The Korean Journal of Applied Statistics (2010)
23(4), 621–633

On Pricing Equity-Linked Investment Products with a Threshold Expense Structure

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(Received April 2010; accepted April 2010)

Abstract

This paper considers a certain expense structure where a vendor of equity-linked investment product will collect its expenses continuously from the investor's account whenever the investment performance exceeds a certain threshold level. Under the Black-Scholes framework, we derive compact convolution formulas for evaluating the total expenses to be collected during the investment period by using the joint Laplace transform of the Brownian motion and its excursion time. We provide numerical examples for illustration.

Keywords: Threshold expense structure, joint Laplace transform, Brownian motion, excursion time.

1. Introduction

In this paper we consider a new method of charging expenses on equity-linked investment products such as mutual funds or variable annuities. Under the new system, which we call a *threshold expense structure*, the expenses are supposed to be withdrawn continuously from the investor's account only when the investment performance exceeds a prespecified threshold level. To fix idea, we denote by $S(t)$ an index of the investment performance at time t or the account value at time t if it were not for any expense. And let $F(t)$ denote the *net account value* at time t after deducting the threshold expenses. The threshold expense structure stipulates that a vendor of the investment product will collect expenses at a continuous rate of c whenever $S(t)$ exceeds the threshold level K. An explicit link between the two processes is given by the relation,

$$F(t) = S(t) \exp\left\{ -c \int_0^t \mathrm{I}\left(S(u) > K\right) du \right\}, \tag{1.1}$$

where $\mathrm{I}(\cdot)$ is an indicator function. In Figure 1, we illustrate a sample path of $S(t)$ and $F(t)$ when $S(0) = F(0) = 100$ and $K = 110$. It is observable from the figure that the discrepancy between the two processes, due to the threshold expense structure, becomes larger as the so-called *excursion time* (or the occupation time) of $S(t)$ staying above K becomes longer.

A motivation behind the threshold expense structure is that, in many cases, investors feel it intolerable to pay expenses for their losing-money investment accounts and our marketing idea of

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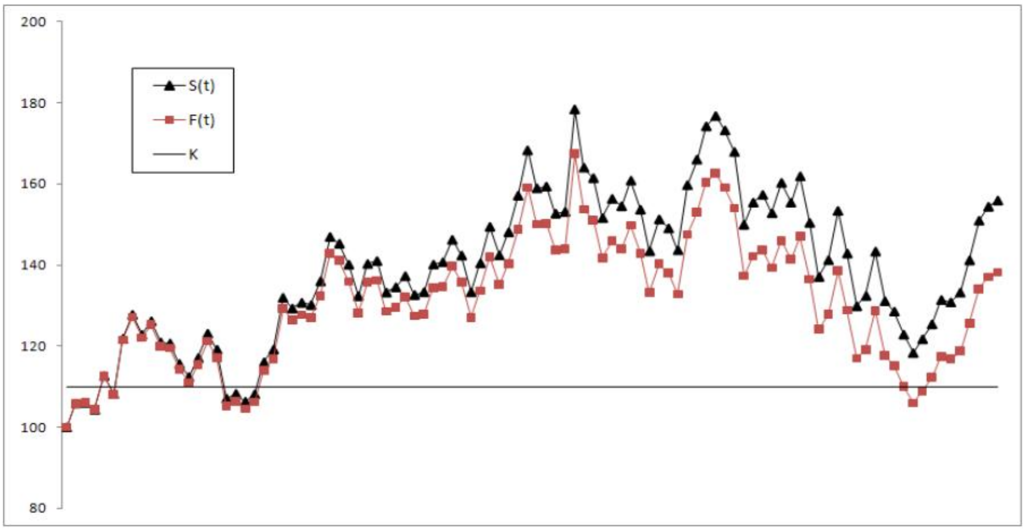

Figure 1.1. A sample path of $S(t)$ and $F(t)$ when $S(0) = F(0) = 100$ and $K = 110$

charging expenses according to the investment performance level might sound appealing to them. The relevant questions then are how one should evaluate the total expenses to be collected for a given pair of (K, c) or how one can find the equivalent pairs of (K, c) for a given target amount of expense. To answer these questions, we derive pricing formulas by employing the risk neutral valuation method. The evaluation strategy is simple: under the Black-Scholes (1973) framework, we assume that $S(t)$ follows a geometric Brownian motion, *i.e.*,

$$S(t) = S(0) \exp\{\mu t + \sigma W(t)\}, \tag{1.2}$$

where $\{W(t)\}$ is a standard Brownian motion, and μ and σ are constants, $\sigma > 0$. Then, we can evaluate the total expenses to be collected during the investment periods as the discounted expectation of the discrepancy between the two processes $S(t)$ and $F(t)$ at maturity T under the risk-neutral measure or the equivalent martingale measure Q.

To those familiar with the classical ruin theory, our problem of pricing the threshold expenses might be reminiscent of an optimal dividend strategy problem. For example, Gerber and Shiu (2006a, 2006b) considered a threshold strategy where dividends are paid at a constant rate whenever the surplus process is above a certain threshold level. In some sense, our problem could be thought of as reconstructing this ruin problem from the perspective of financial pricing. That is, dividend payout (or surplus) corresponds to expense withdrawal (or investment performance). However, there exists a fundamental difference between the two problems: in the ruin problem, the main purpose is to find an optimal dividend barrier in order to maximize the expected discounted dividends until it ruins; in our problem, to evaluate the total expenses to be collected for a fixed duration of investment. Thus the mathematics used are quite different.

In the derivation of our pricing formulas, we exploit some known results (Hugonnier, 1999; Borodin and Salminen, 2002) about the joint Laplace transform of a standard Brownian motion and its excursion time. The readers are referred to Jeanblanc *et al.* (1997, 2009) where the derivation of the joint pdf of Brownian motion and its excursion time or several useful expressions for the

Feynman-Kac formulas of processes associated with Brownian motion can be found. For a more general treatment of these topics, we refer the reader to Karatzas and Shreve (1991) and Shiryayev (1999) among others.

The paper is organized as follows: in Section 2, we introduce some basic preliminaries and reformulate the pricing problem into the one to which the known results about the joint Laplace transform are readily applicable; in Sections 3 and 4, we derive the pricing formulas for a constant threshold or an exponentially growing threshold; in Section 5, we numerically illustrate the results; and, in Appendix, we outline the straightforward proof of proposition 3.2.

2. Modeling Framework

Let us first consider a filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathrm{P}\}$, where the filtration is generated by a standard Brownian motion $\{W(t), t \geq 0\}$. As in (1.2), we assume that $S(t)$ follows a geometric Brownian motion under the physical measure P. We also assume that the continuously compounded risk-free interest rate, r, is constant and that all dividends from the underlying investment product are reinvested. With

$$B(t) = W(t) - \frac{r - \sigma^2/2 - \mu}{\sigma} t,$$

(1.2) becomes

$$S(t) = S(0) \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma B(t) \right\}. \tag{2.1}$$

It is well-known that, under the risk-neutral measure Q, the stochastic process $\{B(t)\}$ is a standard Brownian motion and the discounted process $\{e^{-rt}S(t)\}$ is a martingale. See, *e.g.*, Shreve (2004). We apply the Girsanov theorem to change the measure Q to an equivalent measure $\widetilde{\mathrm{Q}}$ on the filtration $\{\mathcal{F}_t\}$. By denoting $\widetilde{B}(t) = B(t) + \nu t$ and $\nu = \sigma^{-1}(r - \sigma^2/2)$, the process $S(t)$ of (2.1) can be written as

$$S(t) = S(0) \exp\left\{ \sigma \widetilde{B}(t) \right\}. \tag{2.2}$$

The Radon-Nikodym derivative is given by

$$\left. \frac{d\mathrm{Q}}{d\widetilde{\mathrm{Q}}} \right|_t := M(t) = \exp\left\{ \nu B(t) + \frac{\nu^2}{2} t \right\}.$$

Then $\{M(t)\}$ is a $\widetilde{\mathrm{Q}}$-martingale and $\{\widetilde{B}(t)\}$ is a standard Brownian motion under the transformed measure $\widetilde{\mathrm{Q}}$.

Let us denote the excursion time of a standard Brownian motion $W(t)$ above level l by

$$\Lambda(t; l) = \int_0^t \mathrm{I}(W(u) > l) du.$$

Then, by using relation (2.2) and letting $k = \sigma^{-1} \log(K/S(0))$, the net account value at time t in (1.1) can be rewritten as

$$F(t) = S(0) \exp\left\{ \sigma \widetilde{B}(t) - c \int_0^t \mathrm{I}\left(S(u) > K\right) du \right\}. \tag{2.3}$$

Now we apply the fundamental theorem of asset pricing. That is, by calculating the discounted payoff from the net account at maturity T under the risk-neutral measure Q, we can evaluate the net account value at time t as follows:

$$\begin{aligned}\Theta(t;K,c) &= \mathrm{E}_t^{\mathrm{Q}}\left[e^{-r(T-t)}F(T)\right] \\ &= \frac{\mathrm{E}_t^{\widetilde{\mathrm{Q}}}\left[e^{-r(T-t)}F(T)M(T)\right]}{\mathrm{E}_t^{\widetilde{\mathrm{Q}}}[M(T)]} \\ &= S(0)\exp\left\{-\left(r+\frac{\nu^2}{2}\right)(T-t)-\nu\widetilde{B}(t)\right\}\mathrm{E}_t^{\widetilde{\mathrm{Q}}}\left[\exp\left\{(\sigma+\nu)\widetilde{B}(T)-c\Lambda(T;k)\right\}\right], \quad (2.4)\end{aligned}$$

where $\mathrm{E}_t[\cdot]$ represents the conditional expectation operator with respect to $\mathcal{F}_t$.

Note that the process $\Lambda(t;k)$ does not have the Markov property but the joint process $(\widetilde{B}(t),\Lambda(t;k))$ does (*cf.* Meyer, 2001). Hence, denoting the joint Laplace transform of $(\widetilde{B}(T)-\widetilde{B}(t),\ \Lambda(T-t;k-\widetilde{B}(t)))$ under the measure $\widetilde{\mathrm{Q}}$ by

$$L\left[\alpha,\beta;T-t,k-\widetilde{B}(t)\right]=\mathrm{E}_t^{\widetilde{\mathrm{Q}}}\left[\exp\left\{-\alpha\left(\widetilde{B}(T)-\widetilde{B}(t)\right)-\beta\Lambda\left(T-t;k-\widetilde{B}(t)\right)\right\}\right],$$

we can express (2.4) as

$$\Theta\left(t;K,c\right)=\theta(t,T)L\left[-(\sigma+\nu),c;T-t,k-\widetilde{B}(t)\right]. \quad (2.5)$$

Here, $\theta(t,T)=S(0)\exp\{-(r+\nu^2/2)(T-t)+\sigma\widetilde{B}(t)-c\int_0^t \mathrm{I}(\widetilde{B}(u)>k)du\}$ and we slightly abuse the notation by writing $\Lambda(T-t;k-\widetilde{B}(t))=\int_0^{T-t}\mathrm{I}(\widetilde{B}(t+u)-\widetilde{B}(t)>k-\widetilde{B}(t))$ under the conditional expectation operator. Now, reapplying the fundamental theorem of asset pricing, we can evaluate the time-t price of the total expenses to be collected during the time period between t and T as the discounted expectation of $S(T)-F(T)$ under the Q-measure, *i.e.*,

$$\mathrm{E}_t^{\mathrm{Q}}\left[e^{-r(T-t)}\left\{S(T)-F(T)\right\}\right]=S(t)-\Theta\left(t;K,c\right). \quad (2.6)$$

Therefore, our pricing problem reduces to finding an expression for $\Theta\left(t;K,c\right)$ or the joint Laplace transform $L[\alpha,\beta;T-t,k-\widetilde{B}(t)]$.

3. A Constant Threshold Case

In this section, we derive an expression for $\Theta\left(t;K,c\right)$ in convolution form by using the joint Laplace transform of a standard Brownian motion $W(t)$ and its excursion time above level l, $\Lambda(t;l)$.

First, we consider the case when the level l is strictly positive. Note that the joint Laplace transform has a form of the Feynman-Kac formula. In determining the distribution of certain Wiener functionals, Kac (1949) showed the following result: For a Borel function $h:\mathrm{R}\to\mathrm{R}^+$ and $\eta>0$,

$$\int_0^\infty e^{-\eta t}\mathrm{E}\left[h(W(t))\exp\left\{-\lambda\Lambda(t;l)\right\}\right]dt=\int_{-\infty}^\infty h(y)u\left(\eta,\lambda,y\right)dy, \quad (3.1)$$

where $u(x)=u(\eta,\lambda,x)$ is the unique solution of the Sturm-Liouville equation

$$\frac{1}{2}u''(x)=\left(\eta+\lambda\mathrm{I}(x>l)\right)u(x), \quad x\neq 0, \quad (3.2)$$

provided that $u'(x)$ exists and is locally bounded for all $x \neq 0$, and that $u(x)$ vanishes at $\pm\infty$ and that $u'(0+) - u'(0-) = -2$ (*cf.* Theorem 2 of Hugonnier, 1999). Then, considering a Borel function $h(y) = \mathrm{I}(y \in [x, x+dx))$ in (3.1), we obtain the Laplace transform,

$$\begin{aligned} L[\eta] &= \int_0^\infty e^{-\eta t} \mathrm{E}\left[\mathrm{I}\left(W(t) \in [x, x+dx)\right) \exp\left\{-\lambda \Lambda(t; l)\right\}\right] dt \\ &= \int_{-\infty}^\infty \mathrm{I}\left(y \in [x, x+dx)\right) u\left(\eta, \lambda, y\right) dy = u\left(\eta, \lambda, x\right) dx \end{aligned}$$

and the inverse Laplace transform,

$$L_\eta^{-1}\left[u(\eta, \lambda, x) dx\right] = \mathrm{E}\left[\mathrm{I}\left(W(t) \in [x, x+dx)\right) \exp\left\{-\lambda \Lambda(t; k)\right\}\right]. \tag{3.3}$$

The solution to the Sturm-Liouville equation of (3.2) is given as follows.

Lemma 3.1. *Suppose $l \geq 0$, $\lambda \geq 0$ and $\eta \geq 0$. Then,*

$$\begin{aligned} L[\eta] &= u(\eta, \lambda, x) dx \\ &= \begin{cases} \dfrac{\sqrt{2}}{\sqrt{\eta} + \sqrt{\eta+\lambda}} e^{-(x-l)\sqrt{2(\eta+\lambda)}} e^{-l\sqrt{2\eta}} dx, & (x \geq l), \\ \dfrac{\sqrt{\eta} - \sqrt{\eta+\lambda}}{\sqrt{2\eta}\left(\sqrt{\eta} + \sqrt{\eta+\lambda}\right)} e^{-(2l-x)\sqrt{2\eta}} dx + \dfrac{1}{\sqrt{2\eta}} e^{-|x|\sqrt{2\eta}} dx, & (x \leq l). \end{cases} \end{aligned} \tag{3.4}$$

Proof. See Proposition 5 of Hugonnier (1999). □

In what follows, the convolution operator is defined as

$$f(t) * g(t) = \int_0^t f(s) g(t-s) ds = \int_0^t f(t-s) g(s) ds$$

or, following the usual convention, it could be treated as $f(t) * g(t) = \int_{-\infty}^\infty f(s) g(t-s) ds$ but then we would need an extra restriction that f, g are supported on the positive real numbers.

Proposition 3.1. *(Eq. 1.4.7 of Borodin and Salminen, 2002) Suppose $l > 0$ and $\lambda \geq 0$. Then,*

$$\begin{aligned} &\mathrm{E}\left[\mathrm{I}\left(W(t) \in [x, x+dx)\right) \exp\left\{-\lambda \Lambda(t; l)\right\}\right] \\ &= \begin{cases} \dfrac{(x-l)l}{\lambda(2\pi)^{\frac{3}{2}}} \left(\dfrac{1 - e^{-\lambda t}}{t^{\frac{3}{2}}}\right) * \left(\dfrac{e^{-\lambda t}}{t^{\frac{3}{2}}} e^{-\frac{(x-l)^2}{2t}}\right) * \left(\dfrac{1}{t^{\frac{3}{2}}} e^{-\frac{l^2}{2t}}\right) dx, & (x > l), \\ \dfrac{1}{\sqrt{2\pi t}} \left(e^{-\frac{x^2}{2t}} - e^{-\frac{(2l-x)^2}{2t}}\right) dx + \dfrac{2l - x}{2\pi\lambda} \left(\dfrac{1 - e^{-\lambda t}}{t^{\frac{3}{2}}}\right) * \left(\dfrac{1}{t^{\frac{3}{2}}} e^{-\frac{(2l-x)^2}{2t}}\right) dx, & (x \leq l). \end{cases} \end{aligned} \tag{3.5}$$

The formula (3.5) appears in Borodin and Salminen (2002) without proof, so we briefly outline the proof below.

Proof. Consider the following Laplace transform:

$$\begin{aligned} &\int_0^\infty e^{-\eta t} \left(\frac{1 - e^{-\lambda t}}{t^{\frac{3}{2}}}\right) dt = 2\sqrt{\pi}\left(\sqrt{\eta+\lambda} - \sqrt{\eta}\right) \\ &\int_0^\infty e^{-\eta t} \left(\frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}}\right) dt = \frac{1}{\sqrt{2\eta}} e^{-a\sqrt{2\eta}}, \quad (a \geq 0) \\ &\int_0^\infty e^{-\eta t} \left(\frac{a}{\sqrt{2\pi}} \frac{e^{-\frac{a^2}{2t}}}{t^{\frac{3}{2}}}\right) dt = e^{-a\sqrt{2\eta}}, \quad (a \geq 0). \end{aligned} \tag{3.6}$$

Rearranging the terms of (3.4) according to the Laplace transform formulas of (3.6) and then applying the inverse Laplace transform given in (3.3) establish the proposition. □

By performing a series of tedious integrations, we can obtain the following joint Laplace transform of a standard Brownian motion and its excursion time.

Proposition 3.2. *Suppose $l > 0$ and $\lambda \geq 0$. Then,*

$$\begin{aligned} L[\alpha, \lambda; t, l] &= E\left[\exp\{-\alpha W(t) - \lambda\Lambda(t; l)\}\right] \\ &= \int_{-\infty}^{\infty} e^{-\alpha x} \mathrm{E}\left[\mathrm{I}\left(W(t) \in [x, x+dx)\right) \exp\{-\lambda\Lambda(t; l)\}\right] \\ &= \left(F_1 * g_1 * h_1\right)(t) + \left(K_1 * g_1\right)(t) + J_1(t), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} F_1(t; \alpha, \lambda, l) &= l e^{-\alpha l - \lambda t}\left\{\frac{1}{\sqrt{t}} - \sqrt{2\pi}\alpha e^{\frac{\alpha^2 t}{2}} \Phi\left(-\alpha\sqrt{t}\right)\right\}, \\ K_1(t; \alpha, \lambda, l) &= \sqrt{2\pi} e^{-\alpha l}\left\{\frac{e^{-\frac{l^2}{2t}}}{\sqrt{t}} + \sqrt{2\pi}\alpha e^{\frac{\alpha^2 t}{2} - \alpha l} \Phi\left(\frac{\alpha t - l}{\sqrt{t}}\right)\right\}, \\ J_1(t; \alpha, \lambda, l) &= e^{\frac{\alpha^2 t}{2}}\left\{\Phi\left(\frac{\alpha t + l}{\sqrt{t}}\right) - e^{-2\alpha l} \Phi\left(\frac{\alpha t - l}{\sqrt{t}}\right)\right\}, \\ g_1(t; \lambda) &= \frac{1 - e^{-\lambda t}}{\lambda (2\pi t)^{\frac{3}{2}}} \\ h_1(t; l) &= \frac{e^{-\frac{l^2}{2t}}}{t^{\frac{3}{2}}} \end{aligned}$$

and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function.

Proof. See Appendix. □

Here we deliberately omitted the arguments of F, K, J, g and h for notational simplicity.

Next we consider the case when the threshold level l is negative. Because this case can be treated similarly as in the positive threshold case, we only state the corresponding results without proof.

Lemma 3.2. *Suppose $l \leq 0$, $\lambda \geq 0$ and $\eta \geq 0$. Then,*

$$\begin{aligned} L[\eta] &= u(\eta, \lambda, x) dx \\ &= \begin{cases} \dfrac{\sqrt{\eta+\lambda} - \sqrt{\eta}}{\sqrt{2(\eta+\lambda)}\left(\sqrt{\eta+\lambda} + \sqrt{\eta}\right)} e^{(2l - x)\sqrt{2(\eta+\lambda)}} dx + \dfrac{1}{\sqrt{2(\eta+\lambda)}} e^{-|x|\sqrt{2(\eta+\lambda)}} dx, & (x \geq l), \\ \dfrac{\sqrt{2}}{\sqrt{\eta} + \sqrt{\eta+\lambda}} e^{(x-l)\sqrt{2\eta}} e^{l\sqrt{2(\eta+\lambda)}} dx, & (x \leq l). \end{cases} \end{aligned} \tag{3.8}$$

Proposition 3.3. *(Eq. 1.4.7 of Borodin and Salminen, 2002) Suppose $l < 0$ and $\lambda \geq 0$. Then,*

$$\mathrm{E}\left[\mathrm{I}\left(W(t) \in [x, x+dx)\right) \exp\{-\lambda\Lambda(t; l)\}\right]$$

$$= \begin{cases} \dfrac{e^{-\lambda t}}{\sqrt{2\pi t}}\left(e^{-\frac{x^2}{2t}} - e^{-\frac{(2l-x)^2}{2t}}\right)dx + \dfrac{x-2l}{2\pi\lambda}\left(\dfrac{1-e^{-\lambda t}}{t^{\frac{3}{2}}}\right)*\left(\dfrac{e^{-\lambda t}}{t^{\frac{3}{2}}}e^{-\frac{(2l-x)^2}{2t}}\right)dx, & (x \ge l), \\ \dfrac{(x-l)l}{\lambda(2\pi)^{\frac{3}{2}}}\left(\dfrac{1-e^{-\lambda t}}{t^{\frac{3}{2}}}\right)*\left(\dfrac{e^{-\lambda t}}{t^{\frac{3}{2}}}e^{-\frac{l^2}{2t}}\right)*\left(\dfrac{1}{t^{\frac{3}{2}}}e^{-\frac{(l-x)^2}{2t}}\right)dx, & (x < l). \end{cases} \tag{3.9}$$

Proposition 3.4. *Suppose $l < 0$ and $\lambda \ge 0$. Then,*

$$L\left[\alpha, \lambda; t, l\right] = \left(F_2 * g_2 * h_2\right)(t) + \left(K_2 * g_2\right)(t) + J_2(t), \tag{3.10}$$

where

$$\begin{aligned} F_2(t;\alpha,\lambda,l) &= -le^{-\alpha l}\left\{\frac{1}{\sqrt{t}} + \sqrt{2\pi}\alpha e^{\frac{\alpha^2 t}{2}}\Phi\left(\alpha\sqrt{t}\right)\right\}, \\ K_2(t;\alpha,\lambda,l) &= \sqrt{2\pi}e^{-\alpha l-\lambda t}\left\{\frac{e^{-\frac{l^2}{2t}}}{\sqrt{t}} - \sqrt{2\pi}\alpha e^{\frac{\alpha^2 t}{2}-\alpha l}\Phi\left(\frac{l-\alpha t}{\sqrt{t}}\right)\right\}, \\ J_2(t;\alpha,\lambda,l) &= e^{\frac{\alpha^2 t}{2}-\lambda t}\left\{\Phi\left(\frac{-\alpha t - l}{\sqrt{t}}\right) - e^{-2\alpha l}\Phi\left(\frac{-\alpha t + l}{\sqrt{t}}\right)\right\}, \\ g_2(t;\lambda) &= \frac{1-e^{-\lambda t}}{\lambda\left(2\pi t\right)^{\frac{3}{2}}}, \\ h_2(t;\lambda,l) &= \frac{e^{-\lambda t}}{t^{\frac{3}{2}}}e^{-\frac{l^2}{2t}}. \end{aligned}$$

Finally, when $l = 0$, we can solve the problem similarly by inverting the Laplace transform in Lemma 3.1 or Lemma 3.2. Thus we obtain the following.

Proposition 3.5. *(Eq. 1.4.7 of Borodin and Salminen, 2002) Suppose $l = 0$ and $\lambda \ge 0$. Then,*

$$\begin{aligned} &\mathrm{E}\left[\mathrm{I}\left(W(t) \in [x, x+dx)\right)\exp\left\{-\lambda\Lambda(t;l)\right\}\right] \\ &= \begin{cases} \dfrac{x}{2\pi\lambda}\left(\dfrac{1-e^{-\lambda t}}{t^{\frac{3}{2}}}\right)*\left(\dfrac{e^{-\lambda t}}{t^{\frac{3}{2}}}e^{-\frac{x^2}{2t}}\right)dx, & (x > l), \\ \dfrac{-x}{2\pi\lambda}\left(\dfrac{1-e^{-\lambda t}}{t^{\frac{3}{2}}}\right)*\left(\dfrac{1}{t^{\frac{3}{2}}}e^{-\frac{x^2}{2t}}\right)dx, & (x < l), \\ \dfrac{1-e^{-\lambda t}}{\sqrt{2\pi}\lambda t^{\frac{3}{2}}}dx, & (x = l). \end{cases} \end{aligned} \tag{3.11}$$

Proposition 3.6. *Suppose $l = 0$ and $\lambda \ge 0$. Then,*

$$L\left[\alpha, \lambda; t, l\right] = \left(K_1 * g_1\right)(t) + \left(K_2 * g_1\right)(t). \tag{3.12}$$

Here the functions K_1, K_2 and g_1 are defined as in Propositions 3.2 and 3.4.

Now we combine the results of this section with (2.6) to obtain the following expression for the time-t value of the net account:

$$\begin{aligned} \Theta\left(t; K, c\right) &= \theta(t,T)L\left[-(\sigma+\nu), c; T-t, k-\widetilde{B}(t)\right] \\ &= \theta\left(t,T\right)\times\begin{cases} (F_1 * g_1 * h_1)(T-t) + (K_1 * g_1)(T-t) + J_1(T-t), & \text{if } S(t) < K, \\ (F_2 * g_2 * h_2)(T-t) + (K_2 * g_1)(T-t) + J_2(T-t), & \text{if } S(t) > K, \\ (K_1 * g_1)(T-t) + (K_2 * g_1)(T-t), & \text{if } S(t) = K, \end{cases} \end{aligned} \tag{3.13}$$

where $\theta(t,T) = S(0)\exp\{-(r+\nu^2/2)(T-t) + \sigma\widetilde{B}(t) - c\int_0^t \mathrm{I}(\widetilde{B}(u) > k)du\}$.

Remark 3.1.

(i) In some extreme cases, one may check the validity of (3.13) easily. For instance, if $K = \infty$, there would be no expenses to be collected. This, in turn, implies that

$$\Theta(t = 0; K = \infty, c) = S(0) = F(0).$$

It can be immediately shown that (3.13) satisfies this. That is, in this case, the first two terms of (3.13), $F_1 * g_1 * h_1$ and $K_1 * g_1$, vanish and the last term reduces to $S(0)$.

(ii) By using the relation (2.2), $\theta(t, T)$ can be expressed in terms of the process $\{S(t)\}$ or $\{F(t)\}$ as

$$\begin{aligned}\theta(t,T) &= S(t) \exp\left\{-\left(r+\frac{\nu^2}{2}\right)(T-t) - c\int_0^t \mathrm{I}\left(S(u) > K\right) du\right\}\\ &= F(t)\exp\left\{-\left(r+\frac{\nu^2}{2}\right)(T-t)\right\},\end{aligned}$$

which is a function of observables as of time t.

4. An Exponentially Growing Threshold Case

In parallel with Sections 2 and 3, we derive the pricing formulas for the total expenses when the threshold level is an exponentially growing function of time. In this case, the link between $S(t)$ and $F(t)$ is given by

$$F(t) = S(t)\exp\left\{-c\int_0^t \mathrm{I}\left(S(u) > S(0)e^{\beta u}\right) du\right\}, \tag{4.1}$$

for some constant $\beta > 0$. Taking the time value of money into consideration, investors might think this expense structure of (4.1) more attractive than the previous one.

As in Section 2, we apply the Girsanov theorem to change measure from the risk neutral measure Q to an equivalent measure $\hat{\mathrm{Q}}$. Denoting $\hat{B}(t) = B(t) + \omega t$ and $\omega = \sigma^{-1}(r - \sigma^2/2 - \beta)$, we can rewrite the process $S(t)$ of (2.1) as

$$S(t) = S(0)\exp\left\{\sigma\hat{B}(t)\right\}. \tag{4.2}$$

The Radon-Nikodym derivative is given by

$$\left.\frac{d\mathrm{Q}}{d\hat{\mathrm{Q}}}\right|_t := N(t) = \exp\left\{\omega B(t) + \frac{\omega^2}{2}t\right\}.$$

Then $\{N(t)\}$ is a $\hat{\mathrm{Q}}$-martingale and $\{\hat{B}(t)\}$ is a standard Brownian motion under the transformed measure $\hat{\mathrm{Q}}$.

Changing the relation (4.1) into the form of (2.3), we get

$$F(t) = S(0)e^{\beta t}\exp\left\{\sigma\hat{B}(t) - c\int_0^t \mathrm{I}\left(\hat{B}(u) > 0\right) du\right\}.$$

Now it follows from the fundamental theorem of asset pricing that the time-t value of the net account of (4.1) is calculated as

$$\Psi(t;\beta,c) = \mathrm{E}_t^{\mathrm{Q}}\left[e^{-r(T-t)}F(T)\right]$$

$$= \frac{\mathrm{E}_t^{\hat{\mathrm{Q}}}\left[e^{-r(T-t)}F(T)N(T)\right]}{\mathrm{E}_t^{\hat{\mathrm{Q}}}[N(T)]}$$
$$= \psi(t,T)L\left[-(\sigma+\omega), c; T-t, -\hat{B}(t)\right]. \tag{4.3}$$

Here

$$\psi(t,T) = S(0)\exp\left\{\beta T - \left(r+\frac{\omega^2}{2}\right)(T-t) + \sigma\hat{B}(t) - c\int_0^t \mathrm{I}\left(\hat{B}(u)>0\right)du\right\}$$

or

$$\psi(t,T) = S(t)\exp\left\{\beta(T-t) - \left(r+\frac{\omega^2}{2}\right)(T-t) - c\int_0^t \mathrm{I}\left(\hat{B}(u)>0\right)du\right\}$$
$$= F(t)\exp\left\{\beta(T-t) - \left(r+\frac{\omega^2}{2}\right)(T-t)\right\}$$

and $L[-(\sigma+\omega), c; T-t, -\hat{B}(t)]$ denotes the joint Laplace transform as before, *i.e.*,

$$L[-(\sigma+\omega), c; T-t, -\hat{B}(t)] = \mathrm{E}_t^{\hat{\mathrm{Q}}}\left[\exp\left\{(\sigma+\omega)\left(\hat{B}(T)-\hat{B}(t)\right) - c\Lambda\left(T-t; -\hat{B}(t)\right)\right\}\right].$$

Now, combining (4.3) with the joint Laplace transform formulas from Section 3, we obtain the following pricing formula for the net account value at time t:

$$\begin{aligned}
&\Psi(t;\beta,c)\\
&= \psi(t,T)L\left[-(\sigma+\omega), c; T-t, -\hat{B}(t)\right]\\
&= \psi(t,T)\times\begin{cases}(F_1 * g_1 * h_1)(T-t) + (K_1 * g_1)(T-t) + J_1(T-t), & \text{if } S(t) < S(0)e^{\beta t},\\ (F_2 * g_1 * h_2)(T-t) + (K_2 * g_1)(T-t) + J_2(T-t), & \text{if } S(t) > S(0)e^{\beta t},\\ (K_1 * g_1)(T-t) + (K_2 * g_1)(T-t), & \text{if } S(t) = S(0)e^{\beta t}.\end{cases}
\end{aligned} \tag{4.4}$$

As in (2.6), the time-t price of the total expenses to be collected for an exponentially growing threshold can be calculated as

$$\mathrm{E}_t^{\mathrm{Q}}\left[e^{-r(T-t)}\left\{S(T)-F(T)\right\}\right] = S(t) - \Psi(t;\beta,c). \tag{4.5}$$

Remark 4.1. Because the case when $\beta = 0$ is equivalent to the one with $K = S(0)$, we may verify the formula (4.4) through (3.13).

5. Numerical Examples

This section numerically illustrates our previous results. In the following examples, we have assumed that $r = 5\%$ and $S(0) = F(0) = 100$ and used numerical integrations (NIntegrate in Mathematica 6.0) to compute the convolution formulas of (3.13) and (4.4). First, we consider the case where the threshold level is a fixed constant. Table 5.1 shows the numbers calculated from (3.13) for various combinations of K, T, c and σ. As can be expected, the total expenses are an increasing function of both T and c but a decreasing function of K. However, the effect of the volatility on the net account (or the total expenses) is not immediately clear. Notice that, when $K = 110$, the time-0 values of the net account decrease as the volatility increases from 20% to 30%. But, when

Table 5.1. The time-0 values of net account (total expenses): A constant threshold case, $r = 5\%$, $S(0) = F(0) = 100$

		$c = 0.01$		$c = 0.02$		$c = 0.03$		$c = 0.04$	
		$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.2$	$\sigma = 0.3$
$K = 110$	$T = 1$	99.70 (0.30)	99.63 (0.37)	99.41 (0.59)	99.26 (0.74)	99.12 (0.88)	98.90 (1.10)	98.83 (1.17)	98.54 (1.46)
	$T = 3$	98.61 (1.39)	98.48 (1.52)	97.24 (2.76)	96.99 (3.01)	95.91 (4.09)	95.54 (4.46)	94.60 (5.40)	94.12 (5.88)
	$T = 5$	97.29 (2.71)	97.16 (2.84)	94.68 (5.32)	94.42 (5.58)	92.17 (7.83)	91.79 (8.21)	89.75 (10.25)	89.25 (10.75)
$K = 100$	$T = 1$	99.41 (0.59)	99.42 (0.58)	98.83 (1.17)	98.84 (1.16)	98.25 (1.75)	98.27 (1.73)	97.67 (2.33)	97.70 (2.30)
	$T = 3$	98.06 (1.94)	98.10 (1.90)	96.16 (3.84)	96.24 (3.76)	94.31 (5.69)	94.32 (5.68)	92.51 (7.49)	92.66 (7.34)
	$T = 5$	96.59 (3.41)	96.67 (3.33)	93.32 (6.68)	93.48 (6.52)	90.19 (9.81)	90.42 (9.58)	87.18 (12.82)	87.48 (12.52)
$K = 90$	$T = 1$	99.12 (0.88)	99.19 (0.81)	98.27 (1.73)	98.40 (1.60)	97.43 (2.57)	97.63 (2.37)	96.59 (3.41)	96.86 (3.14)
	$T = 3$	97.54 (2.46)	97.71 (2.29)	95.16 (4.84)	95.49 (4.51)	92.84 (7.16)	93.33 (6.67)	90.58 (9.42)	91.22 (8.78)
	$T = 5$	95.94 (4.06)	96.18 (3.82)	92.06 (7.94)	92.54 (7.46)	88.36 (11.64)	89.06 (10.94)	84.83 (15.17)	85.73 (14.27)

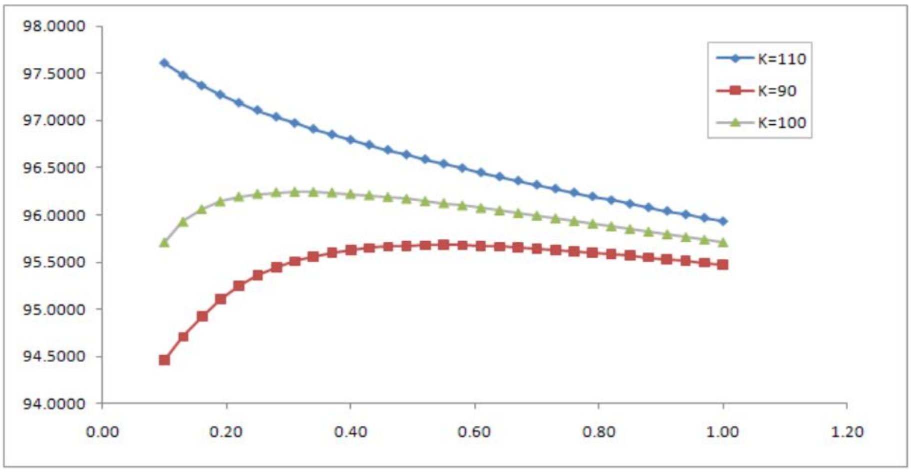

Figure 5.1. The effect of volatility on the time-0 value of net account: $r = 3\%$, $T = 3$, $S(0) = F(0) = 100$ and $c = 2\%$

$K = 90$, the values increase as the volatility increases. For a moderate range of the volatility, this can be explained as follows: if $K = 110$, it is more likely that $S(t)$ goes above the threshold level as the volatility increases and there would be more chance of the expenses being withdrawn from the investor's account. On the other hand, if $K = 90$, then the account is already above the threshold at the beginning and it is less likely that $S(t)$ will stay above the threshold (less expenses) as the volatility increases. Figure 5.1 depicts the effect of volatility on the total expenses with σ ranging from 10% to 100%.

Next we consider the case where the threshold level grows exponentially in the form of (4.1). Table 5.2 shows the values calculated from formula (4.4) with $\beta = 1\%$, 3% and 5%. As can be expected,

Table 5.2. The time-0 values of net account (total expenses): An exponentially growing threshold case, $r = 5\%$, $S(0) = F(0) = 100$ and $c = 2\%$

	$\beta = 0.01$		$\beta = 0.03$		$\beta = 0.05$	
	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.2$	$\sigma = 0.3$
$T = 1$	98.85 (1.15)	98.86 (1.14)	98.90 (1.10)	98.89 (1.11)	98.95 (1.05)	98.93 (1.07)
$T = 3$	96.28 (3.72)	96.33 (3.67)	96.54 (3.46)	96.49 (3.51)	96.80 (3.20)	96.67 (3.33)
$T = 5$	93.56 (6.44)	93.64 (6.36)	94.07 (5.93)	93.99 (6.01)	94.62 (5.38)	94.34 (5.66)

Table 5.3. The time-0 values of net account (total expenses): Expenses are withdrawn irrespective of investment performance, $r = 5\%$, $S(0) = F(0) = 100$

	$c = 0.01$	$c = 0.02$	$c = 0.03$	$c = 0.04$
$T = 1$	99.01 (0.99)	98.02 (1.98)	97.05 (2.95)	96.08 (3.92)
$T = 3$	97.05 (2.95)	94.18 (5.82)	91.39 (8.61)	88.69 (11.31)
$T = 5$	95.12 (4.88)	90.48 (9.52)	86.07 (13.93)	81.87 (18.13)

the total expenses are an increasing function of T but a decreasing function of β. We also comment that a similar argument used in the constant threshold case could be employed to explain the effect of the volatility.

Finally, we consider the case where the expenses are withdrawn at a continuous rate of c irrespective of the investment performance, *i.e.*,

$$F(t) = S(t)e^{-ct}.$$

Table 5.3 shows the time-0 values of the net account (or the total expenses) for some combinations of T and c. In this case, one may find a closed-form pricing formula independent of the volatility level. See, for example, Milevsky and Posner (2001).

6. Conclusion

In this paper, we have derived pricing formulas for the total expenses of equity-linked investment products if the expenses are collected continuously from the investor's account whenever the investment performance exceeds a certain threshold level. Although expressed in convolution form, the pricing formulas did not require too much computational burden. The expense structure suggested has an appealing feature in the sense that, if the investment performs poorly, the investors do not have to pay expenses on their account. Of course, the deficit on the vendor's side would be compensated as a greater incentive for the threshold exceeding performance.

Appendix

Proof of Proposition 3.2. We first let

$$f_1(x,t) = \frac{l(x-l)}{t^{\frac{3}{2}}} e^{-\lambda t} e^{-\frac{(x-l)^2}{2t}},$$

$$j_1(x,t) = \frac{1}{\sqrt{2\pi t}}\left\{e^{-\frac{x^2}{2t}} - e^{-\frac{(2l-x)^2}{2t}}\right\},$$

$$k_1(x,t) = \frac{\sqrt{2\pi}(2l-x)}{t^{\frac{3}{2}}}e^{-\frac{(2l-x)^2}{2t}}.$$

Then it would follow from Proposition 3.1. that

$$L\left[\alpha,\lambda;t,l\right] = \int_l^\infty e^{-\alpha x}\int_0^t\int_0^u f_1(x,s)g_1(u-s)h_1(t-u)ds\,du\,dx \\ + \int_{-\infty}^l e^{-\alpha x}\int_0^t k_1(x,s)g_1(t-s)dsdx + \int_{-\infty}^l e^{-\alpha x}j_1(x,t)dx. \tag{A.1}$$

By changing the order of integration, the first integral of (A.1) is rewritten as

$$\int_0^t\int_0^u\left\{\int_l^\infty e^{-\alpha x}f_1(x,s)dx\right\}g_1(u-s)h_1(t-u)ds\,du = \left(F_1 * g_1 * h_1\right)(t), \tag{A.2}$$

where the term inside the bracket is denoted by $F_1(s;\alpha,\lambda,l)$. Now letting $y = x - l + \alpha t$, we may express $F_1(s;\alpha,\lambda,l)$ as follows:

$$\begin{aligned} F_1(s;\alpha,\lambda,l) &= \int_l^\infty e^{-\alpha x}f_1(x,s)dx \\ &= \frac{l}{s}\sqrt{2\pi}e^{-\lambda s-\alpha l+\frac{\alpha^2 s}{2}}\int_{\alpha s}^\infty \frac{1}{\sqrt{2\pi s}}\left(y-\alpha s\right)e^{-\frac{y^2}{2s}}dy \\ &= le^{-\lambda s-\alpha l}\left\{\frac{1}{\sqrt{t}} - \sqrt{2\pi}\alpha e^{\frac{\alpha^2 s}{2}}\Phi\left(-\alpha\sqrt{s}\right)\right\}, \end{aligned}$$

which is of the form given in Proposition 3.2. Similarly, by changing the order of integration, we get an expression for the second integral of (A.1) as

$$\int_0^t g_1(t-s)\left\{\int_{-\infty}^l e^{-\alpha x}k_1(x,s)dx\right\}ds = \left(K_1 * g_1\right)(t), \tag{A.3}$$

where the term inside the bracket is denoted by $K_1(s;\alpha,\lambda,l)$. Letting $y = (x-2l+\alpha t)/\sqrt{s}$, we can simplify $K_1(s;\alpha,\lambda,l)$ as

$$\begin{aligned} K_1(s;\alpha,\lambda,l) &= \int_{-\infty}^l e^{-\alpha x}k_1(x,s)dx \\ &= \frac{2\pi}{s}e^{\frac{\alpha^2 s}{2}-2\alpha l}\int_{-\infty}^{\frac{(\alpha s-l)}{\sqrt{s}}}\frac{(\alpha s-\sqrt{s}y)}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\ &= \sqrt{2\pi}e^{-\alpha l}\left\{\frac{e^{-\frac{l^2}{2s}}}{\sqrt{s}} + \sqrt{2\pi}\alpha e^{-\alpha l}e^{\frac{\alpha^2 s}{2}}\Phi\left(\frac{\alpha s-l}{\sqrt{s}}\right)\right\}. \end{aligned}$$

The last term of (A.1), denoted by $J_1(t;\alpha,\lambda,l)$, is calculated as

$$J_1(t;\alpha,\lambda,l) = e^{\frac{\alpha^2 t}{2}}\left\{\Phi\left(\frac{\alpha t+l}{\sqrt{t}}\right) - e^{-2\alpha l}\Phi\left(\frac{\alpha t-l}{\sqrt{t}}\right)\right\}. \tag{A.4}$$

Combining the equations of (A.1)–(A.4) establishes the proof of Proposition 3.2.

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