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MULTIPLICATION MODULES WHOSE ENDOMORPHISM RINGS ARE INTEGRAL DOMAINS

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Abstract. In this paper, several properties of endomorphism rings of modules are investigated. A multiplication module M over a commutative ring R induces a commutative ring M^* of endomorphisms of M and hence the relation between the prime (maximal) submodules of M and the prime (maximal) ideals of M^* can be found. In particular, two classes of ideals of M^* are discussed in this paper: one is of the form $G_{M^*}(M, N) = \{f \in M^* \mid f(M) \subseteq N\}$ and the other is of the form $G_{M^*}(N, 0) = \{f \in M^* \mid f(N) = 0\}$ for a submodule N of M.

0. Introduction

Throughout this paper, *unless otherwise specified, we shall assume that all rings are associative with identity and all modules are unitary left modules.*

Let R be a ring and let M be an R-module. Then the set of all R-homomorphisms from M into itself can be given the structure of a ring. We call this ring the *ring of endomorphisms* of M and denote this by M^*.

Let L and N be any two submodules of M. Then the set

$$\{f \in M^* \mid f(L) \subseteq N\}$$

will be considered. This set becomes an additive subgroup of the group $(M^*, +)$. So, we will denote this subgroup by $G_{M^*}(L, N)$.

If we make different choices of L and N, then $G_{M^*}(L, N)$ has different algebraic structures. There are four cases to consider:

$$(1)\ L \supseteq N, \quad (2)\ L \subseteq N, \quad (3)\ L \not\supseteq N, \quad (4)\ L \not\subseteq N.$$

In case of (1), $G_{M^*}(L, N)$ is a subring of the ring M^*. In particular, $G_{M^*}(0, 0) = M^*$, $G_{M^*}(M, 0) = 0$, and $G_{M^*}(M, M) = M^*$.

As special cases of (2), $G_{M^*}(M, M) = M^*$ and for any submodule N of M, $G_{M^*}(0, N) = M^*$.

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In cases of (3) and (4), we do not know the further algebraic structure of $G_{M^*}(L, N)$.

Now, let N be a submodule of M. Then we get $M \supseteq N \supseteq 0$. So, by (1) we get three subrings of M^*: $G_{M^*}(M, N)$, $G_{M^*}(N, N)$, and $G_{M^*}(N, 0)$. We will discuss about these three subrings of M^*. Of course, they have inclusion relation as follows:

$$G_{M^*}(N, 0) \subseteq G_{M^*}(N, N) \supseteq G_{M^*}(M, N).$$

1. Endomorphism rings

Let R be a ring. Let M be an R-module. Define a ring homomorphism $\varphi : R \to M^*$ to be $\varphi(r) = \varphi_r : M \to M$ with $\varphi_r(x) = rx$. Then

$$R/\mathrm{Ann}_R(M) \cong \mathrm{Im}(\varphi) \subseteq M^*.$$

The φ may not be injective. The example of this is given below.

Example 1.1. Take $R = \mathbb{Z}$, $M = \mathbb{Z}/2\mathbb{Z}$. Then $2 \in \mathrm{Ann}_R(M)$.

When M is a faithful R-module, however, φ is injective. If V is a non-zero vector space over a field F, then V is faithful over F. So, $\varphi : F \to V^*$ is injective. Hence, F can be embedded in V^*. If M is a non-zero free module over a commutative ring with identity with finite rank, then M is also faithful over R. So, $\varphi : R \to M^*$ is injective. Hence, R can be embedded in M^*.

Proposition 1.2. *Let R be a ring. Let M be an R-module. If $\varphi : R \to M^*$ is surjective and M^* is a projective R-module with rank 1, then φ is injective and hence $R \cong M^*$.*

Proof. The following exact sequence

$$0 \longrightarrow \mathrm{Ker}(\varphi) \longrightarrow R \xrightarrow{\varphi} M^* \longrightarrow 0$$

splits. So, $R = \mathrm{Ker}(\varphi) \oplus M^*$. Let $\mathfrak{p}$ be any element of $\mathrm{Spec}(R)$. Then $R_\mathfrak{p} = \mathrm{Ker}(\varphi)_\mathfrak{p} \oplus M^*_\mathfrak{p}$. Since $M^*_\mathfrak{p}$ is $R_\mathfrak{p}$-free with rank 1, we have $\mathrm{Ker}(\varphi)_\mathfrak{p} = 0$. This shows that $\mathrm{Ker}(\varphi) = 0$. Hence φ is injective. □

While discussing projective modules [9] with Professor Satya Mandal, we could see incidently that every projective module with positive rank over a reduced Noetherian ring is faithful.

Lemma 1.3. *If R is a reduced Noetherian ring, then every finitely generated projective R-module with positive rank is faithful.*

Proof. Let R be a reduced Noetherian ring and let P be any finitely generated projective R-module with positive rank. Let $\mathfrak{p}$ be any minimal prime ideal of R. Let x be any element of $\mathrm{Ann}_R P$. Then $xP = 0$, and so $(x/1)P_\mathfrak{p} = 0$. $P_\mathfrak{p}$ is a non-zero free $R_\mathfrak{p}$-module. Notice that every non-zero free module with finite rank is faithful. Then $x/1 = 0$, so there exists an element $s \in R \backslash \mathfrak{p}$ such that $sx = 0$. $sx = 0 \in \mathfrak{p}$. Hence, $x \in \mathfrak{p}$. This shows that $\mathrm{Ann}_R P \subseteq \mathfrak{p}$. Thus, $\mathrm{Ann}_R P \subseteq \cap_{\mathfrak{p} \in \mathrm{Min}(R)} \mathfrak{p} = \sqrt{0} = 0$. Therefore, $\mathrm{Ann}_R P = 0$. □

Let R be a commutative ring with identity and let M be an R-module. Then M is called a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$. If R is a commutative ring with identity, then R is a multiplication module over R. If V is a vector space over a field k and if the dimension of V over k is greater than 1, then V is not a multiplication module over k. For otherwise, for a subspace W of V with $\dim_k(W) = 1$, there exists an ideal I of k such that $W = IV$. Since the only ideals of the field k are 0 and k itself, we have $W = 0$ or $W = V$. This is a contradiction.

Let R be a ring and let M be an R-module. Let $f \in M^*$. A submodule N of M such that $f(N) \subseteq N$ is called *f-stable* or *f-invariant*. Further, recall that a submodule N of M is called *fully invariant* if for every $f \in M^*$, N is f-invariant, or equivalently, if $M^* = G_{M^*}(N, N)$.

Let R be a commutative ring with identity and let M be a multiplication module. Let N be any submodule of M. Then there exists an ideal I of R such that $N = IM$. Now, let f be any element of M^*. Then

$$f(N) = f(IM) = If(M) \subseteq IM = N.$$

Hence N is f-invariant. Therefore N is fully invariant. We have proved the following.

Lemma 1.4 ([6, Proposition 7] and [4, Lemma 1])**.** *If M is a multiplication module over a commutative ring with identity, then every submodule of M is fully invariant.*

Let R be a commutative ring with identity. For every R-module M, M^* is a ring with identity. Assume further that M is a multiplication module. Let m be any element of M. Then by Lemma 1.4, Rm is fully invariant. Let f be any element of M^*. Then $f(m) \in f(Rm) \subseteq Rm$. There exists an element $r \in R$ such that $f(m) = rm$. If g is any element of M^*, then by a similar proof we can find an element $s \in R$ such that $g(m) = sm$. Hence

$$(fg)(m) = s(rm) = (sr)m = (rs)m = r(sm) = (gf)(m).$$

Hence $fg = gf$. Therefore, M^* is a commutative ring with identity (see [3, Lemma 2]).

Let R be a ring. An element r of R is called a *zero-divisor* if there exists a non-zero element s in R such that $rs = 0$. From now on we denote the set of all zero-divisors of a ring R by $Z(R)$. A commutative ring R with identity is called an *integral domain* if $Z(R) = 0$.

Theorem 1.5. *If M is a faithful multiplication module over an integral domain, then M^* is an integral domain.*

Proof. M^* is a commutative ring with identity. So, it is sufficient to prove: if $fg = 0$, where $f, g \in M^*$, then either f or g is zero.

There are ideals I, J of R such that $f(M) = IM$, $g(M) = JM$. Then $0 = (fg)(M) = J(IM) = (JI)M = (IJ)M$, so $IJ \subseteq \mathrm{Ann}_R(M) = 0$. Hence,

$IJ = 0$. Since R is an integral domain, either I or J is zero. If $I = 0$, then $f(M) = 0$. If $J = 0$, then $g(M) = 0$. Hence, either f or g is zero. □

Every integral domain is reduced. Hence the next result follows from Lemma 1.3 and Theorem 1.5.

Corollary 1.6. *Let R be a Noetherian domain. If P is a finitely generated projective multiplication R-module with positive rank, then P^* is an integral domain.*

The following result was motivated by [12, Proposition 1.2] and [8, Theorem 2.4].

Lemma 1.7. *Let R be a commutative ring with identity. Let M be a finitely generated R-module.*

(1) *If $f : M \to M$ is an epimorphism, then f satisfies a polynomial of the form*
$$1 + a_1X + a_2X^2 + \cdots + a_nX^n,$$
where the a_i are in R.

(2) *If $f : M \to M$ is an epimorphism, then f is a monomorphism.*

Let R be a commutative ring with identity. Let E be an R-module. An element e of E is said to be *divisible* if, for every r of $R\backslash Z(R)$, there exists $e' \in E$ such that $e = re'$. If every element of E is divisible, then E is said to be a *divisible module*. Alternatively, E is divisible if $E = rE$ whenever r is an element of $R\backslash Z(R)$.

Let R be an integral domain. If E is a non-zero divisible R-module, then the ring homomorphism $\varphi : R \to E^*$ which was discussed in the paragraph just prior to Example 1.1 is injective. In other words, if multiplication by r is zero, then r, as an element of R, is zero.

Theorem 1.8. *If an integral domain admits a non-zero finitely generated injective module, then it is a field.*

Proof. Let R be an integral domain and let E be a non-zero finitely generated injective module. Then E is divisible by [11, Proposition 2.6]. Let r be a non-zero element of R. Then $rE = E$. Hence, multiplication by r is an epimorphism. By Theorem 1.7(1), r satisfies a polynomial of the form
$$1 + a_1X + a_2X^2 + \cdots + a_nX^n,$$
where the a_i are in R. Hence,
$$1 + a_1r + a_2r^2 + \cdots + a_nr^n = 0.$$
This means that $1 + a_1r + a_2r^2 + \cdots + a_nr^n$, as an element of E^*, is zero. By the argument just prior to Theorem 1.8, $1 + a_1r + a_2r^2 + \cdots + a_nr^n$, as an element of R, is zero. Hence, $(-a_1 - a_2r - \cdots - a_nr^{n-1})r = 1$. Therefore, r is invertible. □

Corollary 1.9. *If E is a non-zero finitely generated injective module over an integral domain, E^* is a field.*

Proof. By [8, Theorem 2.1, p. 7], E^* is integral over R. By Theorem 1.8, R is a field. Hence, by [8, Lemma 1, p. 66], E^* is a field. □

Let R be a ring and let M be an R-module. Then we can give M an M^*-module structure as follows:

$$f.m = f(m),$$

where $f \in M^*$ and $m \in M$.

Let f be any element of $\mathrm{Ann}_{M^*}(M)$. Then $f(M) = f.M = 0$ and hence $f = 0$. This shows that $\mathrm{Ann}_{M^*}(M) = 0$. Hence every R-module M can be viewed as a faithful M^*-module.

Lemma 1.10. *Let R be a commutative ring with identity. Let M be an R-module. If M is a multiplication module over R, then M is a faithful multiplication module over M^*.*

Proof. Let M be a multiplication module over R. Let N be any M^*-submodule of the M^*-module M. Then for any $r \in R$ and for any $n \in N$, $rn = \varphi_r(n) = \varphi_r.n \in N$. Hence, N is an R-submodule of M. There exists an ideal I of R such that $N = IM$. Let $\varphi_I = \{\varphi_r \mid r \in I\}$. Then $\varphi_I M^*$ is an ideal of M^* (generated by $\varphi_I \subseteq M^*$) and

$$(\varphi_I M^*).M = \varphi_I.(M^*.M) = \varphi_I.M = \varphi_I(M) = IM = N.$$

Hence, M is also a multiplication module over M^*. □

Every vector space over a field is injective. Hence the next result follows from Corollary 1.9 and Lemma 1.10.

Corollary 1.11. *If E is a non-zero, finitely generated, injective, multiplication module over an integral domain, then it is a non-zero, faithful, finitely generated, injective, multiplication module over the field E^*.*

2. $G_{M^*}(M, N)$

Let N be any submodule of M. The subring $G_{M^*}(M, N)$ of M^* will be considered. This is a right ideal of the ring M^*. However, $G_{M^*}(M, N)$ is not always a left ideal of M^*. The example of this is given below.

Example 2.1. Let R be a ring with identity $\neq 0$ and let M be a free R-module with rank 2. Let $\{e_1, e_2\}$ be an R-free basis for M. Consider the following submodule of M:

$$\nabla = \{ae_1 + ae_2 \mid a \in R\}.$$

Then $G_{M^*}(M, \nabla)$ is not a left ideal of M^*. In fact, define a map $f : M \to M$ by $f(ae_1 + be_2) = ae_1 + ae_2$, where $a, b \in R$. Then $f \in G_{M^*}(M, \nabla)$. Now, define a map $\alpha : M \to M$ by $\alpha(ae_1+be_2) = ae_1$, where $a, b \in R$. Then $\alpha \in M^*$. Further, $\alpha f \notin G_{M^*}(M, \nabla)$. For otherwise,

$$\nabla \ni (\alpha f)(e_1) = \alpha(e_1 + e_2) = e_1.$$

This is a contradiction. Therefore $G_{M^*}(M, \nabla)$ is not a left ideal of M^*.

Lemma 2.2. *Let R be a ring and let M be an R-module. Then for every fully invariant submodule N of M, $G_{M^*}(M, N)$ is a two-sided ideal of M^*.*

Proof. We have already known that $G_{M^*}(M, N)$ is a right ideal of M^*. Now, let $\alpha \in M^*$ and $f \in G_{M^*}(M, N)$. Then $(\alpha f)(M) \subseteq \alpha(N) \subseteq N$. Hence $\alpha f \in G_{M^*}(M, N)$. □

If M is a multiplication module over a commutative ring with identity, then for every submodule N of M, $G_{M^*}(M, N)$ is a two-sided ideal of M^* by Lemma 1.4.

Theorem 2.3. *Let R be a commutative ring with identity. Let M be an R-module. Assume that M is a multiplication R-module. Then P is a prime submodule of M if and only if $G_{M^*}(M, P)$ is a prime ideal of M^*.*

Proof. Recall that M^* is a commutative ring with identity.

Assume that P is a prime submodule of M. Suppose $G_{M^*}(M, P) = M^*$. Consider the identity map $1_M : M \to M$. Then $1_M \in M^* = G_{M^*}(M, P)$, so $M = 1_M(M) \subseteq P$. Hence $P = M$, which implies a contradiction. Hence $G_{M^*}(M, P) \neq M^*$.

Now, assume that $fg \in G_{M^*}(M, P)$, where $f, g \in M^*$. Then since M is a multiplication R-module, there are ideals I and J of R such that $f(M) = IM$ and $g(M) = JM$. So,

$$(IJ)M = (JI)M = J(IM) = J(f(M)) = f(JM) = f(g(M)) \subseteq P.$$

This implies that $IJ \subseteq (P :_R M)$. Since P is a prime submodule of M, it is well-known ([7, p. 2]) that $(P :_R M)$ is a prime ideal of R. Hence

$$I \subseteq (P :_R M) \text{ or } J \subseteq (P :_R M).$$

Assume that $I \subseteq (P :_R M)$. Then $f(M) = IM \subseteq P$, so $f \in G_{M^*}(M, P)$. Or, assume that $J \subseteq (P :_R M)$. Then $g(M) = JM \subseteq P$, so $g \in G_{M^*}(M, P)$.

Therefore, $G_{M^*}(M, P)$ is a prime ideal of M^*.

Conversely, assume that $G_{M^*}(M, P)$ is a prime ideal of M^*. Suppose that $P = M$. Then $G_{M^*}(M, P) = G_{M^*}(M, M) = M^*$. This is a contradiction. Hence $P \neq M$.

Assume that $rm \in P$, where $r \in R$ and $m \in M$. Since M is a multiplication R-module, there exists an ideal I of R such that $Rm = IM$. So,

$$(rI)M = r(IM) = (rR)m \subseteq P.$$

Consider the ring homomorphism $\varphi : R \to M^*$ which was discussed in the paragraph just prior to Example 1.1. Since $G_{M^*}(M, P)$ is a prime ideal of M^*, it follows that $\varphi^{-1}(G_{M^*}(M,P))$ is a prime ideal of R. Further, notice that $\varphi_{rI}(M) = (rI)M \subseteq P$. Then $\varphi(rI) = \varphi_{rI} \subseteq G_{M^*}(M,P)$. This implies that $rI \subseteq \varphi^{-1}(G_{M^*}(M,P))$. Hence $r \in \varphi^{-1}(G_{M^*}(M,P))$ or $I \subseteq \varphi^{-1}(G_{M^*}(M,P))$. Assume that $r \in \varphi^{-1}(G_{M^*}(M,P))$. Then $\varphi(r) \in G_{M^*}(M,P)$, so $rM = \varphi_r(M) \subseteq P$. Hence $r \in (P :_R M)$. Or, assume that $I \subseteq \varphi^{-1}(G_{M^*}(M,P))$. Then $\varphi(I) \subseteq G_{M^*}(M,P)$, so $Rm = IM = \varphi_I(M) \subseteq P$. Hence $m \in P$.

Therefore, P is a prime submodule of M. $\square$

Lemma 2.4. *Let R be a commutative ring with identity and let M be an R-module. If M is a multiplication R-module, then for every submodule N of M, $N = \sum_{f\in G_{M^*}(M,N)} f(M)$.*

Proof. It is obvious that $\sum_{f\in G_{M^*}(M,N)} f(M) \subseteq N$.

Conversely, let x be any element of N. Since M is a multiplication R-module, there exists an ideal I of R such that $Rx = IM$. Further, there are $a_1, a_2, \dots, a_r \in I$ and $m_1, m_2, \dots, m_r \in M$ such that $x = a_1m_1 + a_2m_2 + \cdots + a_rm_r$. Let $\varphi : R \to M^*$ be as before. Then for each $i \in \{1, 2, \dots, r\}$, $\varphi_{a_i}(M) = a_iM \subseteq IM = Rx \subseteq N$ and hence $\varphi_{a_i} \in G_{M^*}(M,N)$. Hence

$$\begin{aligned} x &= a_1m_1 + a_2m_2 + \cdots + a_rm_r \\ &\in \varphi_{a_1}(M) + \varphi_{a_2}(M) + \cdots + \varphi_{a_r}(M) \\ &\subseteq \sum_{f\in G_{M^*}(M,N)} f(M). \end{aligned}$$

Hence $N \subseteq \sum_{f\in G_{M^*}(M,N)} f(M)$. Therefore $N = \sum_{f\in G_{M^*}(M,N)} f(M)$. $\square$

Consider the ring homomorphism $\varphi : R \to M^*$ which was discussed in the paragraph just prior to Example 1.1. $\varphi^{-1}(G_{M^*}(M,N))$ will be denoted by $G_{M^*}(M,N) \cap R$. Then we have the following result.

Proposition 2.5. *For every submodule N of an R-module M,*

$$(N :_R M) = G_{M^*}(M,N) \cap R.$$

Let M be a multiplication R-module and let N be any submodule of M. Then there exists an ideal I of R such that $N = IM$. This implies that $I \subseteq N :_R M)M$. Hence $N = IM \subseteq (N :_R M)M$. Also, $(N :_R M)M \subseteq N$, which is clear from the definition. Hence $N = (N :_R M)M$. This is useful in the proof of the following result.

Theorem 2.6. *Let R be a commutative ring with an identity. Let M be a finitely generated multiplication R-module. Then a submodule N of M is maximal if and only if $(N :_R M)$ is a maximal ideal of R.*

Proof. Let N be a maximal submodule of M. Assume that J is an ideal of R such that $(N :_R M) \subseteq J \subseteq R$. Since M is a multiplication R-module, it follows from the above argument that

$$N = (N :_R M)M \subseteq JM \subseteq M.$$

By the maximality of N, either $JM = N$ or $JM = M$. Assume that $JM = N$. Then $J \subseteq (N :_R M)$. Hence $J = (N :_R M)$. Or, assume that $JM = M$. By the Nakayama Lemma, there exists an element $a \in J$ such that $(1 - a)M = 0$. So, $(1 - a)M = 0 \subseteq N$. This implies that $1 - a \in (N :_R M) \subseteq J$. Hence $1 = a + (1 - a) \in J$. Hence $J = R$. This shows that $(N :_R M)$ is a maximal ideal of R.

Conversely, assume that $(N :_R M)$ is a maximal ideal of R. Let A be a submodule of M such that $N \subseteq A \subseteq M$. Then

$$(N :_R M) \subseteq (A :_R M) \subseteq R.$$

By the maximality of $(N :_R M)$, either $(A :_R M) = (N :_R M)$ or $(A :_R M) = R$. Assume that $(A :_R M) = (N :_R M)$. Since M is a multiplication module, we have $A = (A :_R M)M = (N :_R M)M = N$. Or, if $(A :_R M) = R$, then $M = A$. This shows that N is a maximal submodule of M. □

We have already known that if R is a commutative ring with identity and M is a multiplication module over R, then M^* is a commutative ring with identity. If M, as an R-module, is finitely generated, then M, as an M^*-module, is also finitely generated. Compare the following result with Theorem 2.3.

Corollary 2.7. *Let R be a commutative ring with identity. Let M be a finitely generated multiplication module over R and let N be any submodule of M. Then N is a maximal M^*-submodule of the M^*-module M if and only if $G_{M^*}(M, N)$ is a maximal ideal of M^*.*

Proof. Note that $G_{M^*}(M, N) = (N :_{M^*} M)$. Then it suffices to prove that N is a maximal M^*-submodule of the M^*-module M if and only if $(N :_{M^*} M)$ is a maximal ideal of M^*. Use [5, Theorem 3.1, p. 768] to prove the '*only if part'*. The remainder of the proof is almost the same as that of Theorem 2.6. □

3. $G_{M^*}(N, 0)$

Let R be ring and let N be a submodule of M. Then $G_{M^*}(N, 0)$ is a left ideal of M^*. However, this is not a right ideal of M^*. The example of this is given below.

Example 3.1. Use the same notation as in Example 1.1. Define a map $g : M \to M$ by $g(ae_1 + be_2) = (a - b)e_1$, where $a, b \in R$. Then $g \in G_{M^*}(\nabla, 0)$. Further,

$$(g\alpha)(e_1 + e_2) = g(\alpha(e_1 + e_2)) = g(e_1) = e_1 \neq 0.$$

Hence $g\alpha \notin G_{M^*}(\nabla, 0)$. Hence $G_{M^*}(\nabla, 0)$ is not a right ideal of M^*.

Compare the following lemma with Lemma 2.2.

Lemma 3.2. *Let R be a ring with identity and let M be an R-module. Then for every fully invariant submodule N of M, $G_{M^*}(N, 0)$ is a two-sided ideal of M^**

Proof. We have already known that $G_{M^*}(N,0)$ is a left ideal of M^*. Now, let $\alpha \in M^*$ and $f \in G_{M^*}(N,0)$. Then $(f\alpha)(N) \subseteq f(N) = 0$. Hence $f\alpha \in G_{M^*}(N,0)$. □

If M is a multiplication module over a commutative ring with identity, then for every submodule N of M, $G_{M^*}(N,0)$ is a two-sided ideal of M^*.

Let R be a ring. Let M be an R-module and let N be a submodule of M. Then for each $f \in G_{M^*}(N,0)$, $\mathrm{Ker}(f)$ contains N. Hence

$$\cap_{f\in G_{M^*}(N,0)}\mathrm{Ker}(f) \supseteq N.$$

A submodule N of an R-module M is called to be *tight closed* if

$$\cap_{f\in G_{M^*}(N,0)}\mathrm{Ker}(f) = N.$$

In papers [1] and [2], the name of the submodule in the definition was a "closed submodule", however we call it to be a *tight closed submodule* to avoid confusion with the name in [10]. Moreover, in view of the following Proposition 3.4, it seems like to be reasonable for us to call the submodule a tight closed submodule.

Proposition 3.3. *Let R be a ring and let M be an R-module. Let N be a submodule of M. If there exists an element $f \in M^*$ such that* $\mathrm{Ker}(f) = N$*, then N is tight closed.*

Proof. Assume that there exists an element $f \in M^*$ such that $\mathrm{Ker}(f) = N$. Then $N \subseteq \cap_{g\in G_{M^*}(N,0)}\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f) = N$. Hence $\cap_{g\in G_{M^*}(N,0)}\mathrm{Ker}(g) = N$. Therefore N is tight closed. □

Let R be a ring and let M be an R-module. Then $\mathrm{Ker}(1_M) = 0$ and $\mathrm{Ker}(0_M) = M$. Hence, by Proposition 3.3, the zero submodule of M and M itself are tight closed and for any $f \in M^*$, $\mathrm{Ker}(f)$ is tight closed.

Let V be a finite-dimensional vector space over a field. Let W be any subspace of V. Then there exists a subspace W' of V such that $V = W \oplus W'$. So, we can define a map $f : V \to V$ such that $f|_W = 0$ and $f|_{W'} = 1_{W'}$. Then $f \in V^*$ and $\mathrm{Ker}(f) = W$. Hence W is tight closed in V. Therefore every subspace of a finite-dimensional vector space V over a field is tight closed in V.

Now, let A be an algebra over a field k. Let P be a finitely generated projective A-module. Then there exists an A-module Q and an integer n such that $P \oplus Q = A^n$. So, we can define a map $f : A^n \to A^n$ such that $f|_P = 0$ and $f|_Q = 1_Q$. Then $f \in (A^n)^*$ and $\mathrm{Ker}(f) = P$. Hence, P is tight closed in a free A-module. Therefore every finitely generated projective A-module is tight closed in a free R-module.

Let R be a ring. A submodule K of an R-module M is called *closed* [10, p. 548] if K has no proper essential extension in M.

Proposition 3.4. *Let R be a ring and let M be an R-module such that $Z(M) = 0$. If N is tight closed in M, then N is closed in M.*

Proof. Suppose that N has a proper essential extension E in M. Then there exists an element $e \in E \backslash N$. So, $0 \neq Re \subseteq E$. Since N is essential in E, $Re \cap N \neq 0$. There exists a non-zero element n such that $n \in Re \cap N$. There exists an element $r \in R$ such that $n = re$.

Now, let f be any element in $G_{M^*}(N,0)$. Then $n \in \mathrm{Ker}(f)$. So, $0 = f(n) = rf(e)$. Since $Z(M) = 0$, we have $f(e) = 0$. Hence $e \in \mathrm{Ker}(f)$. This shows that

$$e \in \cap_{f \in G_{M^*}(N,0)} \mathrm{Ker}(f).$$

Since N is tight closed in M, we have $e \in N$. This contradiction shows that N has no proper essential extension. □

Theorem 3.5. *Let R be a ring. Let N be a submodule of an R-module K. If no proper extension of N in K is essential in K, then N is essential in K.*

Proof. Suppose that N is not essential in K. Then there exists a non-zero submodule L of K such that $N \cap L = 0$. By Zorn's lemma, we may assume that L is maximal among such. By assumption, $N \oplus L$ is not essential in K. Then there exists a non-zero submodule F of K such that $(N \oplus L) \cap F = 0$. Hence $N \cap (L \oplus F) = 0$. By the maximality of L, $L = L \oplus F \supseteq F$, so $F = L \cap F = 0$. This is a contradiction. Hence N is essential in K. □

Corollary 3.6. *Let R be a ring. Let N be a submodule of K and let K be a submodule of M. If no proper extension of N in K is essential in K and if N is closed in M, then $N = K$.*

Let R be a ring and let N be a submodule of M. Let

$$K = \cap_{f \in G_{M^*}(N,0)} \mathrm{Ker}(f).$$

Then $N \subseteq K$. If no proper extension of N in K is essential in K and if N is closed in M, then it follows from Corollary 3.6 that N is tight closed in M.

It is well-known [4] that every epimorphism of a multiplication module onto itself is an automorphism. If M is a non-zero multiplication R-module whose endomorphism ring is an integral domain, then we show that every non-zero endomorphism of M is a monomorphism.

Lemma 3.7. *Let M be a non-zero multiplication R-module whose endomorphism ring M^* is an integral domain. Then*

(1) *For every non-zero submodule N of M, $G_{M^*}(N,0) = 0$. Hence every non-zero endomorphism of M is a monomorphism.*
(2) *The only tight closed submodule of M are 0 and M itself.*

Proof. (1) Suppose on the contrary that there exists a non-zero submodule N of M such that $G_{M^*}(N,0) \neq 0$. Then there exists a non-zero f in $G_{M^*}(N,0)$. Since M is a multiplication R-module, there exist ideals I, J of R such that $N = IM$ and $f(M) = JM$. Hence we have

$$0 = f(N) = f(IM) = If(M) = I(JM) = (IJ)M.$$

This implies that $IJ \subseteq \mathrm{Ann}_R(M)$. Since M^* is an integral domain, it follows from the argument just prior to Example 1.1 that $\mathrm{Ann}_R(M)$ is a prime ideal of R. So, we have $I \subseteq \mathrm{Ann}_R(M)$ or $J \subseteq \mathrm{Ann}_R(M)$. If $I \subseteq \mathrm{Ann}_R(M)$, then $N = IM = 0$, a contradiction. Or, if $J \subseteq \mathrm{Ann}_R(M)$, then $f(M) = JM = 0$ and hence $f = 0$, a contradiction. Therefore, for every non-zero submodule N of M, $G_{M^*}(N, 0) = 0$.

Assume that there exists an non-zero endomorphism f of M such that $\mathrm{Ker}(f) \neq 0$. Then by the previous argument we have $f \in G_{M^*}(\mathrm{Ker}(f), 0) = 0$. Thus $f = 0$, a contradiction. Hence every non-zero endomorphism of M is a monomorphism.

(2) Let N be a non-zero tight closed submodule of M. Then by (1), $G_{M^*}(N, 0) = 0$ and so $N = \cap_{f \in G_{M^*}(N,0)} \mathrm{Ker}(f) = \mathrm{Ker}(0) = M$. □

A submodule L of an R-module M is said to be M*-cyclic* if L is isomorphic to M/N for some submodule N of M.

Let L be a submodule of an R-module M. Assume that L is M-cyclic. Then there exists a submodule N of M such that $L \cong M/N$. There exists an isomorphism $g : M/N \to L$. Consider the composite map

$$f : M \xrightarrow{\pi} M/N \xrightarrow{g} L \xrightarrow{\mathrm{inc}} M.$$

Then $f \in M^*$ and $f(M) = (\mathrm{inc} g \pi)(M) = L$.

Conversely, assume that there exists $f \in M^*$ such that $L = f(M)$. Then by the first isomorphism theorem for modules $L = f(M) \cong M/\mathrm{Ker}(f)$.

This shows that L is M-cyclic if and only if there exists $f \in M^*$ such that $L = f(M)$.

An R-module M is said to be *semi-injective* if every homomorphism from an M-cyclic submodule of M to M can be extended to M. Compare the following lemma with [13, Lemma 2.2].

Lemma 3.8. *Let R be a ring and let M be an R-module. Then M is semi-injective if and only if for every $f \in M^*$, $G_{M^*}(\mathrm{Ker}(f), 0)$ is a cyclic left ideal of M^* generated by f.*

Proof. Let f be any element of M^*. For any $g \in M^*$,

$$(gf)(\mathrm{Ker}(f)) = g(f(\mathrm{Ker}(f))) = g(0) = 0.$$

Hence $M^* f \subseteq G_{M^*}(\mathrm{Ker}(f), 0)$. Conversely, let $h \in G_{M^*}(\mathrm{Ker}(f), 0)$. Then $h(\mathrm{Ker}(f)) = 0$, so $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(h)$. Define a map $\varphi : f(M) \to M$ by $\varphi(f(m)) = h(m)$, where $m \in M$. Then

$$f(m) = 0 \Rightarrow m \in \mathrm{Ker}(f) \subseteq \mathrm{Ker}(h) \Rightarrow h(m) = 0.$$

This shows that φ is well-defined. Further, φ is an R-homomorphism and $\varphi f = h$. Now, assume that M is semi-injective. Consider the following diagram:

$$\begin{array}{ccc} 0 \to f(M) & \xrightarrow{\text{inc}} & M \\ & & \Big\downarrow{\scriptstyle \varphi} \\ & & M \end{array}$$

Then $f(M)$ is M-cyclic, so there exists $g \in M^*$ such that $g\text{inc} = \varphi$. Hence

$$h = \varphi f = g\text{inc}f = gf.$$

Thus $h \in M^* f$. This shows that $G_{M^*}(\mathrm{Ker}(f), 0) \subseteq M^* f$. Therefore

$$G_{M^*}(\mathrm{Ker}(f), 0) = M^* f.$$

Assume that for every $f \in M^*$, $G_{M^*}(\mathrm{Ker}(f), 0)$ is a cyclic left ideal of M^* generated by f. Consider the following diagram:

$$\begin{array}{ccc} 0 \to L & \xrightarrow{\text{inc}} & M \\ & & \Big\downarrow{\scriptstyle \varphi} \\ & & M \end{array}$$

where L is M-cyclic. Then there exists $f \in M^*$ such that $L = f(M)$. $\varphi f \in M^*$ and $(\varphi f)(\mathrm{Ker}(f)) = 0$, so $\varphi f \in G_{M^*}(\mathrm{Ker}(f), 0)$. By our assumption, $G_{M^*}(\mathrm{Ker}(f), 0) = M^* f$. So, there exists $g \in M^*$ such that $\varphi f = gf$. Hence, for any $m \in M$,

$$(g\text{inc})(f(m)) = (g\text{inc}f)(m) = (\varphi f)(m) = \varphi(f(m)).$$

This shows that $g\text{inc} = \varphi$. Therefore M is semi-injective. □

Lemma 3.9. *Let R be a ring. Let M be an R-module such that $Z(M^*) = 0$. If f and g are elements of M^* such that $fg = 1_M$, then $gf = 1_M$.*

Proof. Assume $fg = 1_M$. Then $g \neq 0$. Further, $(gf - 1_M)g = 0$. Hence $gf - 1_M = 0$ and thus $gf = 1_M$. □

Theorem 3.10. *Let R be a ring. Let M be a multiplication R-module such that $Z(M^*) = 0$. Then M is semi-injective if and only if M^* is a division ring.*

Proof. Assume that M is semi-injective. Let f be any non-zero element of M^*. Then by Lemma 3.7 (1), $\mathrm{Ker}(f) = 0$. So,

$$M^* f = G_{M^*}(\mathrm{Ker}(f), 0) = G_{M^*}(0, 0) = M^*.$$

By Lemma 3.9, f is an epimorphism. Therefore M^* is a division ring.

Conversely, assume that M^* is a division ring. Let f be any non-zero element of M^*. Then f is an automorphism. Hence

$$G_{M^*}(\mathrm{Ker}(f), 0) = G_{M^*}(0, 0) = M^* = (f).$$

By Lemma 3.8, M is semi-injective. □

An R-module M is said to be *self-cogenerated* if every submodule of M is tight closed. If M is a simple R-module, then by the statement just posterior to Proposition 3.3, M is self-cogenerated.

Theorem 3.11. *Let R be a commutative ring with identity. Let M be a multiplication R-module such that $Z(M^*) = 0$. Then the following statements are equivalent.*

(1) *M is self-cogenerated*;
(2) *For any non-zero $f \in M^*$, f is an epimorphism*;
(3) *M is simple.*

Proof. We have already known that M^* is a commutative ring with identity. Hence by our hypothesis M^* is an integral domain.

(1) $\Rightarrow$ (2). Assume (1). Let f be any non-zero element of M^*. Then $\mathrm{Im}(f)(\subseteq M)$ is tight closed. By Lemma 3.7 (2), $\mathrm{Im}(f) = M$. Hence f is an epimorphism. Hence (2) follows.

(2) $\Rightarrow$ (3). Assume (2). By the statement just prior to Lemma 3.7, M^* is a field.

Now, let N be any non-zero submodule of M. Since M is a multiplication R-module, there exists an ideal I of R such that $N = IM$. Then

$$N = IM = \varphi_I(M) = \varphi(I)(M).$$

Hence $\varphi(I) \neq 0$. There exists an element $r \in I$ such that $\varphi(r) \neq 0$. $\varphi_r = \varphi(r) \neq 0$. By our assumption, φ_r has an inverse φ_r^{-1} in M^*. Further, $\varphi_r^{-1} \in M^*$. By Lemma 1.4, N is fully invariant. So,

$$M = \varphi_r^{-1}\varphi_r(M) \subseteq \varphi_r^{-1}(\varphi_I(M)) = \varphi_r^{-1}(N) \subseteq N.$$

Hence $N = M$. Thus, M is simple. Therefore (3) follows.

(3) $\Rightarrow$ (1). Assume (3). Let N be any submodule of M. Then $N = 0$ or $N = M$. By the statement just posterior to Proposition 3.3, 0 and M are tight closed. Hence N is tight closed. Thus M is self-cogenerated. Therefore, (1) follows. □

Corollary 3.12. *Let R be a commutative ring with identity. Let M be a multiplication R-module such that $Z(M^*) = 0$. Then the following statements are equivalent.*

(1) *M is semi-injective*;
(2) *M^* is a field*;
(3) *M is self-cogenerated*;
(4) *For any non-zero $f \in M^*$, f is an epimorphism*;
(5) *M is simple.*

Proof. M^* is a commutative ring with identity.

(5) $\Rightarrow$ (2) follows from Schur's Lemma. (2) $\Rightarrow$ (5) follows from the proof of Theorem 3.11 (2) $\Rightarrow$ (3). The remainder of the proof follows from Theorem 3.10 and Theorem 3.11. □

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