP synthesis including the ANP approach are classified with two dimensions, which can be resulted in the four application circumstances and additional features for the AHP application.

"Which approach is best to apply or most accurate for all?" is not our intention in this paper. Instead, we note that there are differences, among the approaches we have dealt with, in the way of thinking to apply the AHP. According to the consequence, we can define 4 types of AHP application: (a) Criteria prioritization independent of alternatives and Closed alternative priorities, (b) Criteria prioritization independent of alternatives and Open alternative priorities, (c) Criteria prioritization dependent on alternatives and Closed alternative priorities, and (d) Criteria prioritization dependent on alternatives and Open alternative priorities. (a) and (b) are covered by only the Conventional AHP and the Ideal mode AHP, respectively. If a decision maker is involved in a choice problem with sufficiency of associated information, it is reasonable for him or her to be independent of the given alternatives for the judgments of the related criteria importance. (c) is by the ANP approach and the Referenced AHP. The ANP approach is more recommended than the Referenced AHP when it is uncommon and almost impossible for a decision maker to sum up or take average with such kinds of qualitative alternatives measures. The ANP also can cover trade-off between efficiency in the number of pairwise comparisons and accuracy in results. If accuracy is more prioritized, then the weight vectors with respect to all alternatives are input data into the supermatrix. Under (d), the Dominant AHP and the B-C AHP are differentiated in the circumstance that the criteria prioritization is more based on one alternative or not. Additionally, the ANP approach incorporates the viewpoint of the Dominant AHP to allocate criteria priorities based on each alternative.

We believe that there is no one basic rational decision model, and admit that the definition of rational decision does not lay out a set of rules but rather guides a decision maker in selecting an appropriate set of rules [8]. We recommend that the decision maker be guided by 6 types in 4 categories from the two dimensions of our study. We also provided more guidelines to apply the AHP.

Note that the AHP is not only a method but also a paradigm to think about. Under such a paradigm, we can apply the AHP to the four described circumstances appropriately. It is obvious that the 6 approaches we have treated are representative AHP approaches and their circumstances are clearly classified in this paper. We hope this study contributes to understanding the AHP and the circumstances of its applications, enhancing AHP applications and improving the application quality of the AHP.

References

[1] Barzilai, J. and F. A. Lootsma, "Power relations and group aggregation in the multiplicative AHP and SMART," *Journal of Multi-Criteria Decvision Analysis* 6, 3 (1997), 155-165.

[2] Belton, V. and T. Gear, "On a shortcoming of Saaty's method of Analytic Hierarchies," *Omega International Journal of Management* 11, 3 (1983), 228-230.

[3] Belton, V. and T. Gear, "The legitimacy of rank reversal-a comment," *Omega International Journal of Management* 13, 3 (1985), 143-144.

[4] Dyer, J. S., "Remarks on the Analytic Hierarchy Process," *Management Science* 36, 3 (1990), 249-258.

[5] Dyer, J. S., "A clarification of "Remarks on the Analytic Hierarchy Process," *Management Science* 36, 3 (1990), 274-275.

[6] Forman, E. H., "Ideal and distributed synthesis modes for the analytic hierarchy process," *George Washington University Working Paper* (1993).

[7] Forman, E. H., "AHP is intended for more than expected value calculations," *Decision Sciences* 21, 2 (1990), 670-672.

[8] Forman, E. H. and S. I. Gass, "The Analytic Hierarchy Process-an exposition," *Operations Research* 49, 4 (2001), 469-486.

[9] Harker, P. T. and L. G. Vargas, "The theory of ratio scale estimation: Saaty's Analytic Hierarchy Process," *Management Science* 33, 11 (1987), 1383-1403.

[10] Harker, P. T. and L. G. Vargas, "Reply to 'Remarks on the Analytic Hierarchy Process' by J. S. Dyer," *Management Science* 36, 3 (1990), 269-273.

[11] Kinoshita, E. and M. Nakanishi, "A proposal of new viewpoint in Analytic Hierarchic Process," *Journal of Infrastructure Planning and Management* IV-36-569 (1997), 1-8.

[12] Kinoshita, E. and M. Nakanishi, "Proposal of new AHP model in light of dominant relationship among alternatives," *Journal of the Operations Research Society of Japan* 42, 2 (1999), 180-189.

[13] Kinoshita, E., K. Sekitani, and J. Shi, "Mathematical properties of Dominant AHP and Concurrent Convergence Method," *Journal of the Operations Research Society of Japan* 45, 2 (2002), 198-213.

[14] Ishizaka A., D. Balkenborg, T. Kaplan, "Influence of aggregation and preference scale on ranking a compromise alternative in AHP," *Multidisciplinary Workshop on Advances in Preference Handling*, Italy, 2006.

[15] Perez, J., "Some comments on Saaty's AHP," *Management Science* 41, 6 (1995), 1091-1095.

[16] Saaty, T. L., *The Analytic Hierarchy Process*, McGraw-Hill, New York, 1980.

[17] Saaty, T. L., "Axiomatic foundation of the Analytic Hierarchy Process," *Manage-*

ment Science 32, 7 (1986), 841-855.

[18] Saaty, T. L., "An exposition of the AHP in reply to the paper Remarks on the Analytical Hierarchy Process," *Management Science* 36, 3 (1990), 259-268.

[19] Saaty, T. L., "Highlights and critical points in the theory and application of the Analytic Hierarchy Process," *European Journal of Operational Research* 74 (1994), 426-447.

[20] Saaty, T. L., *The Analytic Network Process*, RWS Publications, Pittsburgh, 1996.

[21] Saaty, T. L., "Decision making-the Analytic Hierarchy and Network Processes (AHP/ANP)," *Journal of Systems Science and Systems Engineering* 13, 1 (2004), 1-35.

[22] Saaty, T. L., *Theory and Applications of the Analytic Network Process*, RWS Publications, Pittsburgh, 2005.

[23] Saaty, T. L. and L. G. Vargas, "The legitimacy of rank reversal," *Omega International Journal of Management Science* 12, 5 (1984), 513-516.

[24] Saaty, T. L., L. G. Vargas, and R. E. Wendell, "FEEDBACK: Assessing attribute weights by ratios," *Omega International Journal of Management Science* 11, 1 (1983), 9-12.

[25] Schoner, B. and W. C. Wedley, "Ambiguous criteria weights in AHP: consequences and solutions," *Decision Sciences* 20, 3 (1989), 462-475.

[26] Schoner, B., W. C. Wedley, and E. U. Choo, "A rejoinder to Forman on AHP, with emphasis on the requirements of composite ratio scales," *Decision Sciences* 23, 3 (1992), 509-517.

[27] Schoner, B., W. C. Wedley, and E. U. Choo, "A unified approach to AHP with linking pins," *European Journal of Operational Research* 64, 3 (1993), 384-392.

[28] Sekitani, K. and H. Ueta, "Remarks the concurrent convergence method for a typical mutual evaluation system," *Journal of the Operations Research Society of Japan* 47, 2 (2004), 82-95.

[29] Stam, A. and A. Pedro Duarte Silva, "On multiplicative priority rating methods for the AHP," *European Journal of Operational Research* 145 (2003), 92-108.

[30] Triantaphyllou, E., *Multi-Criteria Decision Making Methods: A Comparative Study*, Kluwer Academic Publishers, London, 2000.

[31] Triantaphyllou, E., "Two new cases of rank reversals when the AHP and some of its additive variants are used that do not occur with the multiplicative AHP," *Journal of Multi-Criteria Decision Analysis* 10, 1 (2001), 11-25.

[32] Vargas, L. G., "An overview of the Analytic Hierarchy Process and its applications," *European Journal of Operational Research* 48 (1990), 2-8.

[33] Vargas, L. G., "Comments on Barzilai and Lootsma: Why the multiplicative AHP is invalid: A practical esample," *Journal of Multicriteria Decision Analysis* 6, 3 (1997), 169-170.

[34] Watson, S. R. and A. N. S. Freeling, "Assessing attribute weights by ratios," *Omega International Journal of Management Science* 10, 6 (1982), 582-583.

[35] Watson, S. R. and A. N. S. Freeling, "Comment on: Assessing attribute weights by ratios," *Omega International Journal of Management Science* 11, 1 (1983), 13.

[36] Yoon, M. S. and K. S. Whang, "Pooling Partial Pairwise Comparisons in the AHP," *International Journal of Management Science* 4, 1 (1999), 35-56.

[37] Zahedi, F., "The Analytic Hierarchy Process-a survey of the method and its applications," *Interfaces* 16, 4 (1986), 96-108.

[38] Zahir S., "Normalization and rank reversals in the additive analytic hierarchy process: a new analysis," *International Journal of Operational Research* 4, 4 (2009), 446-467.

Appendix

Let us adopt the following notations. Let S be the supermatrix with simple dependency between criteria and alternatives. Let S^+ be the supermatrix where the criteria weights vector (w^+), that is independent of alternatives, is added to S at the first column representing any influence from the ultimate objective. S^+ shows that the null vector is added at the first row of S because there is no feedback from the criteria or alternatives to the objective.

$$S = \begin{pmatrix} \varphi & W \\ V & \varphi \end{pmatrix}, \quad \text{and} \quad S^+ = \begin{pmatrix} 0 & \varphi & \varphi \\ w^+ & \varphi & W \\ \varphi & V & \varphi \end{pmatrix},$$

where the sub-matrix W is composed of vectors of criteria weights with respect to each alternative and the sub-matrix V is of eigenvectors of alternatives priorities with respect to each criterion. φ is a null matrix.

Theorem: If we let $y^T = (w,\ v)$, where w and v are the criteria weights vector and the alternatives priorities vectors, respectively, be the eigenvector of S that satisfies

$Sy = \lambda_{\max} y$, then $y^{+T} = (0,\ y^T)$ is the eigenvector that satisfies $S^+ y^+ = \lambda_{\max} y^+$.

Proof: Since S and S^+ are column stochastic matrices, then $\lambda_{\max} = 1$ in both cases [18].

By the definition,

$$Sy = \begin{pmatrix} \varphi & W \\ V & \varphi \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} Wv \\ Vw \end{pmatrix} = \begin{pmatrix} w \\ v \end{pmatrix} = y$$

Now we need to show only $S^+ y^+ = y^+$ as follows;

$$S^{+}y^{+}=\begin{pmatrix} 0 & \varphi & \varphi \\ w^{+} & \varphi & W \\ \varphi & V & \varphi \end{pmatrix}\begin{pmatrix} 0 \\ w \\ v \end{pmatrix}=\begin{pmatrix} 0 \\ Wv \\ Vw \end{pmatrix}=\begin{pmatrix} 0 \\ w \\ v \end{pmatrix}=\begin{pmatrix} 0 \\ y \end{pmatrix}$$

Accordingly, w^{+} does not affect the eigenvector, y of the supermatrix, S. That is, the criteria weights independent of alternatives measurements remain no longer. From the relationship $w=Wv$ and $v=Vw$, v is determined by $v=VWv$ (without any effect of w^{+}) because VW is a stochastic matrix.