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ON NEAR-RINGS WITH STRONG REGULARITY

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Abstract. Throught this paper, we will investigate some properties of left regular and strongly reduced near-rings.

Mason introduced the notion of left regularity and he characterized left regular zero-symmetric unital near-rings. Also, this concept have been studied by several authors.

The purpose of this paper is to find some characterizations of the strong reducibility in near-rings, and the strong regularity in near-rings which are closely related with strongly reduced near-rings.

1. Introduction

In this paper, our near-ring R is fixed as a right version, that is, a near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $(a+b)c = ac + bc$ for all a, b, c in R.. If R has a unity 1, then R is called *unital*. A near-ring R with the extra axiom $a0 = 0$ for all $a \in R$ is said to be *zero symmetric.* An element d in R is called *distributive* if $d(a+b) = da + db$ for all a and b in R.

Mason [3] introduced the notion of left regularity and characterized left regular zero-symmetric unital near-rings. Also, several authors ([1], [3], [4], [6] etc.) studied them.

We will use the following notations: Given a near-ring R, $R_0 = \{a \in R \mid a0 = 0\}$ which is called the *zero symmetric part* of R, $R_c = \{a \in R \mid a0 = a\}$ which is called the *constant part* of R.

For other notations and basic results, we shall refer to Pilz [5].

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2. Results

A near-ring R is said to be *left regular* if, for each $a \in R$, there exists $x \in R$ such that $a = xa^2$. Right regularity is defined in a symmetric way. Also, we can generalize these concepts as following.

A near-ring R is said to be *left κ-regular* if, for each $a \in R$, there exists a positive integer n and an element $x \in R$ such that $a^n = xa^{n+1}$. Similarly, we can define right κ-regular.

A near-ring R is called *strongly left regular* if R is left regular and regular, similarly, we can define strongly right regular. A strongly left regular and strongly right regular near-ring is called *strongly regular near-ring.*

Also, the concepts of left, strongly left, strongly right and strong regularities are all equivalent conditions [2].

An idempotent element $e^2 = e$ in R is called *left semi-central* if $ea = eae$ for each $a \in R$. Similarly, right semi-centrality is defined in a symmetric way. A near-ring in which every idempotent is left semi-central is called *left semi-central.*

We say that R is *reduced* if R has no nonzero nilpotent elements, that is, for each a in R, $a^n = 0$, for some positive integer n implies $a = 0$. In ring theory, McCoy proved that R is reduced if and only if for each a in R, $a^2 = 0$ implies $a = 0$. A near-ring R is said to be *strongly reduced* if, for $a \in R$, $a^2 \in R_c$ implies $a \in R_c$, that is $a^2 0 = a^2$ implies $a0 = a$. Obviously R is strongly reduced if and only if, for $a \in R$ and any positive integer n, $a^n \in R_c$ implies $a \in R_c$.

Obviously, we get the following examples by the concept of strong reducibility.

Example 1. (1) Every strongly regular near-ring is strongly reduced.

(2) Every right regular near-ring is strongly reduced.

(3) Every commutative integral near-ring is strongly reduced.

Lemma 2. *Let R be a strongly reduced near-ring. Then we have the following conditions.*

(1) *If for any $a, b \in R$ with $ab \in R_c$, then $ba \in R_c$, and $\forall x \in R$, axb, $bxa \in R_c$. Furthermore, $ab^n \in R_c$ implies $ab \in R_c$, for each positive integer n.*

(2) *If for any $a, b \in R$ with $ab = 0$, then $ba = b0 = (ba)^2$. Moreover, $ab^n = 0$ implies $ab = 0$, for any positive integer n.*

Proof. (1) Suppose that $ab \in R_c$. Then $(ba)^2 = baba = bab = bab0 \in R_c$. Since R is strongly reduced, we have $ba \in R_c$.

Next, we see that $xba \in R_c$ for each $x \in R$, whence $(axb)^2 \in R_c$. By the strong reducibility of R, we obtain $axb \in R_c$ for each $x \in R$. Also, since $ba \in R_c$, we obtain $bxa \in R_c$ for each $x \in R$.

Furthermore, assume that $ab^n \in R_c$. Then using the first part of this (1), $(ab)^n \in R_c$. Since R is strongly reduced, we see $ab \in R_c$.

(2) Assume that $ab = 0$. Then $ab \in R_c$ by (1). Hence $(ba)^2 = baba = b0 \in R_c$. Hence $ba \in R_c$. Therefore we obtain that $ba = (ba)^2 = b0$. Moreover, suppose that $ab^n = 0$. Then $ab \in R_c$ by the last part of (1), so that $ab = abb^{n-1} = ab^n = 0$ □

Lemma 3. *Let R be a strongly reduced near-ring. If for any $a, b \in R$ with $ab = 0$ and $a^2 = a0$, then $a = 0$.*

Proof. Suppose that for any $a, b \in R$ with $ab = 0$ and $a^2 = a0$. Then $a^2 = a0 \in R_c$. Strong reducibility implies that $a \in R_c$. Hence we obtain that $a = a0 = a0b = ab = 0$. □

From this Lemma 3, we have the following important statement.

Corollary 4. *Every strongly reduced near-ring is reduced.*

By Reddy and Murty [6], we say that a near-ring R has the property (*) if it satisfies the conditions:

(i) for any $a, b \in R$, $ab = 0$ implies $ba = b0$.

(ii) for $a \in R$, $a^3 = a^2$ implies $a^2 = a$.

Here, clearly we see that strong reducibility is equivalent to the condition (ii) and strong reducibility implies condition (i) by Lemma 2 (2).

According to the Lemmas 2 and 3, we have the following valuable corollaries.

Corollary 5. *Let R be a left (or right) regular near-ring. If for any $a,\ b \in R$ with $ab = 0$, then $(ba)^n = b0$, for all positive integer n. In particular, $ba = b0$.*

Corollary 6. *Let R be a left (or right) regular near-ring. If for any $a, b \in R$ with $ab = 0$ and $a^2 = a0$, then $a = 0$.*

Now, we state another basic properties of strongly reduced near-rings.

Clearly, if R is a zero-symmetric near-ring, then R is strongly reduced if and only if R is reduced. The following example shows that a reduced near-ring which is not necessarily strongly reduced.

Example 7. Let $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$ with addition modulo 6 and define multiplication as follows:

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	3	1	3	1	1
2	0	0	2	0	2	2
3	3	3	3	3	3	3
4	0	0	4	0	4	4
5	3	3	5	3	5	5

Obviously this is a reduced near-ring. The constant part of $\mathbb{Z}_6$ is $\{0,3\}$. Since $1^2 = 3$ is a constant element but 1 is not, this near-ring is not strongly reduced.

We obtain equivalent conditions for a near-ring R to be strongly reduced.

Proposition 8. *The following statements are equivalent for a near-ring R:*

(1) *R is strongly reduced.*

(2) *For $a \in R$, $a^3 = a^2$ implies $a^2 = a$.*

(3) *If $a^{n+1} = xa^{n+1}$ for $a, x \in R$ and some nonnegative integer n, then $a = xa = ax$.*

Proof. (1) $\Longrightarrow$ (2). Assume that $a^3 = a^2$. Then $(a^2 - a)a = 0$, whence $a(a^2 - a) = a0 \in R_c$ by Lemma 2 (2). Then $(a^2 - a)a^2 = (a^3 - a^2)a = 0a = 0$. Again by Lemma 2 (2), $a^2(a^2 - a) = a^2 0 \in R_c$. Hence $(a^2 - a)^2 = a^2(a^2 - a) - a(a^2 - a) = a^2 0 - a0 = (a^2 - a)0 \in R_c$. This implies $a^2 - a \in R_c$. Hence $a^2 - a = (a^2 - a)0 = (a^2 - a)a = 0$.

(2) $\Longrightarrow$ (1). Assume $a^2 \in R_c$. Then $a^3 = a^2 a = a^2$. By hypothesis, this implies $a = a^2 \in R_c$.

(1) $\Longrightarrow$ (3). Suppose $a^{n+1} = xa^{n+1}$ for some $n \geq 0$. Then $(a - xa)a^n = 0$. Hence $(a - xa)a = 0$ by Lemma 2 (2), and so $(a - xa)^2 \in R_c$ by Lemma 2 (1). Since R is strongly reduced, we have $a - xa \in R_c$. Then $a - xa = (a - xa)a = 0$, that is $a = xa$. Now $(a - ax)a = a^2 - axa = a^2 - a^2 = 0 \in R_c$. Hence $(a - ax)^2 = a(a - ax) - ax(a - ax) \in R_c$ by Lemma 2 (1), and so $a - ax \in R_c$. Therefore $a - ax = (a - ax)a = 0$.

(3) $\Longrightarrow$ (2). This is obvious. □

The following is a generalization of [6, Theorem 3].

Proposition 9. *Let R be a strongly reduced near-ring and let $a, x \in R$. If $a^n = xa^{n+1}$ for some positive integer n, then $a = xa^2 = axa$ and $ax = xa$.*

Proof. Assume that $a^n = xa^{n+1}$ for some $n \geq 1$. By Proposition 8 (3), $a = xa^2 = axa$. Then $(ax - xa)a = 0$. Hence, by Lemma 2 (2), $(ax - xa)^2 = ax(ax -$

$xa) - xa(ax - xa) \in R_c$. Since R is strongly reduced, $ax - xa \in R_c$. Hence $ax - xa = (ax - xa)a = 0$. □

Here we give some characterizations of strongly regular near-rings.

Lemma 10 ([2]). *Let R be an arbitrary near-ring. The following statements are equivalent:*

(1) *R is left regular.*
(2) *R is strongly left regular.*
(3) *R is strongly regular.*
(4) *R is strongly right regular.*
(5) *R is left semi-central regular.*

Using Examples 1, Propositions 8, 9 and Lemma 10, we obtain the following conditions.

Proposition 11. *Let R be a near-ring. Then the following statements are equivalent:*

(1) *R is strongly regular.*
(2) *R is strongly reduced and left κ-regular.*
(3) *For each $a \in R$, there exists $x \in R$ such that $a = xa^2xa$.*

Proof. (1) $\Longleftrightarrow$ (2). This follows from Examples 1, a strongly regular near-ring is strongly reduced, and Proposition 9.

(1) $\Longrightarrow$ (3) follows from Proposition 9.

(3) $\Longrightarrow$ (1). By hypothesis, R is strongly reduced. If $a = xa^2xa$, then $xa = xxa^2(xa)$. By Proposition 8, $xa = xaxxa^2$. Thus $a = xa^2xax^2a^2$. This implies that R is strongly regular by Lemma 10. □

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