The Properties of Various Concepts

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Abstract

In this paper, we investigate the relations among formal concepts, attribute oriented concept and object oriented concepts on a complete residuated lattice.

Key Words : Complete residuated lattices, order reverse generating maps, isotone (antitone) Galois connetion, formal (resp. attribute oriented, object oriented) concepts

1. Introduction and preliminaries

Wille [11] introduced the formal concept lattices by allowing some uncertainty in data. Formal concept analysis is an important mathematical tool for data analysis and knowledge processing [1-5,8,9,11]. A fuzzy context consists of (X, Y, R) where X is a set of objects, Y is a set of attributes and R is a relation between X and Y. Bělohlávek [1-4] developed the notion of formal concepts with $R \in L^{X \times Y}$ on a complete residuated lattice L.

In this paper, we show that the classes of formal concepts, attribute oriented concept and object oriented concepts are complete lattices. Moreover, a family of attribute oriented concepts and a family of object oriented concepts are anti-isomorphic, a family of attribute oriented concepts and a family of formal concepts are isomorphic, a family of object oriented concepts and a family of formal concepts are anti-isomorphic.

Definition 1.1. [1-5,10] A triple $(L, \leq, \odot)$ is called a *complete residuated lattice* iff it satisfies the following conditions:

(L1) $L = (L, \leq, 1, 0)$ is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) $(L, \odot, 1)$ is a commutative monoid;

(L3) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i \in \Gamma} a_i) \odot b = \bigvee_{i \in \Gamma} (a_i \odot b).$$

Define an operation $\rightarrow$ as $a \rightarrow b = \bigvee \{c \in L \mid a \odot c \leq b\}$, for each $a, b \in L$.

Let $(L, \leq, \odot)$ be a complete residuated lattice. A order reversing map $^* : L \rightarrow L$ defined by $a^* = a \rightarrow 0$ is called a *strong negation* if $a^{**} = a$ for each $a \in L$.

In this paper, we assume $(L, \leq, \odot, ^*)$ is a complete residuated lattice with a strong negation *.

Definition 1.2. [5,8,9] Let X and Y be two sets. Let $\omega^\rightarrow, \phi^\rightarrow, \xi^\rightarrow : L^X \rightarrow L^Y$ and $\omega^\leftarrow, \phi^\leftarrow, \xi^\leftarrow : L^Y \rightarrow L^X$ be operators.

(1) The pair $(\omega^\rightarrow, \omega^\leftarrow)$ is called *antitone Galois connection* between X and Y if for $\mu \in L^X$ and $\rho \in L^Y$, $\rho \leq \omega^\rightarrow(\mu)$ iff $\mu \leq \omega^\leftarrow(\rho)$.

(2) The pair $(\phi^\rightarrow, \phi^\leftarrow)$ is called an *isotone Galois connection* between X and Y if for $\mu \in L^X$ and $\rho \in L^Y$, $\phi^\rightarrow(\mu) \leq \rho$ iff $\mu \leq \phi^\leftarrow(\rho)$. Moreover, the pair $(\xi^\leftarrow, \xi^\rightarrow)$ is called an *isotone Galois connection* between X and Y if for $\mu \in L^X$ and $\rho \in L^Y$, $\xi^\leftarrow(\rho) \leq \mu$ iff $\rho \leq \xi^\rightarrow(\mu)$.

Definition 1.3. [5,8,9] Let $\omega^\rightarrow, \phi^\rightarrow, \xi^\rightarrow : L^X \rightarrow L^Y$ and $\omega^\leftarrow, \phi^\leftarrow, \xi^\leftarrow : L^Y \rightarrow L^X$ be functions. A pair $(\mu, \rho) \in L^X \times L^Y$ is called:

(1) *formal concept* if $\rho = \omega^\rightarrow(\mu)$ and $\mu = \omega^\leftarrow(\rho)$ where $(\omega^\rightarrow, \omega^\leftarrow)$ is an antitone Galios connection,

(2) *an attribute oriented concept* if $\rho = \phi^\rightarrow(\mu)$ and $\mu = \phi^\leftarrow(\rho)$ where $(\phi^\rightarrow, \phi^\leftarrow)$ is an isotone Galios connection,

(3) *an object oriented concept* if $\rho = \xi^\rightarrow(\mu)$ and $\mu = \xi^\leftarrow(\rho)$ where $(\xi^\leftarrow, \xi^\rightarrow)$ is an isotone Galios connection.

Theorem 1.4. [9] Let $\omega^\rightarrow : L^X \rightarrow L^Y$ and $\omega^\leftarrow : L^Y \rightarrow L^X$ be operators. Let $(\omega^\rightarrow, \omega^\leftarrow)$ be an antitone Galois connection between X and Y. Then the following properties hold:

(1) For each $\mu \in L^X$ and $\mu \in L^X$, $\mu \leq \omega^\leftarrow(\omega^\rightarrow(\mu))$ and $\rho \leq \omega^\rightarrow(\omega^\leftarrow(\rho))$.

(2) If $\mu_1 \leq \mu_2$, then $\omega^\rightarrow(\mu_1) \geq \omega^\rightarrow(\mu_2)$. Moreover, if $\rho_1 \leq \rho_2$, then $\omega^\leftarrow(\rho_1) \geq \omega^\leftarrow(\rho_2)$.

(3) For each $\mu \in L^X$ and $\rho \in L^Y$, $\omega^\leftarrow(\omega^\rightarrow(\omega^\leftarrow(\rho))) = \omega^\leftarrow(\rho)$ and $\omega^\rightarrow(\omega^\leftarrow(\omega^\rightarrow(\mu))) = \omega^\rightarrow(\mu)$.

(4) For each $\mu_i \in L^X$ and $\rho_j \in L^Y$, $\omega^\rightarrow(\bigvee_{i \in I} \mu_i) = \bigwedge_{i \in I} \omega^\rightarrow(\mu_i)$ and $\omega^\leftarrow(\bigvee_{j \in J} \rho_j) = \bigwedge_{j \in J} \omega^\leftarrow(\rho_j)$.

Theorem 1.5. [8] Let $\phi^{\rightarrow} : L^X \to L^Y$ and $\phi^{\leftarrow} : L^Y \to L^X$ be operators. Let $(\phi^{\rightarrow}, \phi^{\leftarrow})$ be an isotone Galois connection between X and Y. For each $\mu, \mu_i \in L^X$ and $\rho, \rho_j \in L^Y$, the following properties hold:

(1) $\mu \leq \phi^{\leftarrow}(\phi^{\rightarrow}(\mu))$ and $\phi^{\rightarrow}(\phi^{\leftarrow}(\rho)) \leq \rho$.

(2) If $\mu_1 \leq \mu_2$, then $\phi^{\rightarrow}(\mu_1) \leq \phi^{\rightarrow}(\mu_2)$. Moreover, if $\rho_1 \leq \rho_2$, then $\phi^{\leftarrow}(\rho_1) \leq \phi^{\leftarrow}(\rho_2)$.

(3) $\phi^{\leftarrow}(\phi^{\rightarrow}(\phi^{\leftarrow}(\rho))) = \phi^{\leftarrow}(\rho)$ and $\phi^{\rightarrow}(\phi^{\leftarrow}(\phi^{\rightarrow}(\mu))) = \phi^{\rightarrow}(\mu)$.

(4) $\phi^{\rightarrow}(\bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} \phi^{\rightarrow}(\mu_i)$ and $\phi^{\leftarrow}(\bigwedge_{j \in J} \rho_j) = \bigwedge_{j \in J} \phi^{\leftarrow}(\rho_j)$.

Definition 1.6. [8,9] An operator $\phi^{\rightarrow} : L^X \to L^Y$ is called a join-generating operator, denoted by $\phi^{\rightarrow} \in J(X,Y)$, if $\phi^{\rightarrow}(\bigvee_{i \in \Gamma} \lambda_i) = \bigvee_{i \in \Gamma} \phi^{\rightarrow}(\lambda_i)$, for $\{\lambda_i\}_{i \in \Gamma} \subset L^X$.

An operator $\psi^{\rightarrow} : L^X \to L^Y$ is called a meet-generating operator, denoted by $\psi^{\rightarrow} \in M(X,Y)$, if $\psi^{\rightarrow}(\bigwedge_{i \in \Gamma} \lambda_i) = \bigwedge_{i \in \Gamma} \psi^{\rightarrow}(\lambda_i)$, for $\{\lambda_i\}_{i \in \Gamma} \subset L^X$.

An operator $\omega^{\rightarrow} : L^X \to L^Y$ is called a order reverse-generating operator, denoted by $\omega^{\rightarrow} \in K(X,Y)$, if $\omega^{\rightarrow}(\bigvee_{i \in \Gamma} \lambda_i) = \bigwedge_{i \in \Gamma} \omega^{\rightarrow}(\lambda_i)$, for $\{\lambda_i\}_{i \in \Gamma} \subset L^X$.

Theorem 1.7. [9] For $\omega^{\rightarrow} \in K(X,Y)$, Define functions $\phi_{\omega}^{\rightarrow}, \xi_{\omega}^{\rightarrow} : L^X \to L^Y$ and $\omega^{\leftarrow}, \phi_{\omega}^{\leftarrow}, \xi_{\omega}^{\leftarrow} : L^Y \to L^X$ as follows: for all $\lambda \in L^X, \rho \in L^Y$,

$$\omega^{\leftarrow}(\rho) = \bigvee \{\lambda \in L^X \mid \omega^{\rightarrow}(\lambda) \geq \rho\}$$

$$\phi_{\omega}^{\rightarrow}(\mu) = (\omega^{\rightarrow}(\mu))^*, \quad \phi_{\omega}^{\leftarrow}(\rho) = \omega^{\leftarrow}(\rho^*)$$

$$\xi_{\omega}^{\leftarrow}(\rho) = \bigwedge \{\lambda \in L^X \mid \omega(\lambda^*) \geq \rho\},$$

$$\xi_{\omega}^{\rightarrow}(\mu) = \bigvee \{\rho \in L^Y \mid \xi_{\omega}^{\leftarrow}(\rho) \leq \mu\}$$

Then the following properties hold:

(1) $\omega^{\leftarrow} \in K(Y,X)$ with $\omega^{\rightarrow}(\lambda) \geq \rho \Leftrightarrow \omega^{\leftarrow}(\rho) \geq \lambda$ for all $\lambda \in L^X$ and $\rho \in L^Y$.

(2) The pair $(\omega^{\rightarrow}, \omega^{\leftarrow})$ is an antitone Galois connection and $(\omega^{\leftarrow}(\omega^{\rightarrow}(\lambda)), \omega^{\rightarrow}(\lambda))$ for all $\lambda \in L^X$ are formal concepts.

(3) The pair $(\phi_{\omega}^{\rightarrow}, \phi_{\omega}^{\leftarrow})$ is an isotone Galois connection and $(\phi_{\omega}^{\leftarrow}(\phi_{\omega}^{\rightarrow}(\lambda)), \phi_{\omega}^{\rightarrow}(\lambda)) = (\omega^{\leftarrow}(\omega^{\rightarrow}(\lambda)), \omega^{\rightarrow}(\lambda)^*)$ for all $\lambda \in L^X$ are attribute oriented concepts.

(4) $\xi_{\omega}^{\leftarrow} : L^Y \to L^X$ is a join-generating function such that $\xi_{\omega}^{\leftarrow}(\rho) = (\omega^{\leftarrow}(\rho))^*$ and $\xi_{\omega}^{\rightarrow} : L^X \to L^Y$ is a meet-generating function such that $\xi_{\omega}^{\rightarrow}(\lambda) = \omega^{\rightarrow}(\lambda^*)$.

(5) The pair $(\xi_{\omega}^{\leftarrow}, \xi_{\omega}^{\rightarrow})$ is an isotone Galois connection and $(\xi_{\omega}^{\leftarrow}(\rho), \xi_{\omega}^{\rightarrow}(\xi_{\omega}^{\leftarrow}(\rho))) = (\omega^{\leftarrow}(\rho)^*, \omega^{\rightarrow}(\omega^{\leftarrow}(\rho)))$ for all $\rho \in L^Y$ are object oriented concepts.

Theorem 1.8. [9] Let (X,Y,R) be a fuzzy context. Define a function $\omega_R^{\rightarrow} : L^X \to L^Y$ as follows:

$$\omega_R^{\rightarrow}(\lambda)(y) = \bigvee_{x \in X} (\lambda(x) \to R(x,y)).$$

Then we have the following properties:

(1) $\omega_R^{\rightarrow} \in K(X,Y)$, $(\omega_R^{\rightarrow}, \omega_R^{\leftarrow})$ is an isotone Galois connections and $(\omega_R^{\leftarrow}(\omega_R^{\rightarrow}(\lambda)), \omega_R^{\rightarrow}(\lambda))$ for all $\lambda \in L^X$ are formal concepts where

$$\omega_R^{\leftarrow}(\rho)(x) = \bigwedge_{y \in Y} (\rho(y) \to R(x,y)).$$

(2) $\phi_{\omega_R}^{\rightarrow}(\mu) = (\omega_R^{\rightarrow}(\mu))^*$ and $\phi_{\omega_R}^{\leftarrow}(\rho) = \omega_R^{\leftarrow}(\rho^*)$ where

$$\phi_{\omega_R}^{\rightarrow}(\mu)(y) = \bigvee_{y \in Y} (\mu(x) \odot R^*(x,y)),$$

$$\phi_{\omega_R}^{\leftarrow}(\rho)(x) = \bigwedge_{y \in Y} (R^*(x,y) \to \rho(y)).$$

Moreover, the pair $(\phi_{\omega_R}^{\rightarrow}, \phi_{\omega_R}^{\leftarrow})$ is an isotone Galois connection and $(\phi_{\omega_R}^{\leftarrow}(\phi_{\omega_R}^{\rightarrow}(\lambda)), \phi_{\omega_R}^{\rightarrow}(\lambda))$ are attribute concepts.

(3) The pair $(\xi_{\omega_R}^{\leftarrow}, \xi_{\omega_R}^{\rightarrow})$ is an isotone Galois connection and $(\xi_{\omega_R}^{\leftarrow}(\rho), \xi_{\omega_R}^{\rightarrow}(\xi_{\omega_R}^{\leftarrow}(\rho)))$ for all $\rho \in L^Y$ are object oriented concepts where

$$\xi_{\omega_R}^{\leftarrow}(\rho)(x) = \bigvee_{y \in Y} (\rho(y) \odot R^*(x,y)),$$

$$\xi_{\omega_R}^{\rightarrow}(\lambda)(y) = \bigwedge_{x \in X} (R^*(x,y) \to \lambda(x))$$

Theorem 1.9. [8] For $\phi^{\rightarrow} \in J(X,Y)$, Define functions $\omega_{\phi}^{\rightarrow}, \xi_{\phi}^{\rightarrow} : L^X \to L^Y$ and $\phi^{\leftarrow}, \omega_{\phi}^{\leftarrow}, \xi_{\phi}^{\leftarrow} : L^Y \to L^X$ as follows: , for all $\lambda \in L^X, \rho \in L^Y$,

$$\phi^{\leftarrow}(\rho) = \bigvee \{\lambda \in L^X \mid \phi^{\rightarrow}(\lambda) \leq \rho\},$$

$$\omega_{\phi}^{\rightarrow}(\lambda) = (\phi^{\rightarrow}(\lambda))^*, \quad \omega_{\phi}^{\leftarrow}(\rho) = \phi^{\leftarrow}(\rho^*),$$

$$\xi_{\phi}^{\leftarrow}(\rho) = \bigwedge \{\lambda \in L^X \mid \phi^{\rightarrow}(\lambda^*) \leq \rho^*\},$$

$$\xi_{\phi}^{\rightarrow}(\lambda) = \bigvee \{\rho \in L^Y \mid \xi_{\phi}^{\leftarrow}(\rho) \leq \lambda\}.$$

Then the following properties hold:

(1) The pair $(\phi^{\rightarrow}, \phi^{\leftarrow})$ is an isotone Galois connection and $(\phi^{\leftarrow}(\phi^{\rightarrow}(\lambda)), \phi^{\rightarrow}(\lambda))$ for each $\lambda \in L^X$ are attribute oriented concepts with $\phi^{\leftarrow} \in M(Y,X)$.

(2) The pair $(\omega_{\phi}^{\rightarrow}, \omega_{\phi}^{\leftarrow})$ is an antitone Galois connection and $(\omega_{\phi}^{\leftarrow}(\omega_{\phi}^{\rightarrow}(\lambda)), \omega_{\phi}^{\rightarrow}(\lambda))$ for $\lambda \in L^X$ are formal concepts.

(3) The pair $(\xi_{\phi}^{\leftarrow}, \xi_{\phi}^{\rightarrow})$ is an isotone Galois connection and $(\xi_{\phi}^{\leftarrow}(\rho), \xi_{\phi}^{\rightarrow}(\xi_{\phi}^{\leftarrow}(\rho)))$ for each $\rho \in L^Y$, are object oriented concepts with $\xi_{\phi}^{\leftarrow}(\rho) = (\phi^{\leftarrow}(\rho^*))^*$ and $\xi_{\phi}^{\rightarrow}(\lambda) = (\phi^{\rightarrow}(\lambda^*))^*$.

2. The properties of various concepts

Theorem 2.1. Let $\mathcal{F}(X,Y) = \{(\mu,\rho) \in L^X \times L^Y\}$ be a family of formal concepts. For all $(\mu_i,\rho_i)_{i \in \Gamma} \in \mathcal{F}(X,Y)$, we define

$$\bigvee_{i \in \Gamma}(\mu_i,\rho_i) = (\omega^{\leftarrow}(\bigwedge_{i \in \Gamma}\rho_i), \bigwedge_{i \in \Gamma}\rho_i),$$

$$\bigwedge_{i \in \Gamma}(\mu_i,\rho_i) = (\bigwedge_{i \in \Gamma}\mu_i, \omega^{\rightarrow}(\bigwedge_{i \in \Gamma}\mu_i)).$$

Then $(\mathcal{F}(X,Y), \vee, \wedge)$ is a complete lattice.

Proof. Since $\bigwedge_{i \in \Gamma}\rho_i = \bigwedge_{i \in \Gamma}\omega^{\rightarrow}(\mu_i) = \omega^{\rightarrow}(\vee_{i \in \Gamma}\mu_i)$, by Theorem 1.4(3), $\omega^{\rightarrow}(\omega^{\leftarrow}(\bigwedge_{i \in \Gamma}\rho_i)) = \omega^{\rightarrow}(\omega^{\leftarrow}(\omega^{\rightarrow}(\vee_{i \in \Gamma}\mu_i))) = \omega^{\rightarrow}(\vee_{i \in \Gamma}\mu_i) = \bigwedge_{i \in \Gamma}\rho_i$. Hence $(\omega^{\leftarrow}(\bigwedge_{i \in \Gamma}\rho_i), \bigwedge_{i \in \Gamma}\rho_i) = (\omega^{\leftarrow}(\bigwedge_{i \in \Gamma}\rho_i), \omega^{\rightarrow}(\omega^{\leftarrow}(\bigwedge_{i \in \Gamma}\rho_i))) \in \mathcal{F}(X,Y)$. Since $\bigwedge_{i \in \Gamma}\mu_i = \bigwedge_{i \in \Gamma}\omega^{\leftarrow}(\rho_i) = \omega^{\leftarrow}(\vee_{i \in \Gamma}\rho_i)$, by Theorem 1.4(3), $\omega^{\leftarrow}(\omega^{\rightarrow}(\bigwedge_{i \in \Gamma}\mu_i)) = \omega^{\leftarrow}(\omega^{\rightarrow}(\omega^{\leftarrow}(\vee_{i \in \Gamma}\rho_i))) = \omega^{\leftarrow}(\vee_{i \in \Gamma}\rho_i) = \bigwedge_{i \in \Gamma}\mu_i$. Thus, $(\bigwedge_{i \in \Gamma}\mu_i, \omega^{\rightarrow}(\bigwedge_{i \in \Gamma}\mu_i)) = (\omega^{\leftarrow}(\omega^{\rightarrow}(\bigwedge_{i \in \Gamma}\mu_i)), \omega^{\rightarrow}(\bigwedge_{i \in \Gamma}\mu_i)) \in \mathcal{F}(X,Y)$. $\square$

Theorem 2.2. Let $\mathcal{G}(X,Y) = \{(\mu,\rho) \in L^X \times L^Y\}$ be a family of attribute oriented concepts. For all $(\mu_i,\rho_i)_{i \in \Gamma} \in \mathcal{G}(X,Y)$, we define

$$\bigvee_{i \in \Gamma}(\mu_i,\rho_i) = (\phi^{\leftarrow}(\bigvee_{i \in \Gamma}\rho_i), \bigvee_{i \in \Gamma}\rho_i),$$

$$\bigwedge_{i \in \Gamma}(\mu_i,\rho_i) = (\bigwedge_{i \in \Gamma}\mu_i, \phi^{\rightarrow}(\bigwedge_{i \in \Gamma}\mu_i)).$$

Then $(\mathcal{G}(X,Y), \vee, \wedge)$ is a complete lattice.

Proof. Since $\bigvee_{i \in \Gamma}\rho_i = \bigvee_{i \in \Gamma}\phi^{\rightarrow}(\mu_i) = \phi^{\rightarrow}(\vee_{i \in \Gamma}\mu_i)$, by Theorem 1.5(3,4), $\phi^{\rightarrow}(\phi^{\leftarrow}(\bigvee_{i \in \Gamma}\rho_i)) = \phi^{\rightarrow}(\phi^{\leftarrow}(\phi^{\rightarrow}(\vee_{i \in \Gamma}\mu_i))) = \phi^{\rightarrow}(\vee_{i \in \Gamma}\mu_i) = \bigvee_{i \in \Gamma}\rho_i$. Hence, $(\phi^{\leftarrow}(\bigvee_{i \in \Gamma}\rho_i), \bigvee_{i \in \Gamma}\rho_i) = (\phi^{\leftarrow}(\bigvee_{i \in \Gamma}\rho_i), \phi^{\rightarrow}(\phi^{\leftarrow}(\bigvee_{i \in \Gamma}\rho_i))) \in \mathcal{G}(X,Y)$. Since $\bigwedge_{i \in \Gamma}\mu_i = \bigwedge_{i \in \Gamma}\phi^{\leftarrow}(\rho_i) = \phi^{\leftarrow}(\wedge_{i \in \Gamma}\rho_i)$, by Theorem 1.5(3,4), $\phi^{\leftarrow}(\phi^{\rightarrow}(\bigwedge_{i \in \Gamma}\mu_i)) = \phi^{\leftarrow}(\phi^{\rightarrow}(\phi^{\leftarrow}(\wedge_{i \in \Gamma}\rho_i))) = \phi^{\leftarrow}(\wedge_{i \in \Gamma}\rho_i) = \bigwedge_{i \in \Gamma}\mu_i$. Hence, $(\bigwedge_{i \in \Gamma}\mu_i, \phi^{\rightarrow}(\bigwedge_{i \in \Gamma}\mu_i)) = (\phi^{\leftarrow}(\phi^{\rightarrow}(\bigwedge_{i \in \Gamma}\mu_i)), \phi^{\rightarrow}(\bigwedge_{i \in \Gamma}\mu_i)) \in \mathcal{G}(X,Y)$. $\square$

Theorem 2.3. Let $\mathcal{H}(X,Y) = \{(\mu,\rho) \in L^X \times L^Y\}$ be a family of object oriented concepts. For all $(\mu_i,\rho_i)_{i \in \Gamma} \in \mathcal{H}(X,Y)$, we define

$$\bigvee_{i \in \Gamma}(\mu_i,\rho_i) = (\bigvee_{i \in \Gamma}\mu_i, \xi^{\rightarrow}(\bigvee_{i \in \Gamma}\mu_i)),$$

$$\bigwedge_{i \in \Gamma}(\mu_i,\rho_i) = (\xi^{\leftarrow}(\bigwedge_{i \in \Gamma}\rho_i), \bigwedge_{i \in \Gamma}\rho_i).$$

Then $(\mathcal{H}(X,Y), \vee, \wedge)$ is a complete lattice.

Proof. Since $\bigwedge_{i \in \Gamma}\rho_i = \bigwedge_{i \in \Gamma}\xi^{\rightarrow}(\mu_i) = \xi^{\rightarrow}(\wedge_{i \in \Gamma}\mu_i)$, by Theorem 1.5(3,4), $\xi^{\rightarrow}(\xi^{\leftarrow}(\bigwedge_{i \in \Gamma}\rho_i)) = \xi^{\rightarrow}(\xi^{\leftarrow}(\xi^{\rightarrow}(\wedge_{i \in \Gamma}\mu_i))) = \xi^{\rightarrow}(\wedge_{i \in \Gamma}\mu_i) = \bigwedge_{i \in \Gamma}\rho_i$. Thus, $(\xi^{\leftarrow}(\bigwedge_{i \in \Gamma}\rho_i), \bigwedge_{i \in \Gamma}\rho_i) \in \mathcal{H}(X,Y)$.

We show $(\bigvee_{i \in \Gamma}\mu_i, \xi^{\rightarrow}(\bigvee_{i \in \Gamma}\mu_i)) \in \mathcal{H}(X,Y)$ from:

Since $\bigvee_{i \in \Gamma}\mu_i = \bigvee_{i \in \Gamma}\xi^{\leftarrow}(\rho_i) = \xi^{\leftarrow}(\vee_{i \in \Gamma}\rho_i)$, by Theorem 1.5(3,4), $\xi^{\leftarrow}(\xi^{\rightarrow}(\bigvee_{i \in \Gamma}\mu_i)) = \xi^{\leftarrow}(\xi^{\rightarrow}(\xi^{\leftarrow}(\vee_{i \in \Gamma}\rho_i))) = \xi^{\leftarrow}(\vee_{i \in \Gamma}\rho_i) = \bigvee_{i \in \Gamma}\mu_i$. Hence, $(\bigvee_{i \in \Gamma}\mu_i, \xi^{\rightarrow}(\bigvee_{i \in \Gamma}\mu_i)) \in \mathcal{H}(X,Y)$. $\square$

Definition 2.4. Let $(L_1, \leq, \vee, \wedge)$ and $(L_2, \leq, \vee, \wedge)$ be two lattices. We call:

(1) L_1 and L_2 are isomorphic, denoted by $L_1 \cong L_2$, if there exists a bijective function $h : L_1 \to L_2$ such that $h(x \vee y) = h(x) \vee h(y)$ and $h(x \wedge y) = h(x) \wedge h(y)$.

(2) L_1 and L_2 are anti-isomorphic, denoted by $L_1 \simeq L_2$, if there exists a bijective function $g : L_1 \to L_2$ such that $g(x \vee y) = g(x) \wedge g(y)$ and $g(x \wedge y) = g(x) \vee g(y)$.

Theorem 2.5. Let $\mathcal{G}(X,Y) = \{(\mu,\rho) \in L^X \times L^Y\}$ and $\mathcal{H}(X,Y) = \{(\mu,\rho) \in L^X \times L^Y\}$ be two families of attribute oriented concepts and object oriented concepts. We define $U : \mathcal{G}(X,Y) \to \mathcal{H}(X,Y)$ as follows:

$$U(\mu,\rho) = \begin{cases} (\xi_\phi^{\leftarrow}(\rho^*), \rho^*), & \text{if } \mu = \phi^{\leftarrow}(\rho), \\ (\mu^*, \xi_\phi^{\rightarrow}(\mu^*)), & \text{if } \rho = \phi^{\rightarrow}(\mu), \end{cases}$$

where $\xi_\phi^{\rightarrow}(\mu) = (\phi^{\rightarrow}(\mu^*))^*$, $\xi_\phi^{\leftarrow}(\rho) = (\phi^{\leftarrow}(\rho^*))^*$ and $(\phi^{\rightarrow}, \phi^{\leftarrow})$ is an isotone Galois connection. Then $\mathcal{G}(X,Y)$ and $\mathcal{H}(X,Y)$ are anti-isomorphic where $(\xi_\phi^{\leftarrow}, \xi_\phi^{\rightarrow})$ is an isotone Galois connection.

Proof. Since $\xi_\phi^{\leftarrow}(\rho) \leq \lambda$ iff $(\phi^{\leftarrow}(\rho^*))^* \leq \lambda$ iff $\phi^{\leftarrow}(\rho^*) \geq \lambda^*$ iff $\phi^{\rightarrow}(\lambda^*) \leq \rho^*$ iff $\xi_\phi^{\rightarrow}(\lambda) \geq \rho$, for all $\lambda \in L^X, \rho \in L^Y$, $(\xi_\phi^{\leftarrow}, \xi_\phi^{\rightarrow})$ is an isotone Galois connection. Since $\rho^* = (\phi^{\rightarrow}(\mu))^* = \xi_\phi^{\rightarrow}(\mu^*)$ and $\mu^* = (\phi^{\leftarrow}(\rho))^* = \xi_\phi^{\leftarrow}(\rho^*)$, then $(\xi_\phi^{\leftarrow}(\rho^*), \rho^*) = (\xi_\phi^{\leftarrow}(\xi_\phi^{\rightarrow}(\mu^*)), \xi_\phi^{\rightarrow}(\mu^*)) \in \mathcal{H}(X,Y)$ and $(\mu^*, \xi_\phi^{\rightarrow}(\mu^*)) = (\xi_\phi^{\leftarrow}(\rho^*), \xi_\phi^{\rightarrow}(\xi_\phi^{\leftarrow}(\rho^*))) \in \mathcal{H}(X,Y)$. Hence U is well defined. For $(\xi_\phi^{\leftarrow}(\rho), \rho) \in \mathcal{H}(X,Y)$, since $\phi^{\rightarrow}(\phi^{\leftarrow}(\rho^*)) = \phi^{\rightarrow}((\xi_\phi^{\leftarrow}(\rho))^*) = (\xi_\phi^{\rightarrow}(\xi_\phi^{\leftarrow}(\rho)))^* = \rho^*$, $(\phi^{\leftarrow}(\rho^*), \rho^*) = (\phi^{\leftarrow}(\rho^*), \phi^{\rightarrow}(\phi^{\leftarrow}(\rho^*))) \in \mathcal{G}(X,Y)$ such that $U(\phi^{\leftarrow}(\rho^*), \rho^*) = (\xi_\phi^{\leftarrow}(\rho), \rho)$.

For $(\mu, \xi_\phi^{\rightarrow}(\mu)) \in \mathcal{H}(X,Y)$, since $\phi^{\leftarrow}(\phi^{\rightarrow}(\mu^*)) = \phi^{\leftarrow}((\xi_\phi^{\rightarrow}(\mu))^*) = (\xi_\phi^{\leftarrow}((\xi_\phi^{\rightarrow}(\mu))))^* = \mu^*$, $(\mu^*, \phi^{\rightarrow}(\mu^*)) \in \mathcal{G}(X,Y)$ such that $U(\mu^*, \phi^{\rightarrow}(\mu^*)) = (\mu, \xi_\phi^{\rightarrow}(\mu))$. Hence U is surjective. Trivially, U is injective.

$$U(\phi^{\leftarrow}(\rho_1), \rho_1) \wedge U(\phi^{\leftarrow}(\rho_2), \rho_2)$$
$$= (\xi_\phi^{\leftarrow}(\rho_1^*), \rho_1^*) \wedge (\xi_\phi^{\leftarrow}(\rho_2^*), \rho_2^*)$$
$$= (\xi_\phi^{\leftarrow}(\rho_1^* \wedge \rho_2^*), \rho_1^* \wedge \rho_2^*) \ \text{(by Theorem 2.3)}$$
$$= ((\phi^{\leftarrow}(\rho_1 \vee \rho_2))^*, (\rho_1 \vee \rho_2)^*)$$
$$= U(\phi^{\leftarrow}(\rho_1 \vee \rho_2), \rho_1 \vee \rho_2)$$
$$= U((\phi^{\leftarrow}(\rho_1), \rho_1) \vee (\phi^{\leftarrow}(\rho_2), \rho_2)) \ \text{(by Theorem 2.2).}$$

$$U(\mu_1, \phi^\rightarrow(\mu_1)) \vee U(\mu_2, \phi^\rightarrow(\mu_2))$$
$$= (\mu_1^*, \xi_\phi^\rightarrow(\mu_1^*)) \vee (\mu_2^*, \xi_\phi^\rightarrow(\mu_2^*))$$
$$= (\mu_1^* \vee \mu_2^*, \xi_\phi^\rightarrow(\mu_1^* \vee \mu_2^*))$$
$$= U(\mu_1 \wedge \mu_2, \phi^\rightarrow(\mu_1 \wedge \mu_2))$$
$$= U((\mu_1, \phi^\rightarrow(\mu_1)) \wedge (\mu_2, \phi^\rightarrow(\mu_2)))$$
(by Theorem 2.2).

Hence, $\mathcal{G}(X, Y)$ and $\mathcal{H}(X, Y)$ are anti-isomorphic.
$\square$

Theorem 2.6. Let $\mathcal{F}(X, Y) = \{(\mu, \rho) \in L^X \times L^Y\}$ and $\mathcal{G}(X, Y) = \{(\mu, \rho) \in L^X \times L^Y\}$ be two families of formal concepts and object oriented concepts. We define $V : \mathcal{F}(X, Y) \rightarrow \mathcal{G}(X, Y)$ as follows:

$$V(\mu, \rho) = \begin{cases} (\phi_\omega^\leftarrow(\rho^*), \rho^*), & \text{if } \mu = \omega^\leftarrow(\rho), \\ (\mu, \phi_\omega^\rightarrow(\mu)), & \text{if } \rho = \omega^\rightarrow(\mu), \end{cases}$$

where $\phi_\omega^\rightarrow(\mu) = (\omega^\rightarrow(\mu))^*$, $\phi_\omega^\leftarrow(\rho) = \omega^\leftarrow(\rho^*)$ and $(\omega^\rightarrow, \omega^\leftarrow)$ is an anti-tone Galois connection. Then $\mathcal{F}(X, Y)$ and $\mathcal{G}(X, Y)$ are isomorphic where $(\phi_\omega^\rightarrow, \phi_\omega^\leftarrow)$ is an isotone Galois connection.

Proof. Since $\phi_\omega^\rightarrow(\mu) \leq \rho$ iff $(\omega^\rightarrow(\mu))^* \leq \rho$ iff $\omega^\rightarrow(\mu) \geq \rho^*$ iff $\omega^\leftarrow(\rho^*) \geq \mu$ iff $\phi_\omega^\leftarrow(\rho) \geq \mu$, for all $\mu \in L^X, \rho \in L^Y$, $(\phi_\omega^\rightarrow, \phi_\omega^\leftarrow)$ is an isotone Galois connection. Since $\phi_\omega^\rightarrow(\phi_\omega^\leftarrow(\rho^*)) = \phi_\omega^\rightarrow(\omega^\leftarrow(\rho)) = (\omega^\rightarrow(\omega^\leftarrow(\rho)))^* = \rho^*$ and $\mu = \omega^\leftarrow(\rho) = \phi_\omega^\leftarrow(\rho^*)$, then $(\phi_\omega^\leftarrow(\rho^*), \rho^*) = (\phi_\omega^\leftarrow(\rho^*), \phi_\omega^\rightarrow(\phi_\omega^\leftarrow(\rho^*))) \in \mathcal{G}(X, Y)$ and $(\mu, \phi_\omega^\rightarrow(\mu)) = (\phi_\omega^\leftarrow(\rho^*), \phi_\omega^\rightarrow(\phi_\omega^\leftarrow(\rho^*))) \in \mathcal{G}(X, Y)$. Hence V is well defined. For $(\phi_\omega^\leftarrow(\rho), \rho) \in \mathcal{G}(X, Y)$, since $\rho = \phi_\omega^\rightarrow(\phi_\omega^\leftarrow(\rho))) = \phi_\omega^\rightarrow(\omega^\leftarrow(\rho^*)) = (\omega^\rightarrow(\omega^\leftarrow(\rho^*)))^*$, $(\omega^\leftarrow(\rho^*), \rho^*) = (\omega^\leftarrow(\rho^*), \omega^\rightarrow(\omega^\leftarrow(\rho^*))) \in \mathcal{F}(X, Y)$ such that $V(\omega^\leftarrow(\rho^*), \rho^*) = (\phi_\omega^\leftarrow(\rho), \rho)$.

For $(\mu, \phi_\omega^\rightarrow(\mu)) \in \mathcal{G}(X, Y)$, since $\mu = \phi_\omega^\leftarrow(\phi_\omega^\rightarrow(\mu)) = \phi_\omega^\leftarrow((\omega^\rightarrow(\mu))^*) = \omega^\leftarrow(\omega^\rightarrow(\mu))$, $(\mu, \omega^\rightarrow(\mu)) = (\omega^\leftarrow(\omega^\rightarrow(\mu)), \omega^\rightarrow(\mu)) \in \mathcal{F}(X, Y)$ such that $V(\mu, \omega^\rightarrow(\mu)) = (\mu, \phi_\omega^\rightarrow(\mu))$. Hence V is surjective. Trivially, V is injective.

$$V(\omega^\leftarrow(\rho_1), \rho_1) \vee V(\omega^\leftarrow(\rho_2), \rho_2)$$
$$= (\phi_\omega^\leftarrow(\rho_1^*), \rho_1^*) \vee (\phi_\omega^\leftarrow(\rho_2^*), \rho_2^*)$$
$$= (\phi_\omega^\leftarrow(\rho_1^* \vee \rho_2^*), \rho_1^* \vee \rho_2^*)$$
$$= V(\omega^\leftarrow(\rho_1 \wedge \rho_2), \rho_1 \wedge \rho_2)$$
$$= V(\omega^\leftarrow(\rho_1), \rho_1) \vee (\omega^\leftarrow(\rho_2), \rho_2)).$$

$$V(\mu_1, \omega^\rightarrow(\mu_1)) \wedge V(\mu_2, \omega^\rightarrow(\mu_2))$$
$$= (\mu_1, \phi_\omega^\rightarrow(\mu_1)) \wedge (\mu_2, \phi_\omega^\rightarrow(\mu_2))$$
$$= (\mu_1 \wedge \mu_2, \phi_\omega^\rightarrow(\mu_1 \wedge \mu_2))$$
$$= V(\mu_1 \wedge \mu_2, \omega^\rightarrow(\mu_1 \wedge \mu_2))$$
$$= V((\mu_1, \omega^\rightarrow(\mu_1)) \wedge (\mu_2, \omega^\rightarrow(\mu_2))).$$

Hence, $\mathcal{F}(X, Y)$ and $\mathcal{G}(X, Y)$ are isomorphic.
$\square$

Theorem 2.7. Let $\mathcal{F}(X, Y) = \{(\mu, \rho) \in L^X \times L^Y\}$ and $\mathcal{H}(X, Y) = \{(\mu, \rho) \in L^X \times L^Y\}$ be two families of formal concepts and attribute oriented concepts. We define $W : \mathcal{F}(X, Y) \rightarrow \mathcal{H}(X, Y)$ as follows:

$$W(\mu, \rho) = \begin{cases} (\xi_\omega^\leftarrow(\rho), \rho), & \text{if } \mu = \omega^\leftarrow(\rho), \\ (\mu^*, \xi_\omega^\rightarrow(\mu^*)), & \text{if } \rho = \omega^\rightarrow(\mu), \end{cases}$$

where $\xi_\omega^\rightarrow(\mu) = \omega^\rightarrow(\mu^*)$, $\xi_\omega^\leftarrow(\rho) = (\omega^\leftarrow(\rho))^*$ and $(\omega^\rightarrow, \omega^\leftarrow)$ is an anti-tone Galois connection. Then $\mathcal{F}(X, Y)$ and $\mathcal{H}(X, Y)$ are anti-isomorphic where $(\xi_\omega^\leftarrow, \xi_\omega^\rightarrow)$ is an isotone Galois connection.

Proof. Since $\xi_\omega^\leftarrow(\rho) \leq \lambda$ iff $(\omega^\leftarrow(\rho))^* \leq \lambda$ iff $\omega^\leftarrow(\rho) \geq \lambda^*$ iff $\omega^\rightarrow(\lambda^*) \geq \rho$ iff $\xi_\omega^\rightarrow(\lambda) \geq \rho$, for all $\lambda \in L^X, \rho \in L^Y$, $(\xi_\omega^\leftarrow, \xi_\omega^\rightarrow)$ is an isotone Galois connection.

Since $\rho = \omega^\rightarrow(\mu) = \xi_\omega^\rightarrow(\mu^*)$ and $\mu = \omega^\leftarrow(\rho) = (\xi_\omega^\leftarrow(\rho))^*$, then $(\xi_\omega^\leftarrow(\rho), \rho) = (\xi_\omega^\leftarrow(\xi_\omega^\rightarrow(\mu^*)), \xi_\omega^\rightarrow(\mu^*)) \in \mathcal{H}(X, Y)$ and $(\mu^*, \xi_\omega^\rightarrow(\mu^*)) = (\xi_\omega^\leftarrow(\rho), \xi_\omega^\rightarrow(\xi_\omega^\leftarrow(\rho))) \in \mathcal{H}(X, Y)$. Hence W is well defined. For $(\xi_\omega^\leftarrow(\rho), \rho) \in \mathcal{H}(X, Y)$, there exists $(\omega^\leftarrow(\rho), \rho) \in \mathcal{F}(X, Y)$ such that $W(\omega^\leftarrow(\rho), \rho) = (\xi_\omega^\leftarrow(\rho), \rho)$.

For $(\mu, \xi_\omega^\rightarrow(\mu)) \in \mathcal{H}(X, Y)$, since $\mu = \xi_\omega^\leftarrow(\xi_\omega^\rightarrow(\mu)) = \xi_\omega^\leftarrow(\omega^\rightarrow(\mu^*)) = (\omega^\leftarrow(\omega^\rightarrow(\mu^*)))^*$, $(\mu^*, \omega^\rightarrow(\mu^*)) = (\omega^\leftarrow(\omega^\rightarrow(\mu^*)), \omega^\rightarrow(\mu^*)) \in \mathcal{F}(X, Y)$ such that $W(\mu^*, \omega^\rightarrow(\mu^*)) = (\mu, \xi_\omega^\rightarrow(\mu))$. Hence W is surjective. Trivially, W is injective.

$$W(\omega^\leftarrow(\rho_1), \rho_1) \wedge W(\omega^\leftarrow(\rho_2), \rho_2)$$
$$= ((\omega^\leftarrow(\rho_1))^*, \rho_1) \wedge ((\omega^\leftarrow(\rho_2))^*, \rho_2)$$
$$= (\xi_\omega^\leftarrow(\rho_1), \rho_1) \wedge (\xi_\omega^\leftarrow(\rho_2), \rho_2)$$
$$= (\xi_\omega^\leftarrow(\rho_1 \wedge \rho_2), \rho_1 \wedge \rho_2)$$
$$= W(\omega^\leftarrow(\rho_1 \wedge \rho_2), \rho_1 \wedge \rho_2)$$
$$= W((\omega^\leftarrow(\rho_1), \rho_1) \vee (\omega^\leftarrow(\rho_2), \rho_2)).$$

$$W(\mu_1, \omega^\rightarrow(\mu_1)) \vee W(\mu_2, \omega^\rightarrow(\mu_2))$$
$$= (\mu_1^*, \xi_\omega^\rightarrow(\mu_1^*)) \vee (\mu_2^*, \xi_\omega^\rightarrow(\mu_2^*))$$
$$= (\mu_1^* \vee \mu_2^*, \xi_\omega^\rightarrow(\mu_1^* \vee \mu_2^*))$$
$$= W(\mu_1 \wedge \mu_2, \omega^\rightarrow(\mu_1 \wedge \mu_2))$$
$$= W((\mu_1, \omega^\rightarrow(\mu_1)) \wedge (\mu_2, \omega^\rightarrow(\mu_2))).$$

Then $\mathcal{F}(X, Y)$ and $\mathcal{H}(X, Y)$ are anti-isomorphic.
$\square$

Definition 2.8. Let $\mathcal{F}(X, Y) = \{(\mu, \rho) \in L^X \times L^Y\}$ (resp. $\mathcal{G}(X, Y)$ or $\mathcal{H}(X, Y)$) be a family of formal concepts(resp. attribute oriented concepts or object oriented concepts). Then:

(1) attribute $a \in Y$ is said to be dispensable if $\mathcal{F}(X, Y) \cong \mathcal{F}(X, Y - \{a\})$ (resp. $\mathcal{G}(X, Y) \cong \mathcal{G}(X, Y - \{a\})$ or $\mathcal{H}(X, Y) \cong \mathcal{H}(X, Y - \{a\}))$, otherwise attribute $a \in Y$ is said to be indispensable.

(2) attributes set $B \subset Y$ is said to be a consistent set if $\mathcal{F}(X, Y) \cong \mathcal{F}(X, B)$ (resp. $\mathcal{G}(X, Y) \cong \mathcal{G}(X, B)$ or $\mathcal{H}(X, Y) \cong \mathcal{H}(X, B))$. Furthermore, B is called otherwise attribute $a \in Y$ is said to be indispensable.

(3) $B \subset Y$ is called a reduct if B is a consistent set and $\mathcal{F}(X, Y) \not\cong \mathcal{F}(X, B - \{b\})$ (resp. $\mathcal{G}(X, Y) \not\cong \mathcal{G}(X, B - \{b\})$ or $\mathcal{H}(X, Y) \not\cong \mathcal{H}(X, B - \{b\})$) for each $b \in B$.

Example 2.9. Let $X = \{a, b\}$ and $Y = \{u, v, w\}$ be sets. Let $(L = \{0, \frac{1}{2}, 1\}, \odot)$ be a complete residuated lattice such that $x \odot y = (x + y - 1) \vee 1$, $x \to y = (1 - x + y) \wedge 1$, $x^* = 1 - x$. For $\lambda \in L^X$, we denote $(\lambda, \omega^\rightarrow(\lambda)) = ((\lambda(a), \lambda(b)), (\omega^\rightarrow(\lambda)(u), \omega^\rightarrow(\lambda)(v), \omega^\rightarrow(\lambda)(w))$, we define;

$$
\begin{array}{lll}
((0,0),(1,1,1)) & ((0,\tfrac{1}{2}),(\tfrac{1}{2},1,1)) & ((0,1),(0,1,\tfrac{1}{2})) \\
((\tfrac{1}{2},0),(\tfrac{1}{2},1,\tfrac{1}{2})) & ((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1,\tfrac{1}{2})) & ((\tfrac{1}{2},1),(0,1,\tfrac{1}{2})) \\
((1,0),(0,\tfrac{1}{2},0)) & ((1,\tfrac{1}{2}),(0,\tfrac{1}{2},0)) & ((1,1),(0,\tfrac{1}{2},0))
\end{array}
$$

Then $\omega^\rightarrow \in K(X, Y)$. By Theorem 1.7, we obtain

$$\omega^\leftarrow(\rho) = \bigvee \{\lambda \in L^X \mid \omega^\rightarrow(\lambda) \geq \rho\}$$

For $\rho \in L^Y$, we denote $(\rho, \omega^\leftarrow(\rho)) = ((\rho(u), \rho(v), \rho(w)), (\omega^\leftarrow(\rho)(a), \omega^\leftarrow(\rho)(b)))$,

$$
\begin{array}{lll}
((1,1,1),(0,0)) & ((1,1,\tfrac{1}{2}),(0,0)) & ((1,1,0),(0,0)) \\
((1,\tfrac{1}{2},1),(0,0)) & ((1,\tfrac{1}{2},\tfrac{1}{2}),(0,0)) & ((1,\tfrac{1}{2},0),(0,0)) \\
((1,0,1),(0,0)) & ((1,0,\tfrac{1}{2}),(0,0)) & ((1,0,0),(0,0)) \\
((\tfrac{1}{2},1,1),(0,\tfrac{1}{2})) & ((\tfrac{1}{2},1,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},1,0),(\tfrac{1}{2},\tfrac{1}{2})) \\
((\tfrac{1}{2},\tfrac{1}{2},1),(0,\tfrac{1}{2})) & ((\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},\tfrac{1}{2},0),(\tfrac{1}{2},\tfrac{1}{2})) \\
((\tfrac{1}{2},0,1),(0,\tfrac{1}{2})) & ((\tfrac{1}{2},0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},0,0),(\tfrac{1}{2},\tfrac{1}{2})) \\
((0,1,1),(0,\tfrac{1}{2})) & ((0,1,\tfrac{1}{2}),(\tfrac{1}{2},1)) & ((0,1,0),(\tfrac{1}{2},\tfrac{1}{2})) \\
((0,\tfrac{1}{2},1),(0,\tfrac{1}{2})) & ((0,\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1)) & ((0,\tfrac{1}{2},0),(1,1)) \\
((0,0,1),(0,\tfrac{1}{2})) & ((0,0,\tfrac{1}{2}),(\tfrac{1}{2},1)) & ((0,0,0),(1,1))
\end{array}
$$

We obtain a formal concept family as follows:

$$\mathcal{F}(X, Y) = \{((0,0),(1,1,1)), ((0,\tfrac{1}{2}),(\tfrac{1}{2},1,1)),$$
$$((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1,\tfrac{1}{2})), ((\tfrac{1}{2},1),(0,1,\tfrac{1}{2})), ((1,1),(0,\tfrac{1}{2},0))\}$$

$v \in Y$ is dispensable for $\mathcal{F}(X, Y)$ because $\mathcal{F}(X, Y) \cong \mathcal{F}(X, Y - \{v\})$. $B = \{u, w\}$ is a consistent set for $\mathcal{F}(X, Y)$ because $\mathcal{F}(X, Y) \cong \mathcal{F}(X, B)$ where

$$\mathcal{F}(X, B) = \{((0,0),(1,1)), ((0,\tfrac{1}{2}),(\tfrac{1}{2},1)),$$
$$((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})), ((\tfrac{1}{2},1),(0,\tfrac{1}{2})), ((1,1),(0,0))\}$$

Moreover, B is a reduct for $\mathcal{F}(X, Y)$ because B is a consistent set and $\mathcal{F}(X, Y) \not\cong \mathcal{F}(X, B - \{u\})$ and $\mathcal{F}(X, Y) \not\cong \mathcal{F}(X, B - \{w\})$.

For $\lambda \in L^X$, we denote $(\lambda, \phi_\omega^\rightarrow(\lambda)) = ((\lambda(a), \lambda(b)), (\phi_\omega^\rightarrow(\lambda)(u), \phi_\omega^\rightarrow(\lambda)(v), \phi_\omega^\rightarrow(\lambda)(w))$, we obtain;

$$
\begin{array}{lll}
((0,0),(0,0,0)) & ((0,\tfrac{1}{2}),(\tfrac{1}{2},0,0)) & ((0,1),(1,0,\tfrac{1}{2})) \\
((\tfrac{1}{2},0),(\tfrac{1}{2},0,\tfrac{1}{2})) & ((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},0,\tfrac{1}{2})) & ((\tfrac{1}{2},1),(1,0,\tfrac{1}{2})) \\
((1,0),(1,\tfrac{1}{2},1)) & ((1,\tfrac{1}{2}),(1,\tfrac{1}{2},1)) & ((1,1),(1,\tfrac{1}{2},1))
\end{array}
$$

For $\rho \in L^Y$, we denote $(\rho, \phi_\omega^\leftarrow(\rho)) = ((\rho(u), \rho(v), \rho(w)), (\phi_\omega^\leftarrow(\rho)(a), \phi_\omega^\leftarrow(\rho)(b)))$,

$$
\begin{array}{lll}
((0,0,0),(0,0)) & ((0,0,\tfrac{1}{2}),(0,0)) & ((0,0,1),(0,0)) \\
((0,\tfrac{1}{2},0),(0,0)) & ((0,\tfrac{1}{2},\tfrac{1}{2}),(0,0)) & ((0,\tfrac{1}{2},1),(0,0)) \\
((0,1,0),(0,0)) & ((0,1,\tfrac{1}{2}),(0,0)) & ((0,1,1),(0,0)) \\
((\tfrac{1}{2},0,0),(0,\tfrac{1}{2})) & ((\tfrac{1}{2},0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},0,1),(\tfrac{1}{2},\tfrac{1}{2})) \\
((\tfrac{1}{2},\tfrac{1}{2},0),(0,\tfrac{1}{2})) & ((\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},\tfrac{1}{2},1),(\tfrac{1}{2},\tfrac{1}{2})) \\
((\tfrac{1}{2},1,0),(0,\tfrac{1}{2})) & ((\tfrac{1}{2},1,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},1,1),(\tfrac{1}{2},\tfrac{1}{2})) \\
((1,0,0),(0,\tfrac{1}{2})) & ((1,0,\tfrac{1}{2}),(\tfrac{1}{2},1)) & ((1,0,1),(\tfrac{1}{2},\tfrac{1}{2})) \\
((1,\tfrac{1}{2},0),(0,\tfrac{1}{2})) & ((1,\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1)) & ((1,\tfrac{1}{2},1),(1,1)) \\
((1,1,0),(0,\tfrac{1}{2})) & ((1,1,\tfrac{1}{2}),(\tfrac{1}{2},1)) & ((1,1,1),(1,1))
\end{array}
$$

We obtain an attribute oriented concept family as follows:

$$\mathcal{G}(X, Y) = \{((0,0),(0,0,0)), ((0,\tfrac{1}{2}),(\tfrac{1}{2},0,0)),$$
$$((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},0,\tfrac{1}{2})), ((\tfrac{1}{2},1),(1,0,\tfrac{1}{2})), ((1,1),(1,\tfrac{1}{2},1))\}$$

$v \in Y$ is dispensable for $\mathcal{G}(X, Y)$ because $\mathcal{G}(X, Y) \cong \mathcal{G}(X, Y - \{v\})$. $B = \{u, w\}$ is a consistent set for $\mathcal{G}(X, Y)$ because $\mathcal{G}(X, Y) \cong \mathcal{G}(X, B)$ where

$$\mathcal{G}(X, B) = \{((0,0),(0,0)), ((0,\tfrac{1}{2}),(\tfrac{1}{2},0)),$$
$$((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})), ((\tfrac{1}{2},1),(1,\tfrac{1}{2})), ((1,1),(1,1))\}$$

Moreover, B is a reduct for $\mathcal{G}(X, Y)$ because B is a consistent set and $\mathcal{G}(X, Y) \not\cong \mathcal{G}(X, B - \{u\})$ and $\mathcal{G}(X, Y) \not\cong \mathcal{G}(X, B - \{w\})$.

We define $V : \mathcal{F}(X, Y) \to \mathcal{G}(X, Y)$ as follows:

$$
\begin{aligned}
V((0,0),(1,1,1)) &= ((0,0),(0,0,0)) \\
V((0,\tfrac{1}{2}),(\tfrac{1}{2},1,1)) &= ((0,\tfrac{1}{2}),(\tfrac{1}{2},0,0)) \\
V((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1,\tfrac{1}{2})) &= ((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},0,\tfrac{1}{2})) \\
V((\tfrac{1}{2},1),(0,1,\tfrac{1}{2})) &= ((\tfrac{1}{2},1),(1,0,\tfrac{1}{2})) \\
V((1,1),(0,\tfrac{1}{2},0)) &= ((1,1),(1,\tfrac{1}{2},1)).
\end{aligned}
$$

Then $\mathcal{F}(X, Y) \cong \mathcal{G}(X, Y)$.

For $\rho \in L^Y$, we denote $(\rho, \xi_\omega^\leftarrow(\rho)) = ((\rho(u), \rho(v), \rho(w)), (\xi_\omega^\leftarrow(\rho)(a), \xi_\omega^\leftarrow(\rho)(b)))$,

$$
\begin{array}{lll}
((0,0,0),(0,0)) & ((0,0,\tfrac{1}{2}),(\tfrac{1}{2},0)) & ((0,0,1),(1,\tfrac{1}{2})) \\
((0,\tfrac{1}{2},0),(0,0)) & ((0,\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},0)) & ((0,\tfrac{1}{2},1),(1,\tfrac{1}{2})) \\
((0,1,0),(\tfrac{1}{2},0)) & ((0,1,\tfrac{1}{2}),(\tfrac{1}{2},0)) & ((0,1,1),(1,\tfrac{1}{2})) \\
((\tfrac{1}{2},0,0),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},0,1),(1,\tfrac{1}{2})) \\
((\tfrac{1}{2},\tfrac{1}{2},0),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},\tfrac{1}{2},1),(1,\tfrac{1}{2})) \\
((\tfrac{1}{2},1,0),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},1,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})) & ((\tfrac{1}{2},1,1),(1,\tfrac{1}{2})) \\
((1,0,0),(1,1)) & ((1,0,\tfrac{1}{2}),(1,1)) & ((1,0,1),(1,1)) \\
((1,\tfrac{1}{2},0),(1,1)) & ((1,\tfrac{1}{2},\tfrac{1}{2}),(1,1)) & ((1,\tfrac{1}{2},1),(1,1)) \\
((1,1,0),(1,1)) & ((1,1,\tfrac{1}{2}),(1,1)) & ((1,1,1),(1,1))
\end{array}
$$

For $\lambda \in L^X$, we denote $(\lambda, \xi_\omega^\rightarrow(\lambda)) = ((\lambda(a), \lambda(b)), (\xi_\omega^\rightarrow(\lambda)(u), \xi_\omega^\rightarrow(\lambda)(v), \xi_\omega^\rightarrow(\lambda)(w))$, we obtain;

$$
\begin{array}{lll}
((0,0),(0,\tfrac{1}{2},0)) & ((0,\tfrac{1}{2}),(0,\tfrac{1}{2},0)) & ((0,1),(0,\tfrac{1}{2},0)) \\
((\tfrac{1}{2},0),(0,1,\tfrac{1}{2})) & ((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1,\tfrac{1}{2})) & ((\tfrac{1}{2},1),(\tfrac{1}{2},1,\tfrac{1}{2})) \\
((1,0),(0,1,\tfrac{1}{2})) & ((1,\tfrac{1}{2}),(\tfrac{1}{2},1,1)) & ((1,1),(1,1,1))
\end{array}
$$

We obtain an object oriented concept family as follows:

$$\mathcal{H}(X,Y) = \{((0,0),(0,\tfrac{1}{2},0)),((\tfrac{1}{2},0),(0,1,\tfrac{1}{2})),$$
$$((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1,\tfrac{1}{2})),((1,\tfrac{1}{2}),(\tfrac{1}{2},1,1)),((1,1),(1,1,1))\}$$

$v \in Y$ is dispensable for $\mathcal{H}(X,Y)$ because $\mathcal{H}(X,Y) \cong \mathcal{H}(X,Y - \{v\})$. $B = \{u,w\}$ is a consistent set for $\mathcal{H}(X,Y)$ because $\mathcal{H}(X,Y) \cong \mathcal{H}(X,B)$ where

$$\mathcal{H}(X,B) = \{((0,0),(0,0)),((\tfrac{1}{2},0),(0,\tfrac{1}{2})),$$
$$((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})),((1,\tfrac{1}{2}),(\tfrac{1}{2},1)),((1,1),(1,1))\}$$

Moreover, B is a reduct for $\mathcal{H}(X,Y)$ because B is a consistent set and $\mathcal{H}(X,Y) \not\cong \mathcal{H}(X,B-\{u\})$ and $\mathcal{H}(X,Y) \not\cong \mathcal{H}(X,B-\{w\})$.

We define $W : \mathcal{F}(X,Y) \to \mathcal{H}(X,Y)$ as follows:

$$W((0,0),(1,1,1)) = ((1,1),(1,1,1))$$
$$W((0,\tfrac{1}{2}),(\tfrac{1}{2},1,1)) = ((1,\tfrac{1}{2}),(\tfrac{1}{2},1,1))$$
$$W((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1,\tfrac{1}{2})) = ((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1,\tfrac{1}{2}))$$
$$W((\tfrac{1}{2},1),(0,1,\tfrac{1}{2})) = ((\tfrac{1}{2},0),(0,1,\tfrac{1}{2}))$$
$$W((1,1),(0,\tfrac{1}{2},0)) = ((1,1),(0,\tfrac{1}{2},0)).$$

Then $\mathcal{F}(X,Y) \cong \mathcal{H}(X,Y)$.

We define $U : \mathcal{G}(X,Y) \to \mathcal{H}(X,Y)$ as follows:

$$U((0,0),(0,0,0)) = ((1,1),(1,1,1))$$
$$U((0,\tfrac{1}{2}),(\tfrac{1}{2},0,0)) = ((1,\tfrac{1}{2}),(\tfrac{1}{2},1,1))$$
$$U((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},0,\tfrac{1}{2})) = ((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1,\tfrac{1}{2}))$$
$$U((\tfrac{1}{2},1),(1,0,\tfrac{1}{2})) = ((\tfrac{1}{2},0),(0,1,\tfrac{1}{2}))$$
$$U((1,1),(1,\tfrac{1}{2},1)) = ((1,1),(0,\tfrac{1}{2},0)).$$

Then $\mathcal{G}(X,Y) \simeq \mathcal{H}(X,Y)$.

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