J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 17, Number 2 (May 2010), Pages 189–197

# ESTIMATION OF DIFFERENCE FROM HÖLDER'S INEQUALITY

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Abstract. We give an upper bound for the estimation of the difference between both sides of the well-known Hölder's inequality. Moreover, an upper bound for the estimation of the difference of the integral form of Hölder's inequality is also obtained. The results of this paper are natural generalizations and refinements of those of [2-4].

## 1. Introduction

It is well-known that the following Hölder's inequality [1]

$$\sum_{i=1}^{m}\left(\prod_{j=1}^{n}a_{ij}\right) \le \prod_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}^{p_j}\right)^{\frac{1}{p_j}}, \tag{1}$$

where $a_{ij} > 0$ for $1 \le i \le m$ and $1 \le j \le n$ and $p_j > 1$ with $\sum_{j=1}^{n} p_j^{-1} = 1$, together with its integral form

$$\int_a^b \left(\prod_{k=1}^{n} f_k(x)\right) dx \le \prod_{k=1}^{n}\left(\int_a^b f_k^{p_k}(x)dx\right)^{\frac{1}{p_k}}, \tag{2}$$

where $f_k \in C([a,b],(0,+\infty))$ for $k = 1, 2, \cdots, n$, plays an important role in the study of inequalities and in the field of applied mathematics. For example, the well-known Cauchy's inequality

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

is a special case of (1) when $n = 2$ and $p_1 = p_2 = 2$. In many cases, we need to know not only the inequalities, but also the estimations of the differences between

Received by the editors March 15, 2010. Accepted May 10, 2010.
2000 *Mathematics Subject Classification.* 26D15, 26D20.
*Key words and phrases.* Hölder's inequality, concave function, estimation of difference.

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the right hand sides and left hand sides of these inequalities. For instance, Cauchy's inequality is a direct consequence of the following Lagrange Identity

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2 = \sum_{1\le i<j\le n} (x_i y_j - x_j y_i)^2.$$

In general, however, such a simple equality does not exist. Sometimes, the estimations of the differences between the right hand sides and left hand sides of these inequalities and the conditions at which these inequalities become equalities play important roles in the study and application of these inequalities. As for Hölder's inequality, many generalizations and refinements have been obtained so far, see, for example, [1–6] and the references therein. In [2, 3, 4], by using Young's inequality and Ozeki's inequality, the authors discussed the maximum of the difference

$$D_p(a,b,w) := \left(\sum_{k=1}^{n} w_k a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} w_k b_k^q\right)^{\frac{1}{q}} - \sum_{k=1}^{n} w_k a_k b_k,$$

where $w = (w_1, w_2, \cdots, w_n)$ is a weight and $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and $a_k,\ b_k,\ k = 1, 2, \cdots, n$ are positive numbers such that

$$0 < m_1 \le a_k \le M_1 \quad \text{and} \quad 0 < m_2 \le b_k \le M_2, \quad k = 1, 2, \cdots, n$$

for some positive constants $m_1, m_2, M_1$ and $M_2$. If we replace $w_k^{1/p} a_k$ and $w_k^{1/q} b_k$ by $\overline{a}_k$ and $\overline{b}_k$ for $k = 1, 2, \cdots, n$ respectively, then we have

$$D_p(a,b,w) = D_p(\overline{a},\overline{b}) = \left(\sum_{k=1}^{n} \overline{a}_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \overline{b}_k^q\right)^{\frac{1}{q}} - \sum_{k=1}^{n} \overline{a}_k \overline{b}_k.$$

Hence we need only to consider the case where $w = (1, 1, \cdots, 1)$.

## 2. Main Results

The main results of this paper are the followings:

**Theorem 1.** *Let $a_{ij} > 0$ for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$ and let $p_k > 1$ for $k = 1, 2, \cdots, n$ with $\sum_{k=1}^{n} p_k^{-1} = 1$. Assume that there exist positive constants $A_j,\ B_j,\ j = 1, 2, \cdots, n$ such that*

$$0 < A_j \le a_{ij} \le B_j, \quad i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, n. \tag{3}$$

*Let*

$$\Delta(a,p) := \prod_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij}^{p_j}\right)^{\frac{1}{p_j}} - \sum_{i=1}^{m} \left(\prod_{j=1}^{n} a_{ij}\right).$$

*Then, we have the following estimation:*

$$0 \le \Delta(a,p) \le m \left(\prod_{j=1}^{n} B_j\right) \cdot \sum_{k=1}^{n} \frac{1}{p_k} \left(\frac{B_k}{A_k}\right)^{p_k} \ln\left(\frac{B_k^{p_k}}{\prod_{j=1}^{n} A_j}\right). \tag{4}$$

*Moreover, the constant in the right hand side of (4) is sharp.*

Similarly, we have the following estimation of the difference of the integral form of Hölder's inequality.

**Theorem 2.** *Let $p_k > 1$ for $k = 1, 2, \cdots, n$ with $\sum_{k=1}^{n} p_k^{-1} = 1$. Assume that $f_k \in C([a,b],(0,+\infty))$ for $k = 1, 2, \cdots, n$ and there exist positive constants $C_k$ and $D_k$ for $k = 1, 2, \cdots, n$ such that*

$$0 < C_k \le f_k(x) \le D_k, \quad k = 1, 2, \cdots, n, \ x \in [a,b]. \tag{5}$$

*Let*

$$\Sigma(f,p) := \prod_{k=1}^{n} \left(\int_a^b f_k^{p_k}(x)dx\right)^{\frac{1}{p_k}} - \int_a^b \left(\prod_{k=1}^{n} f_k(x)\right) dx.$$

*Then we have the following estimate:*

$$0 \le \Sigma(f,p) \le (b-a) \left(\prod_{k=1}^{n} D_k\right) \cdot \sum_{k=1}^{n} \frac{1}{p_k} \left(\frac{D_k}{C_k}\right)^{p_k} \ln\left(\frac{D_k^{p_k}}{\prod_{j=1}^{n} C_j}\right). \tag{6}$$

*Moreover, the constant in the right hand side of (6) is sharp.*

The following corollaries are direct consequences of Theorem 1 and Theorem 2 and so we omit the proofs of them:

**Corollary 1.** *Let $A_k = A$, $B_k = B$ for $k = 1, 2, \cdots, n$ in Theorem 1. Then we have*

$$0 \le \Delta(a,p) \le (mB^n) \cdot \sum_{k=1}^{n} \frac{1}{p_k} \left(\frac{B}{A}\right)^{p_k} \ln\left(\frac{B^{p_k}}{A^n}\right).$$

**Corollary 2.** *Let $C_k = C$, $D_k = D$ for $k = 1, 2, \cdots, n$ in Theorem 2. Then we have*

$$0 \le \Sigma(f,p) \le (b-a)D^n \cdot \sum_{k=1}^{n} \frac{1}{p_k} \left(\frac{D}{C}\right)^{p_k} \ln\left(\frac{D^{p_k}}{C^n}\right).$$

**Corollary 3.** *Let $n = 2$ in Theorem 1. Assume that $a_k, b_k,\ k = 1, 2, \cdots, m$ are positive numbers and that there exist positive constants $C_k,\ D_k,\ k = 1, 2$ such that $0 < C_1 \le a_k \le D_1,\quad 0 < C_2 \le b_k \le D_2$ for $k = 1, 2, \cdots, m$ and that*

$p > 1, q = \frac{p}{p-1} > 1$. *Then we have*

$$
(7) \quad \begin{aligned} 0 &\le \left(\sum_{k=1}^{m} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{m} b_k^q\right)^{\frac{1}{q}} - \sum_{k=1}^{m} a_k b_k \\ &\le m D_1 D_2 \cdot \left[\frac{1}{p}\left(\frac{D_1}{C_1}\right)^p \ln\left(\frac{D_1^p}{C_1 C_2}\right) + \frac{1}{q}\left(\frac{D_2}{C_2}\right)^q \ln\left(\frac{D_2^q}{C_1 C_2}\right)\right]. \end{aligned}
$$

*Moreover, the constant in the right hand side of (7) is the best possible.*

**Corollary 4.** *Let $n = 2$ in Theorem 2. Assume that $f, g \in C([a,b],(0,+\infty))$ and that there exist positive constants $\delta_k, \Delta_k, k = 1, 2$ such that $0 < \delta_1 \le f(x) \le \Delta_1$, $0 < \delta_2 \le g(x) \le \Delta_2$ for all $x \in [a,b]$ and that $p > 1, q = \frac{p}{p-1} > 1$. Then we have*

$$
(8) \quad \begin{aligned} 0 &\le \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \cdot \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} - \int_a^b f(x)g(x)dx \\ &\le (b-a)\Delta_1\Delta_2 \cdot \left[\frac{1}{p}\left(\frac{\Delta_1}{\delta_1}\right)^p \ln\left(\frac{\Delta_1^p}{\delta_1\delta_2}\right) + \frac{1}{q}\left(\frac{\Delta_2}{\delta_2}\right)^q \ln\left(\frac{\Delta_2^q}{\delta_1\delta_2}\right)\right]. \end{aligned}
$$

*Moreover, the constant in the right hand side of (8) is the best possible.*

## 3. PROOFS OF THEOREMS

Consider a continuous function $h : (-\infty, +\infty) \to (0, +\infty)$ defined by

$$
(9) \quad h(t) = \prod_{k=1}^{n} \left[\sum_{i=1}^{m} \left(\prod_{j=1}^{n} a_{ij}\right)^{1-t} a_{ik}^{tp_k}\right]^{\frac{1}{p_k}}.
$$

Then $h \in C^\infty$ and note that (1) is equivalent to the following inequality:

$$
(10) \quad h(0) = \sum_{i=1}^{m} \prod_{j=1}^{n} a_{ij} \le \prod_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij}^{p_j}\right)^{\frac{1}{p_j}} = h(1).
$$

From the above observation, we see that the function $h(t)$ plays a crucial role in proving the Theorem 1. In fact, we have the following:

**Lemma 1.** *Let the function $h$ be defined as in (9). Then $h \in C^\infty$ is a concave function, that is, $h''(t) \ge 0 \;\forall\, t \in \mathbb{R}$ and $h$ takes its minimum at $t = 0$. Moreover, $h''(t_0) = 0$ at some $t_0 \in \mathbb{R}$ if and only if*

$$
(11) \quad \frac{a_{ik}^{p_k}}{\prod_{j=1}^{n} a_{ij}} = \frac{a_{jk}^{p_k}}{\prod_{l=1}^{n} a_{jl}}
$$

*for all* $1 \le i, j \le m$, $1 \le k \le n$. *In this case,* $h(t) \equiv h(0) =$ *constant and* $h'(t) \equiv h''(t) \equiv 0$. *If (11) does not hold, then* $h$ *is a strictly concave function, i.e.,* $h''(t) > 0\ \forall\, t \in \mathbb{R}$ *and* $h'(t)t > 0$ *for all* $t \neq 0$.

*Proof.* Let

$$b_i = \prod_{j=1}^{n} a_{ij}, \quad d_{ik} = \frac{a_{ik}^{p_k}}{b_i} = \frac{a_{ik}^{p_k}}{\prod_{j=1}^{n} a_{ij}}, \quad 1 \le i \le m, \quad 1 \le k \le n. \tag{12}$$

Then $h(t)$ can be written as

$$h(t) = \prod_{k=1}^{n} \left[ \sum_{i=1}^{m} b_i d_{ik}^t \right]^{\frac{1}{p_k}}.$$

Define a $C^\infty$ function $H : \mathbb{R} \to \mathbb{R}$ by $H(t) = \ln h(t)$. Then we have

$$H(t) = \sum_{k=1}^{n} \frac{1}{p_k} \ln \left( \sum_{i=1}^{m} b_i d_{ik}^t \right),$$

$$H'(t) = \sum_{k=1}^{n} \frac{1}{p_k} \frac{\sum_{i=1}^{m} b_i d_{ik}^t \ln d_{ik}}{\sum_{i=1}^{m} b_i d_{ik}^t},$$

and

$$H''(t) = \sum_{k=1}^{n} \frac{1}{p_k} \frac{\sum_{1 \le i < j \le m} b_i b_j d_{ik}^t d_{ij}^t (\ln d_{ik} - \ln d_{jk})^2}{(\sum_{i=1}^{m} b_i d_{ik}^t)^2} \ge 0.$$

From (12) and the relation $\sum_{k=1}^{n} p_k^{-1} = 1$ and the expression of $H'(t)$, we have

$$\begin{aligned}
H'(0) &= \sum_{k=1}^{n} \frac{1}{p_k} \frac{\sum_{i=1}^{m} b_i \ln d_{ik}}{\sum_{i=1}^{m} b_i} \\
&= \left( \sum_{i=1}^{m} b_i \right)^{-1} \sum_{k=1}^{n} \frac{1}{p_k} \sum_{i=1}^{m} b_i \ln d_{ik} \\
&= \left( \sum_{i=1}^{m} b_i \right)^{-1} \sum_{i=1}^{m} b_i \sum_{k=1}^{n} \frac{1}{p_k} \ln d_{ik} \\
&= \left( \sum_{i=1}^{m} b_i \right)^{-1} \sum_{i=1}^{m} b_i \sum_{k=1}^{n} \frac{1}{p_k} \ln \frac{a_{ik}^{p_k}}{b_i} \\
&= \left( \sum_{i=1}^{m} b_i \right)^{-1} \sum_{i=1}^{m} b_i \left( \sum_{k=1}^{n} \ln a_{ik} - \sum_{k=1}^{n} \frac{1}{p_k} \ln b_i \right) \\
&= \left( \sum_{i=1}^{m} b_i \right)^{-1} \sum_{i=1}^{m} b_i \left( \ln b_i - \ln b_i \right) = 0.
\end{aligned}$$

Hence we have $h'(0) = h(0)H'(0) = 0$.

Moreover, we see that $H''(t_0) = 0$ for some $t_0 \in \mathbb{R}$ if and only if

$$d_{ik} = d_{jk}, \quad 1 \le i < j \le m, \quad k = 1, 2, \cdots, n$$

which is equivalent to (11) by (12). In this case, $H''(t) \equiv 0$, and so $H'(t) \equiv H'(0) = 0$. Since $h'(t) = h(t)H'(t)$ and

$$h''(t) = h(t)\left[H''(t) + \left(H'(t)\right)^2\right],$$

we also have $h''(t) = h'(t) = 0\ \forall t \in \mathbb{R}$. If (11) does not hold, then $H''(t) > 0\ \forall t \in \mathbb{R}$ and hence we have $h''(t) > 0\ \forall\, t \in \mathbb{R}$ and $h'(t)t > 0\ \forall\, t \neq 0$ □

*Proof of Theorem 1.* If the equality (11) holds, then by Lemma 1, we have $h(t) \equiv h(0)$ =constant, in particular, we have $h(1) = h(0)$, and hence $\Delta(a, p) = 0$. Otherwise, $h'(t)t > 0\ \forall t \neq 0$ and $h''(t) > 0\ \forall t \in \mathbb{R}$. This implies that there exists a $\tau \in (0, 1)$ such that $\Delta(a, p) = h(1) - h(0) = h'(\tau) < h'(1)$. But it follows from $h'(t) = h(t)H'(t)$ that $h'(1) = h(1)H'(1)$. Since the equality can be achieved, the right hand side of (4) is sharp. Now the conclusion of Theorem 1 follows from the expressions of $h(1)$, $H'(1)$ and the assumption (3). □

The proof of Theorem 2 goes in parallel to that of Theorem 1 and so we omit the details of it with the following Lemmma 2, from which it follows immediately. Let the conditions of Theorem 2 hold. Consider a $C^\infty$ function $g : \mathbb{R} \to (0,\ +\infty)$ defined by

$$g(t) = \prod_{k=1}^{n}\left[\int_a^b \left(\prod_{j=1}^{n} f_j(x)\right)^{1-t} f_k^{tp_k}(x)dx\right]^{\frac{1}{p_k}}. \tag{13}$$

Then (2) is equivalent to $g(0) \le g(1)$. Now we have the following:

**Lemma 2.** *Let the function $g$ be defined as in (13). Then $g \in C^\infty$ is a concave function, i.e., $g''(t) \ge 0\ \forall t \in \mathbb{R}$ and it takes its minimum at $t = 0$. Moreover, $g''(t_0) = 0$ for some $t_0 \in \mathbb{R}$ if and only if*

$$f_k(x) \equiv constant, \quad \forall\, x \in [a, b],\ 1 \le k \le n.$$

*In this case, $g(t) \equiv g(0) = constant$ and $g'(t) \equiv g''(t) \equiv 0$. Otherwise, $g$ is a strictly concave function, i.e., $g''(t) > 0\ \forall t \in \mathbb{R}$ and $g'(t)t > 0$ for all $t \neq 0$.*

*Proof.* Let

$$F(x) = \prod_{k=1}^{n} f_k(x), \quad F_k(x) = \frac{f_k^{p_k}(x)}{F(x)}, \quad x \in [a,\ b], \quad k = 1, 2, \cdots, n. \tag{14}$$

Then $g(t)$ can be written as

$$g(t)=\prod_{k=1}^{n}\left[\int_a^b F(x)F_k^t(x)dx\right]^{\frac{1}{p_k}}.$$

Define a function $G:\mathbb{R}\to\mathbb{R}$ by $G(t)=\ln g(t)$. Then $G\in C^\infty$ and

$$g'(t)=G'(t)g(t),\;\; g''(t)=g(t)\left[G''(t)+\big(G'(t)\big)^2\right].$$

As in the proof of Lemma 1, we can show that $g'(0)=0$ and

$$G'(t)=\sum_{k=1}^{n}\frac{1}{p_k}\frac{\displaystyle\int_a^b F(x)F_k^t(x)\ln F_k(x)dx}{\displaystyle\int_a^b F(x)F_k^t(x)dx},$$

$$G''(t)=\sum_{k=1}^{n}\frac{1}{p_k}\frac{\displaystyle\int_a^b\int_a^b F(x)F(y)F_k^t(x)F_k^t(y)\left(\ln F_k(x)-\ln F_k(y)\right)^2dxdy}{2\left(\displaystyle\int_a^b F(x)F_k^t(x)dx\right)^2}\geq 0.$$

Moreover, it follows from the expression of $G''(t)$ that the equality $G''(t_0)=0$ holds for some $t_0\in\mathbb{R}$ if and only if $F_k(x)=F_k(y)$ $\forall x,y\in[a,\ b]$, that is, $f_k^{p_k}(x)/F(x)=f_k^{p_k}(y)/F(y)$ $\forall x,y\in[a,\ b],\ k=1,2,\cdots,n$, which is equivalent to $F_k(x)=$constant $\forall x\in[a,\ b], k=1,2,\cdots,n$. In this case, we get from the relation of $g''$ and $G''$ that $g''(t)=g'(t)=g'(0)\equiv 0$ and $g(t)=g(0)=$constant. Otherwise, $g''(t)>0$ for all $t\in\mathbb{R}$ and it follows from $g'(0)=0$ that $g'(t)t>0\;\forall\, t\neq 0$. □

**Example 1.** Let $p=q=2$ and replace $m$ by $n$ in Corollary 3. Then we obtain

$$\begin{aligned}0 &\leq\left[\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}b_k^2\right)\right]^{\frac{1}{2}}-\sum_{k=1}^{n}a_kb_k\\ &\leq\frac{n}{2}D_1D_2\left[\left(\frac{D_1}{C_1}\right)^2\ln\left(\frac{D_1^2}{C_1C_2}\right)+\left(\frac{D_2}{C_2}\right)^2\ln\left(\frac{D_1^2}{C_1C_2}\right)\right].\end{aligned}\tag{15}$$

In particular, if $D_1=D_2=D$ and $C_1=C_2=C$, we have

$$0\leq\left[\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}b_k^2\right)\right]^{\frac{1}{2}}-\sum_{k=1}^{n}a_kb_k\leq\frac{2nD^4}{C^2}\ln\left(\frac{D}{C}\right).\tag{16}$$

Moreover, the right hand sides of (15) and (16) are sharp.

**Example 2.** Let $p = q = 2$ in Corollary 4. Then we have

$$\begin{aligned} 0 &\leq \left[\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)\right]^{\frac{1}{2}} - \int_a^b f(x)g(x)dx \\ &\leq \frac{(b-a)}{2}\cdot\Delta_1\Delta_2\left[\left(\frac{\Delta_1}{\delta_1}\right)^2 \ln\left(\frac{\Delta_1^2}{\delta_1\delta_2}\right) + \left(\frac{\Delta_2}{\delta_2}\right)^2 \ln\left(\frac{\Delta_2^2}{\delta_1\delta_2}\right)\right]. \end{aligned} \tag{17}$$

In particular, if $\Delta_1 = \Delta_2 = \Delta$ and $\delta_1 = \delta_2 = \delta$, we have

(18)

$$0 \leq \left[\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)\right]^{\frac{1}{2}} - \int_a^b f(x)g(x)dx \leq 2(b-a)\frac{\Delta^4}{\delta^2}\ln\left(\frac{\Delta}{\delta}\right).$$

Moreover, the right hand sides of (17) and (18) are sharp.

**Example 3.** Consider the weighted difference

$$D_p(a,b,w) = \left(\sum_{k=1}^n w_k a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n w_k b_k^q\right)^{\frac{1}{q}} - \sum_{k=1}^n w_k a_k b_k$$

where $w = (w_1, w_2, \cdots, w_n)$ is a weight and $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and $a_k, b_k, w_k, k = 1, 2, \cdots, n$ are positive numbers such that

$$0 < m_1 \leq a_k \leq M_1, \quad 0 < m_2 \leq b_k \leq M_2, \quad 0 < \gamma \leq w_k \leq \Gamma, \quad k = 1, 2, \cdots, n$$

for some positive constants $m_1, m_2, M_1, M_2, \gamma$ and $\Gamma$. Setting $\overline{a}_k = w_k^{\frac{1}{p}} a_k$ and $\overline{b}_k = w_k^{\frac{1}{q}} b_k$ for $k = 1, 2, \cdots, n$, then it follows from Corollary 3 that $0 \leq D_p(a,b,w)$ and

(19)

$$D_p(a,b,w) \leq \frac{n\Gamma^2 M_1 M_2}{\gamma}\left[\frac{1}{p}\left(\frac{M_1}{m_1}\right)^p \ln\left(\frac{\Gamma M_1^p}{\gamma m_1 m_2}\right) + \frac{1}{q}\left(\frac{M_2}{m_2}\right)^q \ln\left(\frac{\Gamma M_2^q}{\gamma m_1 m_2}\right)\right].$$

Moreover, the right hand side of (19) is sharp.

**Remark.** From Theorem 1 and Theorem 2 as well as example 3, we see that the results of this paper are natural generalizations and refinements of the results of [2-4].

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