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DISTRIBUTION OF ROOTS OF CUBIC EQUATIONS

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Abstract. In this note the distribution of roots of cubic equations in contrast to 0 is given, which is useful to discuss eigenvalues for qualitative properties of differential equations.

Cubic equations are frequently encountered in many important problems in physics, applied mathematics, and engineering. More concretely, one needs to deal with cubic equations for eigenvalues in order to exhibit qualitative properties and bifurcation phenomena for higher dimensional differential equations of dynamic models. Solving cubic equations is a basic problem but has attracted research interests of many mathematicians. The well-known Cardano method for obtaining roots of cubic equations was first published early in 1545 by Italian mathematician Girolamo Cardano in his algebra book *Ars Magna*. Although this method remains the simplest and most useful way to solve cubics, the algebraic form in which the solutions normally appear is so awkward and clumsy in most use. Later, methods of graphical determination were given in [1, 3]. In 1958 Pennisi [6] gave a method for finding real roots of cubic equations by using the slide rule. In 1984 McKelvey [5] showed how these solutions can be expressed in simpler transcendental form so as to be more suitable for subsequent algebraic manipulation. In 1991 Lebedev [4], representing in terms of the cubic Chebyshev polynomial of the first kind, derived a universal formula for roots of the equation $ax^3 + bx^2 + cx + d = 0$ with either complex or real coefficients and obtained explicit expressions for parameters of this formula in the case of complex coefficients. In 1994 Fedorov [2] investigated the equation $z^3 - az + b = 0$ with real $a > 0$ and b and obtained approximate algebraic expressions for the three roots as functions of a and b for the irreducible case ($b^2/a^3 \le \frac{4}{27} = c_0^2$).

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These expressions are much simpler than the trigonometric modification of the exact Cardano solution.

Although there are some difficulties in giving simple expressions of cubics, in many applications (for example, finding eigenvalues to show qualitative properties of differential equations) we need not to calculate the exact values of those eigenvalues but only need to see their distribution in contrast to 0. The purpose of this note is to show the distribution.

Consider the general cubic equation

$$x^3 + ax^2 + bx + c = 0 \tag{1}$$

with real coefficients a, b, c. It is known that equation (1) has three roots

$$\begin{aligned} x_1 &= \sqrt[3]{-\tfrac{Q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\tfrac{Q}{2} - \sqrt{\Delta}} - \tfrac{a}{3}, \\ x_2 &= \sqrt[3]{-\tfrac{Q}{2} + \sqrt{\Delta}}\,\omega^2 + \sqrt[3]{-\tfrac{Q}{2} - \sqrt{\Delta}}\,\omega - \tfrac{a}{3}, \\ x_3 &= \sqrt[3]{-\tfrac{Q}{2} + \sqrt{\Delta}}\,\omega + \sqrt[3]{-\tfrac{Q}{2} - \sqrt{\Delta}}\,\omega^2 - \tfrac{a}{3} \end{aligned} \tag{2}$$

in complex field $\mathbb{C}$, where $\omega := (-1 + \sqrt{3}i)/2$, $Q := 2a^3/27 + c - ab/3$ and $\Delta = (4a^3c - a^2b^2 + 4b^3 - 18abc + 27c^2)/108$. One can check easily that equation (1) has exactly one real root x_1 and a pair of complex conjugate roots x_2 and x_3 when $\Delta > 0$, three real roots x_1, x_2, x_3 when $\Delta \le 0$. In particular, $x_2 = x_3$ when $\Delta = 0$ but x_1, x_2, x_3 are distinct when $\Delta < 0$. Let $\Re x_j$ and $\Im x_j$ denote the real part and the imaginary part of x_j.

Theorem. *The distribution of roots of the cubic equation* (1) *is shown in the following table:*

Case	Possibilities of a, b, c	Distribution of roots
I1	$a^2 < 4b, a <(>) c = 0$	$x_1 = 0, \Re x_{2,3} >(<) 0, \Im x_{2,3} \neq 0$
I2	$b > 0,\ ab = c >$(resp. $=, <$)0	$x_1 <$(resp. $=, >$)$0, \Re x_{2,3} = 0, \Im x_{2,3} \neq 0$
I3	$\Delta > 0,\ ab >(<) c >(<) 0$	$x_1 <(>) 0, \Re x_{2,3} <(>) 0, \Im x_{2,3} \neq 0$
I4	$\Delta > 0,\ c > \max(< \min)\{ab, 0\}$	$x_1 <(>) 0, \Re x_{2,3} >(<) 0, \Im x_{2,3} \neq 0$
II1	$a^2 = 4b,\ ab <(>) c = 0$	$x_1 = 0, x_2 = x_3 >(<) 0$
II2	$a^2 = -b,\ ab = c >(<) 0$	$x_1 <(>) 0, x_2 = x_3 >(<) 0$
II3	$a >$(resp. $=, <$)$b = c = 0$	$x_1 <$(resp. $=, >$)$x_2 = x_3 = 0$
II4	$\Delta = 0,\ ab <(>) c <(>) 0$	$x_1 >(<) 0, x_2 = x_3 >(<) 0$
II5	$\Delta = 0,\ c > \max(< \min)\{ab, 0\}$	$x_1 <(>) 0, x_2 = x_3 >(<) 0$

III1	$a^2 > 4b,\ a <(>)0, b > 0, c = 0$	$x_1 >(=)0, x_2 >(<)0, x_3 =(<)0$
III2	$b < 0, c = 0$	$x_1 > 0, x_2 = 0, x_3 < 0$
III3	$\Delta < 0,\ a \geq(\leq)0, c <(>)0$	$x_1 > 0, x_2 <(>)0, x_3 < 0$
III4	$\Delta < 0,\ a <(>)0, b >(<)0, c <(>)0$	$x_1 > 0, x_2 > 0, x_3 >(<)0$
III5	$\Delta < 0,\ a <(>)0, b <(>)0, c <(>)0$	$x_1 >(<)0, x_2 < 0, x_3 < 0$

Proof. We separately discuss in the three cases: $\Delta > 0$, $\Delta = 0$ and $\Delta < 0$.

In the case $\Delta > 0$, by the relations between roots and coefficients we get from equation (1) that

$$a = -(x_1 + 2\Re x_2),\ b = 2x_1\Re x_2 + (\Re x_2^2 + \Im x_2^2),\ c = -x_1(\Re x_2^2 + \Im x_2^2). \tag{3}$$

Obviously, $\Re x_2^2 + \Im x_2^2 > 0$ since $\Im x_2$ does not vanish. By the third equality, the sign of x_1 is opposite to the sign of c. When $c = 0$, we have $x_1 = 0$. Then, $\Re x_2 = -a/2$, implying the corresponding results indicated in the Table. When $c \neq 0$, we have

$$c - ab = 2\Re x_2((x_1 + \Re x_2)^2 + \Im x_2^2),$$

implying that $\Re x_2$ and $c - ab$ have the same sign since $(x_1 + \Re x_2)^2 + \Im x_2^2 > 0$. If $c - ab = 0$, i.e., $\Re x_2 = 0$, then $x_1 = -a = -c/\Im x_2^2$. Note that $\Delta = b^2(4b - a^2)/108$ as $c = 0$. Then, in this case the fact $\Delta > 0$ is equivalent to $a^2 < 4b$. Furthermore, $\Delta = b(a^2 + b)^2/27$ when $ab = c$. Then, in this case the fact $\Delta > 0$ is equivalent to $b > 0$. These give the distribution of roots of (1) for $\Delta > 0$.

In the case $\Delta = 0$, we similarly have

$$a = -(x_1 + 2x_2),\ b = 2x_1x_2 + x_2^2,\ c = -x_1x_2^2,\ c - ab = 2x_2(x_1 + x_2)^2$$

since $x_2 = x_3$. Then, the sign of x_1 is opposite to that of c but the sign of x_2 is the same as $c - ab$. If $c = 0$ and $c - ab \neq 0$ then $x_1 = 0$; if $c \neq 0$ and $c - ab = 0$ then $x_1 = -x_2 \neq 0$; if c and $c - ab$ both vanish then $x_1 = -a$ and $x_2 = 0$. Thus, the Theorem is obtained for $\Delta = 0$.

In the case $\Delta < 0$, we have $x_1 > x_2 > x_3$. We discuss in the three subcases: $c = 0$, $c < 0$ and $c > 0$.

When $c = 0$, equation (1) is of the form $(x^2 + ax + b)x = 0$, which has only two distinct nonzero real roots $(-a \pm \sqrt{a^2 - 4b})/2$. Hence we can get the distribution of roots by discussing the signs of a and b. Obviously, $b \neq 0$; otherwise, the multiplicity of 0 is at least 2, which makes a contradiction.

When $c < 0$, there are only two circumstances: either $x_1 > 0, x_2 > 0$ and $x_3 > 0$ or $x_1 > 0$, $x_2 < 0$ and $x_3 < 0$. In fact, let $\phi(x)$ denote the left-hand side of (1). Clearly, $\phi(x) \to +\infty$ when $x \to +\infty$ because the leading coefficient of ϕ is positive.

Since $\phi(0) = c < 0$, the continuity implies that $\phi(x)$ has at least one positive root. We claim that the other two real roots have the same sign. Clearly, neither of them is equal to 0 since $c < 0$. If their signs are different, then the negative one is an extreme point of $\phi(x)$ because $\phi(0) = c < 0$ and $\phi(x) \to -\infty$ when $x \to -\infty$, showing that it is a root with multiplicity 2, a contradiction to the fact that (1) has three distinct real roots in this case. That is why just the above two circumstances appear. Furthermore, the derivative $\phi'(x) = 3x^2 + 2ax + b$, which has two different roots $x_\pm = (-a \pm \sqrt{a^2 - 3b})/3$. We can easily determine which circumstance appears by the sign of x_-. Moreover, $x_- > 0$ if $a < 0, b > 0$; $x_- < 0$ if either $a < 0, b < 0$ or $a \geq 0$.

When $c > 0$, we similarly see that either $x_1 < 0, x_2 < 0, x_3 < 0$ if $a > 0, b > 0$ or $x_1 > 0, x_2 > 0, x_3 < 0$ if either $a > 0, b < 0$ or $a \leq 0$. Note that the fact $\Delta < 0$ is equivalent to that $a^2 > 4b$ as $c = 0$. Thus the distribution of roots for this case is obtained. □

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