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ON COMPLETE SPACELIKE $(r-1)$-MAXIMAL HYPERSURFACES IN THE ANTI-DE SITTER SPACE $\mathbb{H}_1^{n+1}(-1)$

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Abstract. In this paper we investigate complete spacelike $(r-1)$-maximal (i.e., $H_r \equiv 0$) hypersurfaces with two distinct principal curvatures in the anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$. We give a characterization of the hyperbolic cylinder.

1. Introduction

Let $\overline{M}_1^{n+1}(c)$ be an $(n+1)$-dimensional Lorenztian space form with constant sectional curvature c. When $c > 0$, $M_1^{n+1}(c) = \mathbb{S}_1^{n+1}(c)$ is called $(n+1)$-dimensional de Sitter space; when $c = 0$, $M_1^{n+1}(c) = \mathbb{L}^{n+1}$ is called $(n+1)$-dimensional Lorentz-Minkowski space; when $c < 0$, $M_1^{n+1}(c) = \mathbb{H}_1^{n+1}(c)$ is called $(n+1)$-dimensional anti-de Sitter space. A hypersurface M^n is said to be spacelike if the induced metric on M^n from that of the ambient space is Riemannian. The spacelike hypersurfaces in the anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$ are very interesting geometrical objects that were investigated by many geometers.

T. Ishihara [4] proved the following well-known result:

Theorem 1.1 ([4]). *Let M^n be an n-dimensional complete maximal spacelike hypersurface in the anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$, and let S be square of the norm of the second fundamental form. Then,*

$$S \leq n, \tag{1.1}$$

and $S = n$ if and only if $M^n = \mathbb{H}^m(-\frac{n}{m}) \times \mathbb{H}^{n-m}(-\frac{n}{n-m})$ $(1 \leq m \leq n-1)$.

Recently, Cao and Wei [2] studied n-dimensional complete maximal spacelike hypersurfaces with two distinct principal curvatures in an $(n+1)$-dimensional anti-de Sitter space and gave a characterization of hyperbolic cylinders in the anti-de Sitter space. The author and Liu [10] extended their result and proved the following result:

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Theorem 1.2 ([10])**.** *Let M^n be an n-dimensional $(n \geq 3)$ complete spacelike hypersurface with constant mean curvature H immersed in an anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$. Suppose in addition that M has two distinct principal curvatures λ and μ with the multiplicities $(n-1)$ and 1, respectively, and satisfying* $\inf(\lambda-\mu)^2 > 0$, *then M^n is the hyperbolic cylinder $\mathbb{H}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$.*

In this paper we will investigate complete $(r-1)$-maximal spacelike hypersurfaces with two principal curvature in the anti-de Sitter spacetime $\mathbb{H}_1^{n+1}(-1)$ and obtain the following main result:

Theorem 1.3. *Let M^n be an n-dimensional $(n \geq 3)$ connected, complete $(r-1)$-maximal $(1 \leq r \leq n)$ spacelike hypersurface immersed in anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$. Suppose in addition that M^n has two distinct principal curvatures λ and μ with the multiplicities $n-1$ and 1, respectively. Then*

(i) *$\lambda \equiv 0$ and $r \geq 2$. Furthermore, the normalized scalar curvature $R = -1$ and M^n is both 1-maximal and $(n-1)$-maximal, or*

(ii) *S satisfies inequality*

$$S \geq \frac{n(r^2-2r+n)}{r(n-r)}, \tag{1.2}$$

provided that $\inf(\lambda-\mu)^2 > 0$, *and $S = \frac{n(r^2-2r+n)}{r(n-r)}$ if and only if M is the hyperbolic cylinder $\mathbb{H}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$.*

Corollary 1.4. *Let M^n be an n-dimensional $(n \geq 3)$ connected, complete $(r-1)$-maximal spacelike hypersurface immersed in the anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$. Suppose in addition that M has two distinct principal curvatures $\lambda \neq 0$ and μ with the multiplicities $n-1$ and 1, respectively, satisfying* $\inf(\lambda-\mu)^2 > 0$, *and*

$$S \leq \frac{n(r^2-2r+n)}{r(n-r)}. \tag{1.3}$$

Then $S = \frac{n(r^2-2r+n)}{r(n-r)}$ and M is hyperbolic cylinder $\mathbb{H}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$.

Remark 1.5. When $r = 1$, since $H = \frac{1}{n}\{(n-1)\lambda + \mu\} \equiv 0$, but $\lambda \neq \mu$, then $\lambda \neq 0$. By Theorem 1.3, we have $S \geq n$. Therefore, using Theorem 1.1, we can obtain Theorem 1.2 in [2] from Theorem 1.3. Hence we extend Cao and Wei's result in [2] from another perspective.

2. Preliminaries

Let M^n be a complete hypersurface in anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$. For any $p \in M$, we can choose a local orthonormal frame fields $e_1, \ldots, e_n, e_{n+1}$ in a neighborhood U of M such that $e_1, \ldots, e_n$ are tangential to M^n and e_{n+1} is normal to M^n. Let $\omega_1, \ldots, \omega_n, \omega_{n+1}$ be the corresponding dual frame so that the pseudo-Riemannian metric of $\mathbb{H}_1^{n+1}(-1)$ is given by $d\bar{s}^2 = \sum_i \omega_i^2 - \omega_{n+1}^2$. The smooth connection 1-forms are denoted by ω_{ij}.

A well-known argument shows that the forms ω_{in+1} may be expressed as $\omega_{in+1} = \sum_j h_{ij}\omega_j$, $h_{ij} = h_{ji}$. The square of the length of the second fundamental form $h = \sum h_{ij}\omega_i \otimes \omega_j$ is given by $S = |h|^2 = \sum_{i,j} h_{ij}^2$.

Associated to the second fundamental form h of M^n one has n invariants S_r, given by the equality

$$\det(tI - A) = \sum_{k=0}^{n}(-1)^k S_k t^{n-k},$$

where A is the shape operator of M^n. If $p \in M$ and $\{e_k\}_{1\le k\le n}$ is a basis of T_pM formed by eigenvectors of the shape operator A_p, with corresponding eigenvalues λ_k's, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[x_1, \ldots, x_n]$ is the r-th elementary symmetric polynomial on the indeterminates $x_1, \ldots, x_n$. The r-th mean curvature of M is given by

$$H_r = \frac{1}{\binom{n}{r}} S_r.$$

In particular, when $r = 1$

$$H_1 = \frac{1}{n}\sum_i \lambda_i = \frac{1}{n} S_1 = H$$

is nothing but the mean curvature of M.

A spacelike hypersurface M^n in Lorentzian space form $\overline{M}_1^{n+1}(c)$ is called $(r-1)$-maximal if $H_r \equiv 0$. In particular, an 0-maximal spacelike hypersurface is precisely ordinary maximal spacelike one.

The Gauss equations are [7]

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.1}$$

$$R_{ij} = -(n-1)\delta_{ij} - nHh_{ij} + \sum_k h_{ik}h_{kj}, \tag{2.2}$$

$$n(n-1)(R+1) = -n^2H^2 + S = -n(n-1)H_2, \tag{2.3}$$

where R is the normalized scalar curvature of M^n.

The Codazzi equation is

$$h_{ijk} = h_{ikj}, \tag{2.4}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \tag{2.5}$$

We also have the Simons formula [3]

$$\frac{1}{2}\triangle S = |\nabla A|^2 + S^2 - n(S - nH^2) - nH\mathrm{tr}A^3 + n\sum_i \lambda_i H_{;ii}, \tag{2.6}$$

where $H_{;ii} = e_i(e_i(H))$.

3. Some lemmas

Let M^n be an $(r-1)$-maximal spacelike hypersurface with two distinct principal curvatures λ and μ (which means that $\lambda(p) \neq \mu(p)$, $\forall p \in M$) in anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$. In addition we assume that the multiplicities of the principal curvatures λ and μ are $n-1$ and 1, respectively, i.e., $\lambda_1 = \cdots = \lambda_{n-1} = \lambda$, $\lambda_n = \mu$. In the following we shall make use of the following convention on the ranges of indices:

$$1 \leq i, j, k, l \leq n, \ 1 \leq a, b, \ldots \leq n-1.$$

Then $\lambda_a = \lambda$, $\lambda_n = \mu$, and

$$\begin{aligned} S_r &= \binom{n}{r} H_r = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \lambda_{i_1} \cdots \lambda_{i_r} \\ &= \binom{n-1}{r} \lambda^r + \binom{n-1}{r-1} \lambda^{r-1} \mu = 0, \end{aligned}$$

hence

$$\lambda^{r-1}[(n-r)\lambda + r\mu] = 0. \tag{3.1}$$

Letting $U = \{p \in M \mid \lambda(p) \neq 0\}$, $V = \{q \in M \mid (n-r)\lambda(q) + r\mu(q) = 0\}$. Since these principal curvatures λ and μ are continuous, U is an open set in M, V is a closed set in M. We claim that $U = V$. In fact, if $p \in U$, i.e., $\lambda(p) \neq 0$, by (3.1), $(n-r)\lambda(p) + r\mu(p) = 0$, and $p \in V$. Hence $U \subseteq V$. On the other hand, if $q \in V$, then $(n-r)\lambda(q) + r\mu(q) = 0$. Since λ and μ are distinct, so $\lambda(q) \neq 0$, otherwise $\lambda(q) = 0 = \mu(q)$. That is $q \in U$, then we have $V \subseteq U$. Therefore $U = V$. Note that if M is connected, then $U = \emptyset$ or $U = M$. This is, λ is always 0 or λ is never 0.

(i) If $U = \emptyset$, then $\lambda \equiv 0$. Meanwhile, $r \geq 2$, otherwise, by $H = \frac{1}{n}((n-1)\lambda + \mu) = 0$, it implies that $\lambda = \mu = 0$ which is a contradiction with assumption that λ and μ are distinct. Furthermore, $\binom{n}{2} H_2 = \lambda(\binom{n-1}{2}\lambda + \binom{n-1}{1}\mu) = 0$, i.e., $R = -1$ by (2.3), and $H_n = \lambda^{n-1}\mu = 0$. Thus M^n is both 1-maximal and $(n-1)$-maximal.

(ii) If $U = M$, then $\lambda(p) \neq 0$, $\forall p \in M$. By (3.1), we obtain

$$(n-r)\lambda + r\mu = 0. \tag{3.2}$$

We notice that

$$(n-1)\lambda + \mu = nH, \tag{3.3}$$

$$(n-1)\lambda^2 + \mu^2 = S, \tag{3.4}$$

$$(n-1)\lambda^3 + \mu^3 = \mathrm{tr}A^3. \tag{3.5}$$

Solving the above system of equations, we have

$$\mu = -\frac{n-r}{r}\lambda, \tag{3.6}$$

$$\lambda-\mu = \frac{n}{r}\lambda, \tag{3.7}$$

$$H = \frac{r-1}{r}\lambda, \tag{3.8}$$

$$S = \frac{n(r^2-2r+n)}{r^2}\lambda^2, \tag{3.9}$$

$$\mathrm{tr}A^3 = \frac{n(r^3-3r^2+3rn-n^2)}{r^3}\lambda^3. \tag{3.10}$$

In following we suppose $\lambda(p)\neq 0$, $\forall p\in M$. Firstly, By making use of similar methods to the ones in [6], we will prove the following result.

Lemma 3.1. *let M^n be an n-dimensional $(n\geq 3)$ $(r-1)$-maximal spacelike hypersurface immersed in the anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$. Suppose in addition that M^n has two distinct principal curvatures $\lambda\neq 0$ and μ with the multiplicities $(n-1)$ and 1, respectively. Then*

$$\lambda_{;nn} = \frac{r+n}{n}\frac{(\lambda_{;n})^2}{\lambda} - \frac{n}{r}\lambda + \frac{n(n-r)}{r^2}\lambda^3, \tag{3.11}$$

$$\lambda_{;aa} = -\frac{r}{n}\frac{(\lambda_{;n})^2}{\lambda}. \tag{3.12}$$

Proof. Using (2.5) we have

$$\Sigma_k h_{ijk}\omega_k = \delta_{ij}d\lambda_i + (\lambda_i-\lambda_j)\omega_{ij}. \tag{3.13}$$

Thus

$$\begin{aligned} &h_{abk}=0,\ a\neq b,\\ &h_{aab}=h_{aba}=0\Rightarrow \lambda_{;a}=\mu_{;a}=0=h_{nna},\\ &h_{aan}=\lambda_{;n},\\ &h_{nnn}=\mu_{;n}=-\frac{n-r}{r}\lambda_{;n}. \end{aligned} \tag{3.14}$$

Letting $i=a,\ j=n$ in (3.13), we get

$$\omega_{an} = \frac{\lambda_{;n}}{\lambda-\mu}\omega_a = \frac{r}{n}\frac{\lambda_{;n}}{\lambda}\omega_a. \tag{3.15}$$

By the definition of covariant derivative, we have

$$\sum \lambda_{;ij}\omega_j = d\lambda_{;i} + \Sigma_j\lambda_{;j}\omega_{ji} = d\lambda_{;i} + \lambda_{;n}\omega_{ni}, \tag{3.16}$$

$$d\lambda_{;n} = \sum_j \lambda_{;nj}\omega_j. \tag{3.17}$$

From (3.15), we obtain

$$\begin{aligned} dw_{an} &= \frac{r}{n}\left(\frac{1}{\lambda}d\lambda_{;n}\wedge\omega_a - \frac{\lambda_{;n}}{\lambda^2}d\lambda\wedge\omega_a + \frac{\lambda_{;n}}{\lambda}d\omega_a\right) \\ &= \frac{r}{n}\left(\frac{1}{\lambda}\sum_j \lambda_{;nj}\omega_j\wedge\omega_a - \frac{(\lambda_{;n})^2}{\lambda^2}\omega_n\wedge\omega_a + \frac{\lambda_{;n}}{\lambda}\sum_j \omega_{aj}\wedge\omega_j\right) \\ &= \frac{r}{n}\left(\frac{1}{\lambda}\sum_b \lambda_{;nb}\omega_b\wedge\omega_a + \frac{1}{\lambda}\lambda_{;nn}\omega_n\wedge\omega_a - \frac{(\lambda_{;n})^2}{\lambda^2}\omega_n\wedge\omega_a \right. \\ &\quad \left. +\frac{\lambda_{;n}}{\lambda}\sum_b \omega_{ab}\wedge\omega_b + \frac{\lambda_{;n}}{\lambda}\omega_{an}\wedge\omega_n\right). \end{aligned}$$

On the other hand, by the structure equations of M, we have

$$\begin{aligned} dw_{an} &= \sum_j \omega_{aj}\wedge\omega_{jn} - \frac{1}{2}\sum_{k,l} R_{ankl}\omega_k\wedge\omega_l \\ &= \frac{r}{n}\frac{\lambda_{;n}}{\lambda}\sum_b \omega_{ab}\wedge\omega_b - \frac{1}{2}\sum_{k,l} R_{ankl}\omega_k\wedge\omega_l. \end{aligned}$$

By (2.1) we can see

$$\begin{aligned} R_{anbk} &= -(\delta_{ab}\delta_{nk} - \delta_{ak}\delta_{nb}) - \lambda_a\lambda_n(\delta_{ab}\delta_{nk} - \delta_{ak}\delta_{nb}) \\ &= -\delta_{ab}\delta_{nk} - \lambda_a\lambda_n\delta_{ab}\delta_{nk}, \end{aligned}$$

$$R_{anbc} = 0,\ R_{anbn} = -\delta_{ab} - \lambda\mu\delta_{ab}.$$

Noting that M^n is $(r-1)$-maximal, i.e., $H_r = 0$, but $\lambda \neq 0$. Using (3.6), we have

$$\begin{aligned} dw_{an} &= \frac{r}{n}\frac{\lambda_{;n}}{\lambda}\sum_b \omega_{ab}\wedge\omega_b + (1+\lambda\mu)\omega_a\wedge\omega_n \\ &= \frac{r}{n}\frac{\lambda_{;n}}{\lambda}\sum_b \omega_{ab}\wedge\omega_b + \left(1 - \frac{n-r}{r}\lambda^2\right)\omega_a\wedge\omega_n. \end{aligned}$$

So we have

$$\lambda_{;nn} = \frac{r+n}{n}\frac{(\lambda_{;n})^2}{\lambda} - \frac{n}{r}\lambda + \frac{n(n-r)}{r^2}\lambda^3.$$

Let $i = a$ in (3.16). Then

$$\sum \lambda_{;aj}\omega_j = d\lambda_{;a} + \Sigma_j\lambda_{;j}\omega_{ja} = \lambda_{;n}\omega_{na} = -\frac{r}{n}\frac{(\lambda_{;n})^2}{\lambda}\omega_a,$$

so

$$\lambda_{;aa} = -\frac{r}{n}\frac{(\lambda_{;n})^2}{\lambda}.$$ □

Secondly, based on Lemma 3.1, we have the following key lemma.

Lemma 3.2. *let M^n be an n-dimensional $(n \geq 3)$ $(r-1)$-maximal spacelike hypersurface immersed in anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$. Suppose in addition that M^n has two distinct principal curvatures $\lambda \neq 0$ and μ with the multiplicities $(n-1)$ and 1, respectively. Then*

$$\frac{1}{2}\triangle S = \frac{(3n-2)r^2 - 2rn + n^2 - (r-1)[(n-2)r^2 + n^2]}{4n(r^2-2r+n)}\frac{|\nabla S|^2}{S} - \frac{n}{r}S + \frac{n-r}{r^2-2r+n}S^2. \tag{3.18}$$

Proof. By Lemma 3.1, we have

$$\begin{aligned}
&\sum_i \lambda_i (nH)_{;ii} \\
= &\sum_a \lambda (nH)_{;aa} + \mu (nH)_{;nn} \\
= &\frac{n(r-1)}{r}\lambda \sum_a \lambda_{;aa} - \frac{n(n-r)(r-1)}{r^2}\lambda\lambda_{;nn} \\
= &-(n-1)(r-1)(\lambda_{;n})^2 - \frac{n(n-r)(r-1)}{r^2}\left[\frac{r+n}{n}(\lambda_{;n})^2 - \frac{n}{r}\lambda^2 + \frac{n(n-r)}{r^2}\lambda^4\right] \\
= &-\frac{(r-1)[(n-2)r^2+n^2]}{r^2}(\lambda_{;n})^2 + \frac{n^2(n-r)(r-1)}{r^4}\lambda^2[r-(n-r)\lambda^2].
\end{aligned} \tag{3.19}$$

By (3.9), we obtain

$$|\nabla S|^2 = \sum_i (S_{;i})^2 = \frac{4n^2(r^2-2r+n)^2}{r^4}\lambda^2(\lambda_{;n})^2. \tag{3.20}$$

By (3.14), we have

$$|\nabla A|^2 = \sum_{i,j,k} h_{ijk}^2 = h_{nnn}^2 + 3\sum_i h_{iin}^2 = \frac{(3n-2)r^2 - 2rn + n^2}{r^2}(\lambda_{;n})^2. \tag{3.21}$$

From (3.8)–(3.10), it is easy to see that

$$\lambda^2 = \frac{r^2}{n(r^2-2r+n)}S, \tag{3.22}$$

$$S - nH^2 = \frac{n-1}{r^2-2r+n}S, \tag{3.23}$$

$$nH\mathrm{tr}A^3 = \frac{(r-1)(r^3-3r^2+3rn-n^2)}{(r^2-2r+n)^2}S^2. \tag{3.24}$$

Substituting (3.22) into (3.20), we obtain

$$(\lambda_{;n})^2 = \frac{r^2}{4n(r^2-2r+n)}\frac{|\nabla S|^2}{S}. \tag{3.25}$$

Furthermore

$$|\nabla A|^2 = \frac{(3n-2)r^2 - 2rn + n^2}{4n(r^2-2r+n)} \frac{|\nabla S|^2}{S}. \tag{3.26}$$

Using (3.22) and (3.25), and substituting (3.19), (3.23), (3.24) and (3.26) into (2.6), we can get

$$\begin{aligned}
\frac{1}{2}\triangle S &= |\nabla A|^2 + S^2 - n(S - nH^2) - nH\mathrm{tr}A^3 + \sum_i \lambda_i (nH)_{;ii} \\
&= \frac{(3n-2)r^2 - 2rn + n^2}{4n(r^2-2r+n)} \frac{|\nabla S|^2}{S} + S^2 - \frac{n(n-1)}{r^2-2r+n} S \\
&\quad - \frac{(r-1)(r^3 - 3r^2 + 3rn - n^2)}{(r^2-2r+n)^2} S^2 - \frac{(r-1)[(n-2)r^2+n^2]}{4n(r^2-2r+n)} \frac{|\nabla S|^2}{S} \\
&\quad + \frac{n(n-r)(r-1)}{r(r^2-2r+n)} S - \frac{(n-r)^2(r-1)}{(r^2-2r+n)^2} S^2 \\
&= \frac{(3n-2)r^2 - 2rn + n^2 - (r-1)[(n-2)r^2+n^2]}{4n(r^2-2r+n)} \frac{|\nabla S|^2}{S} \\
&\quad - \frac{n}{r} S + \frac{n-r}{r^2-2r+n} S^2.
\end{aligned}$$

□

We need the following lemma (see [6, 9]).

Lemma 3.3 (Omori [6], Yau [9]). *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let F be a C^2 function on M which is bounded from below on M. Then there exists a sequence of points p_k in M such that*

$$\lim_{k\to\infty} F(p_k) = \inf(F), \ \lim_{k\to\infty} |\nabla F(p_k)| = 0, \ \lim_{k\to\infty} \triangle F(p_k) \geq 0. \tag{3.27}$$

4. Proof of Theorem 1.3

The case for $\lambda \equiv 0$, from what has been discussed in Section 4, we know that M^n is both 1-maximal and $(n-1)$-maximal.

If $\lambda(p) \neq 0$, $\forall p \in M$, from Gauss equation (2.1) we have

$$\begin{aligned}
R_{ii} &= -(n-1) - nH\lambda_i + \lambda_i^2 \\
&\geq -(n-1) - \frac{n^2H^2}{4} + \left(\lambda_i - \frac{nH}{2}\right)^2 \\
&\geq -(n-1) - \frac{n^2H^2}{4} \\
&= -(n-1) - \frac{n^2(r-1)^2}{4r^2}\lambda^2. \qquad (4.1)
\end{aligned}$$

Now we let $W_1 = \{p \in M \mid \lambda^2(p) > \frac{r}{n-r}\}$, $W_2 = \{p \in M \mid \lambda^2(p) \leq \frac{r}{n-r}\}$. Since

$$S = \frac{n(r^2-2r+n)}{r^2}\lambda^2 > \frac{r^2-2r+n}{n} \inf(\lambda-\mu)^2 > 0 \tag{4.2}$$

if $p \in W_1$, obviously, we have (1.2). There is nothing to prove. Thus we consider on W_2, i.e., $\lambda^2(p) \leq \frac{r}{n-r}$. From (4.1), we know that the Ricci curvature is bounded from below on W_2. Consequently, applying Omori and Yau's generalized maximum principle to function S on W_2, it is possible to find in W_2 a sequence of points $p_k,\ k \in N$, such that

$$\lim_{k\to\infty} S(p_k) = \inf_{W_2}(S) := \widetilde{\alpha},\ \lim_{k\to\infty} |\nabla S(p_k)| = 0,\ \lim_{k\to\infty} \triangle S(p_k) \geq 0. \tag{4.3}$$

Taking limit of both sides of (3.18), we obtain

$$\widetilde{\alpha}\left[\widetilde{\alpha} - \frac{n(r^2-2r+n)}{r(n-r)}\right] \geq 0. \tag{4.4}$$

Using assumption $\inf(\lambda-\mu)^2 > 0$, we have $\widetilde{\alpha} \geq \alpha := \inf_M(S) > 0$. Hence

$$\widetilde{\alpha} \geq \frac{n(r^2-2r+n)}{r(n-r)}, \tag{4.5}$$

then we get $\widetilde{\alpha} = \frac{n(r^2-2r+n)}{r(n-r)}$ by the definition of W_2. Hence we prove the inequality (1.2). If $S = \frac{n(r^2-2r+n)}{r(n-r)}$, then λ is constant by (3.9), the same as μ by (3.6). Therefore M is isoparametric with two constant distinct principal curvatures. According to Theorem 1 in [5] or by the congruence theorem of Abe, Koike, and Yamaguchi [1], M is the hyperbolic cylinder $\mathbb{H}^{n-1}(c_1)\times\mathbb{H}^1(c_2)$. This finishes the proof of Theorem 1.3.

Corollary 1.4 is obvious by Theorem 1.3.

Remark 4.1. Wei [8] studied n-dimensional compact hypersurfaces with $H_r \equiv 0$ and with two distinct principal curvatures in $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}(1)$.

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