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APPROXIMATELY QUADRATIC DERIVATIONS AND GENERALIZED HOMOMORPHISMS

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Abstract. Let $\mathcal{A}$ be a unital Banach algebra. If $f : \mathcal{A} \to \mathcal{A}$ is an approximately quadratic derivation in the sense of Hyers-Ulam-J.M. Rassias, then $f : \mathcal{A} \to \mathcal{A}$ is an exactly quadratic derivation. On the other hands, let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Any approximately generalized homomorphism $f : \mathcal{A} \to \mathcal{B}$ corresponding to Cauchy, Jensen functional equation can be estimated by a generalized homomorphism.

1. Introduction

In 1940, S. M. Ulam [26] proposed the following question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In next year, D.H. Hyers [10] answers the problem of Ulam under the assumption that the groups are Banach spaces: *if $\varepsilon > 0$ and $f : \mathcal{X} \to \mathcal{Y}$ is a mapping with $\mathcal{X}$ a normed space, $\mathcal{Y}$ a Banach space such that*

$$||f(x + y) - f(x) - f(y)|| \le \varepsilon$$

for all $x, y \in \mathcal{X}$, then there exists a unique additive mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$||f(x) - T(x)|| \le \varepsilon$$

for all $x \in \mathcal{X}$.

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A generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [21] by introducing the unbounded Cauchy difference. Since then, the stability problems of several functional equation have been extensively investigated by a number of authors (for instance, [1, 3, 6, 23]).

On the other hand, J.M. Rassias [19] generalized the Hyers' stability result by presenting a weaker condition controlled by (or involving) a product of different powers of norms (from the right-hand side of assumed conditions). That is, *assume that there exist constants $\varepsilon \geq 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f : X \to Y$ is a mapping with X a normed space, Y a Banach space such that the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \|x\|^{p_1} \|y\|^{p_2}$$

for all $x, y \in X$, then there exist a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{\varepsilon}{2 - 2^p} \|x\|^p$$

for all $x \in X$. If, in addition, $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed x in X, then T is linear.

A counter-example for a singular case of this result was given by P. Găvruţă [7].

Particularly, one of the important functional equations studied is the following functional equation:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

The quadratic function $f(x) = ax^2$ is a solution of this functional equation, and so one usually is said the above functional equation to be quadratic [1, 13, 20].

The Hyers-Ulam stability problem of the quadratic functional equation was first proved by F. Skof [25] for functions between a normed space and a Banach space. Afterwards, her result was extended by P.W. Cholewa [4] and S. Czerwik [5]:

If $p \neq 2$ and $f : \mathcal{X} \to \mathcal{Y}$ is a mapping with $\mathcal{X}$ a normed space, $\mathcal{Y}$ a Banach space such that

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in \mathcal{A}$, then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq c + k\varepsilon \|x\|^p$$

for all $x \in \mathcal{X}$ if $p \geq 0$ and for all $x \in \mathcal{X} \setminus \{0\}$ if $p < 0$, where: when $p < 2$, $c = \frac{\|f(0\|}{3}$, $k = \frac{2}{4 - 2^p}$ and when $p > 2$, $c = 0$, $k = \frac{2}{2^p - 4}$.

Let $\mathcal{A}$ be an algebra over the real or complex field $\mathbb{F}$. An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a *ring derivation* if the functional equation $d(xy) = xd(y)+d(x)y$ holds for all $x, y \in \mathcal{A}$.

T. Miura *et al.* [18] investigated the stability of ring derivations on Banach algebras:

Suppose that $\mathcal{A}$ is a Banach algebra, $p \geq 0$ and $\varepsilon \geq 0$. If $p \neq 1$ and $f : \mathcal{A} \to \mathcal{A}$ is a mapping such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in \mathcal{A}$ and

$$\|f(xy) - xf(y) - f(x)y\| \leq \varepsilon\|x\|^p\|y\|^p$$

for all $x, y \in \mathcal{A}$, then there exists a unique ring derivation $d : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - d(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|x\|^p$$

for all $x \in \mathcal{A}$. In particular, if $\mathcal{A}$ is a Banach algebra without order, then f is an ring derivation.

Several results for the stability of derivations have been obtained by many authors (for instances, [2, 16, 17, 24]).

We here introduce the following mapping:

A quadratic mapping $D : \mathcal{A} \to \mathcal{A}$ is said to be a *quadratic derivation* if the functional equation $D(xy) = x^2D(y) + D(x)y^2$ holds for all $x, y \in \mathcal{A}$. As a simple example, let us consider the algebra of 2×2 matrices

$$\mathcal{A} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : \ a, b \in \mathbb{C} \right\},$$

where $\mathbb{C}$ is a complex field. Then it is easy to see that the mapping $D : \mathcal{A} \to \mathcal{A}$ defined by

$$D\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a^2 \\ 0 & 0 \end{bmatrix}$$

is a quadratic derivation. Here it is natural to ask that there exists an approximately quadratic derivation which is not an exactly quadratic derivation. The following example is a slight modification of an example due to [18].

Example. Let X be a compact Hausdorff space and let $C(X)$ be the commutative Banach algebra of real-valued continuous functions on X under pointwise operations and the supremum norm $\|\cdot\|_\infty$. We define $f : C(X) \to C(X)$ by

$$f(a)(x) = \begin{cases} a(x)^2 \log|\,a(x)| & \text{if } a(x) \neq 0, \\ 0 & \text{if } a(x) = 0 \end{cases}$$

for all $a \in C(X)$ and $x \in X$. It is easy to see that

$$f(ab) = a^2 f(b) + f(a)b^2$$

for all $a, b \in C(X)$.

Note that the following inequality holds for all $u, v \in \mathbb{R} \setminus \{0\}$ with $u + v \neq 0$, where $\mathbb{R}$ is a real field,

$$\big|(u+v)^2 \log|u+v| + (u-v)^2 \log|u-v| - 2u^2 \log|u| - 2v^2 \log|v|\big| \leq 4|u|\,|v|$$

In fact, fix $u, v \in \mathbb{R} \setminus \{0\}$, $u + v \neq 0$ arbitrarily. Since $\log(1 + x) \leq x$ for all $x \geq 0$,

$$\begin{aligned}
&\big|(u+v)^2 \log|u+v| + (u-v)^2 \log|u-v| - 2u^2 \log|u| - 2v^2 \log|v|\big| \\
&\leq \big|(u+v)^2 \log(|u|+|v|) + (u-v)^2 \log(|u|+|v|) - 2u^2 \log|u| - 2v^2 \log|v|\big| \\
&= \big|2(u^2+v^2)\log(|u|+|v|) - 2u^2 \log|u| - 2v^2 \log|v|\big| \\
&\leq 2|u|^2 \left|\log \frac{|u|+|v|}{|u|}\right| + 2|v|^2 \left|\log \frac{|u|+|v|}{|v|}\right| \\
&\leq 2|u|^2 \log\left(1 + \frac{|v|}{|u|}\right) + 2|v|^2 \log\left(1 + \frac{|u|}{|v|}\right) \\
&\leq 2|u|^2 \frac{|v|}{|u|} + 2|v|^2 \frac{|u|}{|v|} = 4|uv|
\end{aligned}$$

which gives

$$\|f(a+b) + f(a-b) - 2f(a) - 2f(b)\|_\infty \leq 4\|ab\|_\infty$$

for all $a, b \in C(X)$. Hence we may regard f as an approximately quadratic derivation on $C(X)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $h : \mathcal{A} \to \mathcal{B}$ be a linear mapping. Define the bilinear mapping H by $H(x, y) = h(xy) - h(x)h(y)$ for all $x, y \in \mathcal{A}$. We say that h is a *generalized homomorphism* if H is continuous in $x \in \mathcal{A}$ for each fixed $y \in \mathcal{A}$ and in $y \in \mathcal{A}$ for each fixed $x \in \mathcal{A}$, respectively. The mapping was introduced by B.E. Johnson [12].

By an *approximately generalized homomorphism* corresponding to a functional equation $\mathcal{E}(f) = 0$, we mean a mapping $f : \mathcal{A} \to \mathcal{B}$ such that

$$\|\mathcal{E}(f)\| \leq \varepsilon$$

and the mapping $F : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ defined by

$$F(x, y) = f(xy) - f(x)f(y) \tag{1}$$

for all $x, y \in \mathcal{A}$, is continuous in $x \in \mathcal{A}$ for each fixed $y \in \mathcal{A}$ and in $y \in \mathcal{A}$ for each fixed $x \in \mathcal{A}$, respectively.

In Section 2, we prove the stability in the sense of Hyers-Ulam-J.M. Rassias and the superstability of quadratic derivations on Banach algebras as in the case of ring derivations. In Section 3 and 4, the stability of generalized homomorphisms on Banach algebras via Cauchy, Jensen equations is established, respectively.

2. Stability of Quadratic Derivations

In this section, $\mathbb{Q}$ and $\mathbb{N}$ will denote the set of the rational and the natural numbers, respectively.

Lemma 2.1. *Suppose that $\mathcal{A}$ is a Banach algebra. Let $\delta, \varepsilon \geq 0$ and let $p, q \geq 0$ with either $p < 1$, $q < 2$ or $p > 1$, $q > 2$. If $f : \mathcal{A} \to \mathcal{A}$ is a mapping such that*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \delta\|x\|^p\|y\|^p \tag{2}$$

for all $x, y \in \mathcal{A}$ and

$$\|f(xy) - x^2f(y) - f(x)y^2\| \leq \varepsilon\|x\|^q\|y\|^q \tag{3}$$

for all $x, y \in \mathcal{A}$, then there exists a unique quadratic derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - D(x)\| \leq k\delta\|x\|^{2p} \tag{4}$$

for all $x \in \mathcal{A}$, where $k = \frac{1}{4-4^p}$ if $p < 1$ and $k = \frac{1}{4^p-4}$ if $p > 1$.

Proof. Assume that either $p < 1$, $q < 2$ or $p > 1$, $q > 2$. Set $\tau = 1$ if $p < 1$, $q < 2$ and $\tau = -1$ if $p > 1$, $q > 2$. In (2), put $x = y = 0$ to see that $f(0) = 0$. Hence, following Czerwik's process [5] using the direct method, we obtain from (2)

$$\|4^{-n}f(2^nx) - f(x)\| \leq \varepsilon\|x\|^{2p}\sum_{k=1}^{n} 2^{2(k-1)p}\,4^{-k}$$

for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$ if $p < 1$, and

$$\|f(x) - 4^nf(2^{-n}x)\| \leq \left(\frac{\varepsilon}{4}\right)\|x\|^{2p}\sum_{k=1}^{n} 2^{-2k(p-1)}$$

for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$ if $p > 1$. Using these inequalities and Czerwik's process, we see that there exists a unique quadratic mapping $D : \mathcal{A} \to \mathcal{A}$ defined by $D(x) = \lim_{n\to\infty} 4^{-\tau n} f(2^{\tau n}x)$ for all $x \in \mathcal{A}$ such that

$$\|f(x) - D(x)\| \le k\delta\|x\|^{2p}$$

for all $x \in \mathcal{A}$, where $k = \frac{1}{4-4^p}$ if $p < 1$ and $k = \frac{1}{4^p-4}$ if $p > 1$.

We claim that

$$D(xy) = x^2D(y) + D(x)y^2$$

for all $x, y \in \mathcal{A}$. Since D is quadratic, we see that $D(x) = 4^{-\tau n}D(2^{\tau n}x)$ for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. First, it follows from (4) that

$$\begin{aligned}\|4^{-\tau n} f(2^{\tau n}x) - D(x)\| &= 4^{-\tau n}\|f(2^{\tau n}x) - D(2^{\tau n}x)\| \\ &\le 4^{-\tau n}k\delta\|2^{\tau n}x\|^{2p} \\ &= 4^{\tau(p-1)n}k\delta\|x\|^{2p}\end{aligned}$$

for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. Since $\tau(p-1) < 0$, we have

$$\|4^{-\tau n} f(2^{\tau n}x) - D(x)\| \to 0 \quad \text{as } n \to \infty. \tag{5}$$

Following the similar argument as the above, we obtain

$$\|4^{-2\tau n} f(2^{2\tau n}xy) - D(xy)\| \le 4^{\tau(p-1)n}k\delta\|xy\|^{2p}$$

for all $x, y \in \mathcal{A}$ and all $n \in \mathbb{N}$, and so

$$\|4^{-2\tau n} f(2^{2\tau n}xy) - D(xy)\| \to 0 \quad \text{as } n \to \infty. \tag{6}$$

Since f satisfies (3), we get

$$\begin{aligned}&\|4^{-2\tau n} f(2^{2\tau n}xy) - 4^{-\tau n}x^2 f(2^{\tau n}y) - f(2^{\tau n}x)4^{-\tau n}y^2\| \\ &= 4^{-2\tau n}\|f((2^{\tau n}x)(2^{\tau n}y)) - (2^{\tau n}x)^2 f(2^{\tau n}y) - f(2^{\tau n}x)(2^{\tau n}y)^2\| \\ &\le 4^{-2\tau n}\varepsilon\|2^{\tau n}x\|^q\|2^{\tau n}y\|^q = 2^{\tau n(q-2)}\varepsilon\|x\|^q\|y\|^q\end{aligned}$$

for all $x, y \in \mathcal{A}$ and all $n \in \mathbb{N}$. Invoking $\tau(q-2) < 0$, we obtain

$$\|4^{-2\tau n} f(2^{2\tau n}xy) - 4^{-\tau n}x^2 f(2^{\tau n}y) - f(2^{\tau n}x)4^{-\tau n}y^2\| \to 0 \quad \text{as } n \to \infty. \tag{7}$$

Using (5), (6) and (7), we now see that

$$\begin{aligned}&\|D(xy) - x^2D(y) - D(x)y^2\| \\ &\le \|D(xy) - 4^{-2\tau n} f(2^{2\tau n}xy)\|\end{aligned}$$

$$
\begin{aligned}
&\quad + \|4^{-2\tau n} f(2^{2\tau n} xy) - 4^{-\tau n} x^2 f(2^{\tau n} y) - 4^{-\tau n} f(2^{\tau n} x) y^2\| \\
&\quad + \|4^{-\tau n} x^2 f(2^{\tau n} y) - x^2 D(y)\| + \|4^{-\tau n} f(2^{\tau n} x) y^2 - D(x) y^2\| \\
&\le \|D(xy) - 4^{-2\tau n} f(2^{2\tau n} xy)\| \\
&\quad + \|4^{-2\tau n} f(2^{2\tau n} xy) - 4^{-\tau n} x^2 f(2^{\tau n} y) - 4^{-\tau n} f(2^{\tau n} x) y^2\| \\
&\quad + \|x^2\| \|4^{-\tau n} f(2^{\tau n} y) - D(y)\| + \|f(2^{\tau n} x) 4^{-\tau n} - D(x)\| \|y^2\| \to 0 \quad \text{as } n \to \infty
\end{aligned}
$$

which implies that $D(xy) = x^2 D(y) + D(x) y^2$ for all $x \in \mathcal{A}$. Namely, D is a quadratic derivation, as claimed and the proof is complete. □

Lemma 2.2. *Suppose that $\mathcal{A}$ is a unital Banach algebra. Let $\delta, \varepsilon \ge 0$ and let $p, q \ge 0$ with either $p < 1$, $q < 2$ or $p > 1$, $q > 2$. If $f : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying* (2) *and* (3), *then we have we have*

$$f(rx) = r^2 f(x)$$

for all $x \in \mathcal{A}$ and all $r \in \mathbb{Q}$.

Proof. In the case when $r = 0$, it is trivial since $f(0) = 0$. Let e be a unit element of $\mathcal{A}$ and $r \in \mathbb{Q} \setminus \{0\}$ arbitrarily. Put $\tau = 1$ if $p < 1$, $q < 2$ and $\tau = -1$ if $p > 1$, $q > 2$. Hence it follows that $\tau(p-1) < 0$ and $\tau(q-2) < 0$. By Lemma 2.1, there exists a unique quadratic derivation $D : \mathcal{A} \to \mathcal{A}$ such that (4) is true. Recall that D is quadratic, and hence it is easy to see that $D(rx) = r^2 D(x)$ for all $x \in \mathcal{A}$. Then we get

$$
\begin{aligned}
&\|D((2^{\tau n} e)(rx)) - r^2 2^{2\tau n} e f(x) - f(2^{\tau n} e) r^2 x^2\| \\
&\le r^2 \|D(2^{\tau n} ex) - f(2^{\tau n} ex)\| + r^2 \|f(2^{\tau n} ex) - 4^{\tau n} e f(x) - f(2^{\tau n} e) x^2\|
\end{aligned}
\tag{8}
$$

for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. Now the inequalities (3), (4) and (8) yields that

$$
\begin{aligned}
&\|D((2^{\tau n} e)(rx)) - r^2 2^{2\tau n} e f(x) - f(2^{\tau n} e) r^2 x^2\| \\
&\le r^2 4^{\tau n p} k\delta \|x\|^{2p} + r^2 2^{\tau n q} \varepsilon \|x\|^q
\end{aligned}
\tag{9}
$$

for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$.

It follows from (4) and (9) that

$$
\begin{aligned}
&\|f((2^{\tau n} e)(rx)) - r^2 2^{2\tau n} e f(x) - f(2^{\tau n} e) r^2 x^2\| \\
&\le \|f((2^{\tau n} e)(rx)) - D((2^{\tau n} e)(rx))\| \\
&\quad + \|D((2^{\tau n} e)(rx)) - r^2 2^{2\tau n} e f(x) - f(2^{\tau n} e) r^2 x^2\| \\
&\le k\delta 4^{\tau n p} (r^{2p} + r^2) \|x\|^{2p} + r^2 2^{\tau n q} \varepsilon \|x\|^q
\end{aligned}
$$

for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. That is, we have

$$\begin{aligned} &\|f((2^{\tau n}e)(rx)) - r^2 2^{2\tau n} ef(x) - f(2^{\tau n}e)r^2x^2\| \\ &\leq k\delta 4^{\tau np}(r^{2p} + r^2)\|x\|^{2p} + r^2 2^{\tau nq}\varepsilon\|x\|^q \end{aligned} \tag{10}$$

for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. From (3) and (10), we obtain

$$\begin{aligned} &\|4^{\tau n}\{f(rx) - r^2 f(x)\}\| \\ &= \|4^{\tau n}e\{f(rx) - r^2 f(x)\}\| \\ &\leq \|2^{2\tau n}ef(rx) + f(2^{\tau n}e)r^2x^2 - f((2^{\tau n}e)(rx))\| \\ &\quad + \|f((2^{\tau n}e)(rx)) - r^2 2^{2\tau n}ef(x) - f(2^{\tau n}e)r^2x^2\| \\ &\leq \varepsilon\|2^{\tau n}e\|^q\|rx\|^q + k\delta 4^{\tau np}(r^{2p} + r^2)\|x\|^{2p} + r^2 2^{\tau nq}\varepsilon\|x\|^q \\ &= 2^{\tau nq}(r^q + r^2)\varepsilon\|x\|^q + k\delta 4^{\tau np}(r^{2p} + r^2)\|x\|^{2p} \end{aligned}$$

for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. This means that

$$\begin{aligned} &\|f(rx) - r^2 f(x)\| \\ &\leq 2^{\tau(q-2)n}(r^q + r^2)\varepsilon\|x\|^q + k\delta 4^{\tau(p-1)n}(r^{2p} + r^2)\|x\|^{2p} \end{aligned} \tag{11}$$

for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. If we take $n \to \infty$ in (11), then we arrive at

$$f(rx) = r^2 f(x)$$

for all $x \in \mathcal{A}$. This completes the proof since $r \in \mathbb{Q} \setminus \{0\}$ was arbitrary. □

Now we are ready to prove the main result in this section.

Theorem 2.3. *Suppose that $\mathcal{A}$ is a unital Banach algebra. Let $\delta, \varepsilon \geq 0$ and let $p, q \geq 0$ with either $p < 1$, $q < 2$ or $p > 1$, $q > 2$. If $f : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying* (2) *and* (3)*, then $f : \mathcal{A} \to \mathcal{A}$ is a quadratic derivation.*

Proof. Let D be a unique quadratic derivation as in Lemma 2.2. Put $\tau = 1$ if $p < 1$, $q < 2$ and $\tau = -1$ if $p > 1$, $q > 2$. Since $f(2^{\tau n}x) = 4^{\tau n}f(x)$ for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$ by Lemma 2.2, it follows from (4) that

$$\begin{aligned} \|f(x) - D(x)\| &= \|4^{-\tau n}f(2^{\tau n}x) - 4^{-\tau n}D(2^{\tau n}x)\| \\ &\leq 4^{-\tau n}k\delta\|2^{\tau n}x\|^{2p} \\ &= 4^{\tau(p-1)n}k\delta\|x\|^{2p} \end{aligned}$$

for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. Namely,

$$\|f(x) - D(x)\| \leq 4^{\tau(p-1)n}k\delta\|x\|^{2p} \tag{12}$$

for all $x \in \mathcal{A}$ and all $n \in \mathbb{N}$. Since $\tau(p-1) < 0$, if we let $n \to \infty$ in (12), then we conclude that $f(x) = D(x)$ for all $x \in \mathcal{A}$ which implies that f is a quadratic derivation. □

3. Stability of Generalized Homomorphisms via Cauchy Equation

We begin with our investigation establishing the stability of generalized homomorphisms via Cauchy equation. From now on, $\mathcal{A}$ and $\mathcal{B}$ denote Banach algebras.

Theorem 3.1. *Let $\varepsilon \geq 0$. For each approximately generalized homomorphism $f : \mathcal{A} \to \mathcal{B}$ corresponding to the Cauchy inequality*

$$\|f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)\| \leq \varepsilon, \tag{13}$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U} = \{\mu \in \mathbb{C} : |\mu| = 1\}$, there exists a unique generalized homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - h(x)\| \leq \varepsilon \tag{14}$$

for all $x \in \mathcal{A}$.

Proof. Let us the second variable of F be fixed. Then, by hypothesis, for each fixed $z \in \mathcal{A}$, the mapping $F : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ satisfies the inequality

$$\begin{aligned}
&\|F(\alpha x + \beta y, z) - \alpha F(x, z) - \beta F(y, z)\| \\
&\leq \|f(\alpha xz + \beta yz) - f(\alpha x + \beta y) f(z) \\
&\quad - \alpha f(xz) + \alpha f(x) f(z) - \beta f(yz) + \beta f(y) f(z)\| \\
&\leq \|f(\alpha xz + \beta yz) - \alpha f(xz) - \beta f(yz)\| \\
&\quad + \|f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)\| \|f(z)\| \\
&\leq (1 + \|f(z)\|)\varepsilon,
\end{aligned}$$

that is, we obtain the inequality

$$\|F(\alpha x + \beta y, z) - \alpha F(x, z) - \beta F(y, z)\| \leq (1 + \|f(z)\|)\varepsilon \tag{15}$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$.

Putting $\alpha = \beta = 1$ in (15) and utilizing the Hyers' direct method [10], there is an additive mapping in the first variable $S : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ such that

$$\|F(x, z) - S(x, z)\| \leq (1 + \|f(z)\|)\varepsilon \tag{16}$$

for all $x \in \mathcal{A}$, where

$$S(x,z) = \lim_{n\to\infty} \frac{F(2^n x, z)}{2^n} \tag{17}$$

for all $x \in \mathcal{A}$. Replacing x, y by $2^n x, 2^n y$ in (15), we get

$$\|2^{-n}F(2^n(\alpha x + \beta y), z) - \alpha 2^{-n}F(2^n x, z) - \beta 2^{-n}F(2^n y, z)\| \le 2^{-n}(1 + \|f(z)\|)\varepsilon$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$. Taking limits as $n \to \infty$, we obtain

$$S(\alpha x + \beta y, z) = \alpha S(x,z) + \beta S(y,z) \tag{18}$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$.

Clearly, $S(0x, z) = 0 = 0S(x,z)$ for all $x \in \mathcal{A}$. Now, let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$), and let $M \in \mathbb{N}$ greater than $|\lambda|$. By applying a geometric argument, we see that there exists $\alpha_1, \beta_2 \in \mathbb{U}$ such that $2\frac{\lambda}{M} = \alpha_1 + \beta_2$. By the additivity of $S(\cdot, z)$, we get $S(\frac{1}{2}x, z) = \frac{1}{2}S(x,z)$ for all $x \in \mathcal{A}$. Therefore

$$\begin{aligned} S(\lambda x, z) &= S\left(\frac{M}{2} \cdot 2 \cdot \frac{\lambda}{M} x, z\right) = MS\left(\frac{1}{2} \cdot 2 \cdot \frac{\lambda}{M} x, z\right) = \frac{M}{2} S((\alpha_1 + \beta_2)x, z) \\ &= \frac{M}{2}(\alpha_1 + \beta_2) S(x,z) = \frac{M}{2} \cdot 2 \cdot \frac{\lambda}{M} S(x,z) = \lambda S(x,z) \end{aligned} \tag{19}$$

for all $x \in \mathcal{A}$, so that the mapping $S : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear in the first variable.

From the Hyers' theorem [10], the inequality (13) with $\alpha = \beta = 1$ guarantees that there exists a *unique* additive mapping $h : \mathcal{A} \to \mathcal{B}$ defined by

$$h(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in \mathcal{A}$ satisfying the inequality (14). Applying a similar approach of (15)$\sim$(19) to (13), we see that h is $\mathbb{C}$-linear.

For each fixed $x \in \mathcal{A}$, we note that the mapping $F : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ satisfies the inequality

$$\begin{aligned} &\|2^{-n}F(2^n x, \alpha y + \beta z) - \alpha 2^{-n}F(2^n x, y) - \beta 2^{-n}F(2^n x, z)\| \\ &\le \|2^{-n}f(\alpha 2^n(xy) + \beta 2^n(xz)) - 2^{-n}f(2^n x)f(\alpha y + \beta z) \\ &\quad - \alpha 2^{-n}f(2^n(xy)) + \alpha 2^{-n}f(2^n x)f(y) - \beta 2^{-n}f(2^n(xz)) + \beta 2^{-n}f(2^n x)f(z)\| \\ &\le 2^{-n}\|f(\alpha 2^n(xy) + \beta 2^n(xz)) - \alpha f(2^n(xy)) - \beta f(2^n(xz))\| \\ &\quad + 2^{-n}\|f(2^n x)\| \, \|f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)\| \\ &\le 2^{-n}\varepsilon + 2^{-n}\|f(2^n x)\|\varepsilon, \end{aligned}$$

Letting $n \to \infty$ in this inequality, it follows from (17) that the inequality

$$\|S(x, \alpha y + \beta z) - \alpha S(x, y) - \beta S(x, z)\| \leq \|h(x)\|\varepsilon \tag{20}$$

holds for all $y, z \in \mathcal{A}$. Following the same process as (15)∼(19) with (20), it follows that the mapping $H : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ defined by

$$H(x, y) = \lim_{n\to\infty} \frac{S(x, 2^n y)}{2^n} \tag{21}$$

for all $z \in \mathcal{A}$, is $\mathbb{C}$-linear in second variable. Since S was $\mathbb{C}$-linear in first variable, H is also $\mathbb{C}$-linear in first variable. Hence, we conclude that H is $\mathbb{C}$-bilinear.

From (13), we obtain

$$\frac{F(2^n x, y)}{2^n} = \frac{f(2^n(xy))}{2^n} - \frac{f(2^n x)}{2^n} f(y) \tag{22}$$

for all $x, y \in \mathcal{A}$, and so taking $n \to \infty$ in (22) yields

$$S(x, y) = h(xy) - h(x)f(y) \tag{23}$$

for all $x, y \in \mathcal{A}$. Replacing y by $2^n y$ in (23), we get

$$\frac{S(x, 2^n y)}{2^n} = h(xy) - h(x)\frac{f(2^n y)}{2^n} \tag{24}$$

for all $x, y \in \mathcal{A}$. Now, setting $n \to \infty$ in the both sides of (24) gives

$$H(x, y) = h(xy) - h(x)h(y) \tag{25}$$

for all $x, y \in \mathcal{A}$.

To show the continuity of H in $x \in \mathcal{A}$ for each fixed $y \in \mathcal{A}$ we use the way of [10].

Assume that F is continuous in $x \in \mathcal{A}$ for each fixed $y \in \mathcal{A}$. If S is not continuous at a point $x \in \mathcal{A}$ for some fixed $y_0 \in \mathcal{A}$, then there exist a positive integer η and a sequence $\{x_n\}$ in $\mathcal{A}$ converging to zero such that

$$\|S(x_n, y_0)\| > \frac{1}{\eta}$$

for all $n \in \mathbb{N}$. Let k be an integer greater than $3\eta\delta$, where $\delta = (\|1 + f(y_0)\|)\varepsilon$. Then we have

$$\|S(kx_n, y_0) - S(0, y_0)\| = \|S(kx_n, y_0)\| > 3\delta$$

for all $n \in \mathbb{N}$.

But, from (16), we obtain the inequality

$$\begin{aligned} \|S(kx_n, y_0) - S(0, y_0)\| &\leq \|S(kx_n, y_0) - F(kx_n, y_0)\| \\ &\quad + \|F(kx_n, y_0) - F(0, y_0)\| \\ &\quad + \|F(0, y_0) - S(0, y_0)\| \leq 3\delta \end{aligned} \tag{26}$$

for sufficiently large n, since $F(kx_n, y_0) \to F(0, y_0)$ as $n \to \infty$. This contradiction means that S is continuous in $x \in \mathcal{A}$ for each fixed $y \in \mathcal{A}$. Hence, the relation (21) tells us that H is continuous in $x \in \mathcal{A}$ for each fixed $y \in \mathcal{A}$.

To prove that the mapping H defined by (25) is continuous in $y \in \mathcal{A}$ for each fixed $x \in \mathcal{A}$, let us the first variable of F be fixed. By the similar one to the manner obtaining the inequality (15), we see that for each fixed $x \in \mathcal{A}$, the mapping $F : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ satisfies the inequality

$$\|F(x, \alpha y + \beta z) - \alpha F(x, y) - \beta F(x, z)\| \leq (1 + \|f(x)\|)\varepsilon$$

for all $y, z \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$. Now, the remainder of the proof carries over almost verbatim among (16)$\sim$(26). So we conclude that H is continuous in $y \in \mathcal{A}$ for each fixed $x \in \mathcal{A}$. Consequently, h is a generalized homomorphism. □

4. Stability of Generalized Homomorphisms via Jensen Equation

Consider the Jensen equation

$$2f\Big(\frac{x+y}{2}\Big) = f(x) + f(y).$$

It is well known that a function f between vector spaces with $f(0) = 0$ satisfies the Jensen equation if and only if it is additive. In this section, we obtain the stability result of generalized homomorphisms via the Jensen equation.

Theorem 4.1. *Let $\varepsilon \geq 0$ and let $f : \mathcal{A} \to \mathcal{B}$ be an approximately generalized homomorphism corresponding to the Jensen inequality*

$$\left\| 2f\Big(\frac{\alpha x + \beta y}{2}\Big) - \alpha f(x) - \beta f(y) \right\| \leq \varepsilon, \tag{27}$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{I} = \{1, \mathbf{i}\}$. For each fixed $z \in \mathcal{A}$ (resp. $x \in \mathcal{A}$), there is a positive number r_z (resp. r_x) such that the real functions $t \mapsto \|F(tx, z)\|$ (resp. $t \mapsto \|F(x, tz)\|$) is bounded on the interval $[0, r_z]$ (resp. $[0, r_x]$). Then there exists a unique generalized homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - h(x)\| \leq \varepsilon \tag{28}$$

for all $x \in \mathcal{A}$.

Proof. By hypothesis, for each fixed $z \in \mathcal{A}$, the mapping $F : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ satisfies the inequality

$$\begin{aligned}
&\Big\|2F\Big(\frac{\alpha x+\beta y}{2}, z\Big) - \alpha F(x,z) - \beta F(y,z)\Big\| \\
&\le \Big\|2f\Big(\frac{\alpha xz+\beta yz}{2}\Big) - 2f\Big(\frac{\alpha x+\beta y}{2}\Big)f(z) \\
&\qquad - \alpha f(xz) + \alpha f(x)f(z) - \beta f(yz) + \beta f(y)f(z)\Big\| \\
&\le \Big\|2f\Big(\frac{\alpha xz+\beta yz}{2}\Big) - \alpha f(xz) - \beta f(yz)\Big\| \\
&\qquad + \Big\|2f\Big(\frac{\alpha x+\beta y}{2}\Big) - \alpha f(x) - \beta f(y)\Big\|\|f(z)\| \\
&\le (1+\|f(z)\|)\varepsilon,
\end{aligned}$$

that is, we obtain the inequality

$$\Big\|2F\Big(\frac{\alpha x+\beta y}{2}, z\Big) - \alpha F(x,z) - \beta F(y,z)\Big\| \le (1+\|f(z)\|)\varepsilon \tag{29}$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{I}$.

Putting $\alpha = \beta = 1$ in (29) and using the Jung's result [14], there is an additive mapping in the first variable $S : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ such that

$$\|F(x,z) - S(x,z)\| \le (1+\|f(z)\|)\varepsilon \tag{30}$$

for all $x \in \mathcal{A}$, where

$$S(x,z) = \lim_{n\to\infty} \frac{F(2^n x, z)}{2^n} \tag{31}$$

for all $x \in \mathcal{A}$. By replacing x by $2^{n+1}x$ and letting $y = 0$ in (29), we get

$$2^{-(n+1)}\Big\|2F\Big(\frac{2^{n+1}}{2}\mathbf{i}x, z\Big) - \mathbf{i}F(2^{n+1}x, z) - F(0,z)\Big\| \le 2^{-(n+1)}(1+\|f(z)\|)\varepsilon$$

for all $x \in \mathcal{A}$. Taking limits as $n \to \infty$, we obtain

$$S(\mathbf{i}x, z) = \mathbf{i}S(x,z) \tag{32}$$

for all $x \in \mathcal{A}$. To prove the homogeneous property in the first variable of S, let us $g \in \mathcal{A}^*$, where $\mathcal{A}^*$ is the dual of $\mathcal{A}$, and define the additive function $\Upsilon : \mathbb{R} \to \mathbb{R}$ by

$\Upsilon(t) = g(S(tx, z))$. The function is bounded since

$$\begin{aligned} |\Upsilon(t)| &\le \|g\|\|S(tx, z)\| \\ &\le \|g\|(S(tx, z) - F(tx, z)\| + \|F(tx, z)\|) \\ &\le \|g\|\big((1 + \|f(z)\|)\varepsilon + \sup\{\|F(tx, z)\| : t \in [0, r_z]\}\big). \end{aligned} \tag{33}$$

It follows from Corollary 2.5 of [1] that $\Upsilon(t) = \Upsilon(1)t$ for all $t \in \mathbb{R}$. Hence we get

$$g(S(tx, z)) = g(tS(x, z))$$

for all $t \in \mathbb{R}$ and all $g \in \mathcal{A}^*$ which implies that $S(tx, z) = tS(x, z)$ for all $t \in \mathbb{R}$.

Now, for each complex number $\lambda = u + \mathbf{i}v$, we have

$$\begin{aligned} S(\lambda x, z) &= S(ux + \mathbf{i}vx, z) \\ &= S(ux, z) + S(\mathbf{i}vx, z) \\ &= uS(x, z) + \mathbf{i}vS(x, z) = \lambda S(x, z), \end{aligned} \tag{34}$$

that is, the mapping $S : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear in the first variable.

From Jung's result [14], the inequality (27) with $\alpha = \beta = 1$ implies that there exists a *unique* additive mapping $h : \mathcal{A} \to \mathcal{B}$ defined by

$$h(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in \mathcal{A}$ satisfying the inequality (28). Applying a similar approach to (29)∼(34) to (27), we see that h is $\mathbb{C}$-linear. The remainder of the proof follows the similar argument as in the proof of Theorem 2.1. Therefore, h is a generalized homomorphism. □

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