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# ON STRONGLY $\theta$-$e$-CONTINUOUS FUNCTIONS

Murad Özkoç and Gülhan Aslim

Abstract. A new class of generalized open sets in a topological space, called $e$-open sets, is introduced and some properties are obtained by Ekici [6]. This class is contained in the class of $\delta$-semi-preopen (or $\delta$-$\beta$-open) sets and weaker than both $\delta$-semiopen sets and $\delta$-preopen sets. In order to investigate some different properties we introduce two strong form of $e$-open sets called $e$-regular sets and $e$-$\theta$-open sets. By means of $e$-$\theta$-open sets we also introduce a new class of functions called strongly $\theta$-$e$-continuous functions which is a generalization of $\theta$-precontinuous functions. Some characterizations concerning strongly $\theta$-$e$-continuous functions are obtained.

## 1. Introduction

The concept of strong $\theta$-continuity which is stronger than $\delta$-continuity [14] is introduced by Noiri [14]. Some properties of strongly $\theta$-continuous functions defined by $\theta$-open sets are studied by Long and Herrington [11]. Recently, four generalizations of strong $\theta$-continuity are obtained by Jafari and Noiri [9], Noiri [15], Noiri and Popa [16] and Park [17]. In this paper, we introduce and investigate some fundamental properties of strongly $\theta$-$e$-continuous functions defined via $e$-open sets introduced by Ekici [6] in a topological space. It turns out that strong $\theta$-$e$-continuity is stronger than strong $\theta$-$\beta$-continuity [16] and weaker than strong $\theta$-precontinuity [15].

## 2. Preliminaries

Throughout the present paper, spaces $X$ and $Y$ always mean topological spaces. Let $X$ be a topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $cl(A)$ and $int(A)$, respectively. A subset $A$ is said to be regular open (resp. regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The $\delta$-interior [22] of a subset $A$ of $X$ is the union of all regular open sets of $X$ contained in $A$ and is denoted by $int_\delta(A)$. The subset $A$ is called $\delta$-open [22] if $A = int_\delta(A)$, i.e., a set is $\delta$-open if it is the union of

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regular open sets. The complement of a $\delta$-open set is called $\delta$-closed. Alternatively, a set $A \subset (X, \tau)$ is called $\delta$-closed [22] if $A = cl_\delta(A)$, where $cl_\delta(A) = \{x | \, x \in U \in \tau \Rightarrow int(cl(A)) \cap A \neq \emptyset\}$. The family of all $\delta$-open (resp. $\delta$-closed) sets in $X$ is denoted by $\delta O(X)$ (resp. $\delta C(X)$).

The $e$-interior [6] of a subset $A$ of $X$ is the union of all $e$-open sets of $X$ contained in $A$ and is denoted by $e\text{-}int(A)$. The $e$-closure [6] of a subset $A$ of $X$ is the intersection of all $e$-closed sets of $X$ containing $A$ and is denoted by $e\text{-}cl(A)$.

A subset $A$ of $X$ is called semiopen [10] (resp. $\alpha$-open [13], $\delta$-semiopen [18], preopen [12], $\delta$-preopen [19], $e$-open [6], semi-preopen [2] (or $\beta$-open [1]), $\delta$-semi-preopen (or $\delta$-$\beta$-open [8]) if $A \subset cl(int(A))$ (resp. $A \subset int(cl(int(A)))$, $A \subset cl(int_\delta(A))$, $A \subset int(cl(A))$, $A \subset int(cl_\delta(A))$, $A \subset int(cl_\delta(A)) \cup cl(int_\delta(A))$, $A \subset cl(int(cl(A)))$, $A \subset cl(int(cl_\delta(A))))$ and the complement of a semiopen (resp. $\alpha$-open, $\delta$-semiopen, preopen, $\delta$-preopen, semi-preopen, $\delta$-semi-preopen) set are called semiclosed (resp. $\alpha$-closed, $\delta$-semiclosed, preclosed, $\delta$-preclosed, semi-preclosed, $\delta$-semi-preclosed). A subset $A$ is called $\delta$-semi regular [18] (resp. $\delta$-pre-regular [19]) if it is $\delta$-semiopen and $\delta$-semiclosed (resp. $\delta$-preopen and $\delta$-preclosed). The intersection of all semiclosed (resp. preclosed, $\delta$-semiclosed, $\delta$-preclosed) sets of $X$ containing $A$ is called the semi-closure [5] (resp. pre-closure [12], $\delta$-semi-closure [18], $\delta$-pre-closure [19]) of $A$ and is denoted by $scl(A)$ (resp. $pcl(A)$, $\delta\text{-}scl(A)$, $\delta\text{-}pcl(A)$). Dually, the semi-interior (resp. pre-interior, $\delta$-semi-interior, $\delta$-pre-interior) of $A$ is defined to be the union of all semiopen (resp. preopen, $\delta$-semiopen, $\delta$-preopen) sets contained in $A$ and is denoted by $sint(A)$ (resp. $pint(A)$, $\delta\text{-}sint(A)$, $\delta\text{-}pint(A)$). The family of all $\delta$-semiopen (resp. $\delta$-preopen, $\delta$-semi-preopen (or $\delta$-$\beta$-open)) sets in $X$ is denoted by $\delta SO(X)$ (resp. $\delta PO(X), \delta\beta O(X)$).

**Lemma 2.1** ([18], [19]). *Let $A$ be a subset of a space $X$. Then the following hold*:

(a) $\delta\text{-}scl(A) = A \cup int(cl_\delta(A))$, $\delta\text{-}sint(A) = A \cap cl(int_\delta(A))$,

(b) $\delta\text{-}pcl(A) = A \cup cl(int_\delta(A))$, $\delta\text{-}pint(A) = A \cap int(cl_\delta(A))$.

**Lemma 2.2** ([18], [19]). *Let $A$ be a subset of a space $X$. Then the following hold*:

(a) $\delta\text{-}scl(\delta\text{-}sint(A)) = \delta\text{-}sint(A) \cup int(cl(int_\delta(A)))$,

(b) $\delta\text{-}pcl(\delta\text{-}pint(A)) = \delta\text{-}pint(A) \cup cl(int_\delta(A))$.

**Lemma 2.3** ([22]). *Let $A$ and $B$ be any subsets of a space $X$. Then the following hold*:

(a) $A \in \delta O(X)$ *if and only if* $A = int_\delta(A)$,

(b) $cl_\delta(X \backslash A) = X \backslash int_\delta(A)$,

(c) $int_\delta(A \cap B) = int_\delta(A) \cap int_\delta(B)$,

(d) *If $A_\alpha$ is $\delta$-open in $X$ for each $\alpha \in \Lambda$, then $\cup_{\alpha \in \Lambda} A_\alpha$ is $\delta$-open in $X$.*

The family of all $e$-open ($e$-closed) sets in $X$ will be denoted by $eO(X)$ ($eC(X)$), respectively.

## 3. $e$-regular sets and $e$-$\theta$-open sets

In this section we introduce some strong types of $e$-open sets, called $e$-regular sets and $e$-$\theta$-open sets. Using these sets we give a characterization of $e$-open sets and some properties.

**Definition 3.1.** A subset $A$ of a topological space $X$ is $e$-regular if it is $e$-open and $e$-closed. The family of all $e$-regular sets in $X$ will be denoted by $eR(X)$. The family of all $e$-regular sets which contain $x$ in $X$ will be denoted by $eR(X, x)$.

**Theorem 3.1.** *For a subset $A$ of a topological space $X$, the following properties hold*:

(a) $A \in eO(X)$ *if and only if* $e\text{-}cl(A) \in eR(X)$,

(b) $A \in eC(X)$ *if and only if* $e\text{-}int(A) \in eR(X)$.

*Proof.* We will prove only the first statement. The second one can be proved similarly.

*Necessity.* Let $A \in eO(X)$.

$$\begin{aligned} A \in eO(X) &\Rightarrow A \subset int(cl_\delta(A)) \cup cl(int_\delta(A)) \\ &\Rightarrow e\text{-}cl(A) \subset e\text{-}cl(int(cl_\delta(A)) \cup cl(int_\delta(A))) \\ &\Rightarrow e\text{-}cl(A) \subset \delta\text{-}scl(int(cl_\delta(A)) \cup cl(int_\delta(A))) \cap \\ &\qquad \delta\text{-}pcl(int(cl_\delta(A)) \cup cl(int_\delta(A))) \end{aligned}$$

From Lemma 2.1 we have

$$\begin{aligned} & e\text{-}cl(A) \\ \subset\ & [(int(cl_\delta(A)) \cup cl(int_\delta(A))) \cup int(cl_\delta(int(cl_\delta(A)) \cup cl(int_\delta(A))))] \cap \\ & [(int(cl_\delta(A)) \cup cl(int_\delta(A))) \cup cl(int_\delta(int(cl_\delta(A)) \cup cl(int_\delta(A))))] \\ \subset\ & [(int(cl_\delta(A)) \cup cl(int_\delta(A))) \cup (int(cl_\delta(int(cl_\delta(A)))) \cup cl_\delta(cl(int_\delta(A))))] \cap \\ & [(int(cl_\delta(A)) \cup cl(int_\delta(A))) \cup (cl(int(cl_\delta(A)) \cup cl(int_\delta(A))))] \\ \subset\ & [(int(cl_\delta(A)) \cup cl(int_\delta(A))) \cup (int(cl_\delta(A) \cup cl_\delta(cl_\delta(int_\delta(A)))))] \cap \\ & [(int(cl_\delta(A)) \cup cl(int_\delta(A))) \cup (cl(int(cl_\delta(A))) \cup cl(cl(int_\delta(A))))] \\ \subset\ & [(int(cl_\delta(A)) \cup cl(int_\delta(A))) \cup (int(cl_\delta(A)) \cup cl_\delta(int_\delta(A)))] \cap \\ & [(int(cl_\delta(A)) \cup cl(int_\delta(A))) \cup (cl(int(cl_\delta(A))) \cup cl(int_\delta(A)))] \\ \subset\ & [(int(cl_\delta(A)) \cup cl(int_\delta(A))) \cup (int(cl_\delta(A)) \cup cl(int_\delta(A)))] \cap \\ & [cl(int(cl_\delta(A))) \cup cl(int_\delta(A))] \\ \subset\ & [(int(cl_\delta(A)) \cup cl(int_\delta(A)))] \cap [cl(int(cl_\delta(A))) \cup cl(int_\delta(A))] \\ \subset\ & int(cl_\delta(A)) \cup cl(int_\delta(A)). \end{aligned}$$

Since $A \subset e\text{-}cl(A)$, we have $e\text{-}cl(A) \subset int(cl_\delta(e\text{-}cl(A))) \cup cl(int_\delta(e\text{-}cl(A)))$. This shows that $e\text{-}cl(A)$ is an $e$-open set. On the other hand, $e\text{-}cl(A)$ is always an $e$-closed set. Therefore $e\text{-}cl(A)$ is an $e$-regular set.

*Sufficiency.* This follows from [6, Theorem 2.15]. □

**Theorem 3.2.** *For a subset $A$ of a topological space $X$, the following are equivalent*:
(a) $A \in eR(X)$,
(b) $A = e\text{-}cl(e\text{-}int(A))$,
(c) $A = e\text{-}int(e\text{-}cl(A))$.

*Proof.* The proofs of the implications (a) ⇒(b) and (a)⇒(c) are obvious.
(b) ⇒ (a): Since $e\text{-}cl(A)$ is $e$-closed, by Theorem 3.1(a) we have $e\text{-}int(e\text{-}cl(A)) \in eR(X)$ and $A \in eR(X)$.
(c) ⇒ (a): Since $e\text{-}int(A)$ is $e$-open, by Theorem 3.1(b) we have $e\text{-}cl(e\text{-}int(A)) \in eR(X)$ and $A \in eR(X)$. □

**Definition 3.2.** A point $x$ of $X$ is called an $e$-$\theta$-cluster point of $A$ if $e\text{-}cl(U) \cap A \neq \emptyset$ for every $U \in eO(X, x)$. The set of all $e$-$\theta$-cluster points of $A$ is called $e$-$\theta$-closure of $A$ and is denoted by $e\text{-}cl_\theta(A)$. A subset $A$ is said to be $e$-$\theta$-closed if $A = e\text{-}cl_\theta(A)$. The complement of an $e$-$\theta$-closed set is said to be $e$-$\theta$-open.

**Theorem 3.3.** *For any subset $A$ of a space $X$, we have*

$$\begin{aligned} e\text{-}cl_\theta(A) &= \cap \{ V \mid A \subset V \text{ and } V \text{ is } e\text{-}\theta\text{-closed} \} \\ &= \cap \{ V \mid A \subset V \text{ and } V \in eR(X) \} . \end{aligned}$$

*Proof.* We prove only the first equality since the other is similarly proved. First, suppose that $x \notin e\text{-}cl_\theta(A)$. Then there exists $V \in eO(X, x)$ such that $e\text{-}cl(V) \cap A = \emptyset$. By Theorem 3.1, $X \backslash e\text{-}cl(V)$ is $e$-regular and hence $X \backslash e\text{-}cl(V)$ is an $e$-$\theta$-closed set containing $A$ and $x \notin X \backslash e\text{-}cl(V)$. Therefore, we have $x \notin \cap \{ V \mid A \subset V$ and $V$ is $e$-$\theta$-closed$\}$ . Conversely, suppose that $x \notin \cap \{ V \mid A \subset V$ and $V$ is $e$-$\theta$-closed$\}$. There exists an $e$-$\theta$-closed set $V$ such that $A \subset V$ and $x \notin V$. There exists $U \in eO(X)$ such that $x \in U \subset e\text{-}cl(U) \subset X \backslash V$. Therefore, we have $e\text{-}cl(U) \cap A \subset e\text{-}cl(U) \cap V = \emptyset$. This shows that $x \notin e\text{-}cl_\theta(A)$. □

**Theorem 3.4.** *Let $A$ and $B$ be any subsets of a space $X$. Then the following properties hold*:
(a) $x \in e\text{-}cl_\theta(A)$ *if and only if* $U \cap A \neq \emptyset$ *for each* $U \in eR(X, x)$,
(b) *If* $A \subset B$, *then* $e\text{-}cl_\theta(A) \subset e\text{-}cl_\theta(B)$,
(c) $e\text{-}cl_\theta(e\text{-}cl_\theta(A)) = e\text{-}cl_\theta(A)$,
(d) *If* $A_\alpha$ *is $e$-$\theta$-closed in* $X$ *for each* $\alpha \in \Lambda$, *then* $\cap_{\alpha \in \Lambda} A_\alpha$ *is $e$-$\theta$-closed in* $X$.

*Proof.* The proofs of (a) and (b) are obvious.
(c) Generally we have $e\text{-}cl_\theta(A) \subset e\text{-}cl_\theta(e\text{-}cl_\theta(A))$. Suppose that $x \notin e\text{-}cl_\theta(A)$. There exists $U \in eR(X, x)$ such that $U \cap A = \emptyset$. Since $U \in eR(X)$, we have $e\text{-}cl_\theta(A) \cap U = \emptyset$. This shows that $x \notin e\text{-}cl_\theta(e\text{-}cl_\theta(A))$. Therefore, we obtain $e\text{-}cl_\theta(e\text{-}cl_\theta(A)) \subset e\text{-}cl_\theta(A)$.

(d) Let $A_\alpha$ be $e$-$\theta$-closed in $X$ for each $\alpha \in \Lambda$. For each $\alpha \in \Lambda, A_\alpha = e\text{-}cl_\theta(A_\alpha)$. Hence we have

$$e\text{-}cl_\theta(\cap_{\alpha\in\Lambda} A_\alpha) \subset \cap_{\alpha\in\Lambda} e\text{-}cl_\theta(A_\alpha) = \cap_{\alpha\in\Lambda} A_\alpha \subset e\text{-}cl_\theta(\cap_{\alpha\in\Lambda} A_\alpha).$$

Therefore, we obtain $e\text{-}cl_\theta(\cap_{\alpha\in\Lambda} A_\alpha) = \cap_{\alpha\in\Lambda} A_\alpha$. This shows that $\cap_{\alpha\in\Lambda} A_\alpha$ is $e$-$\theta$-closed. □

*Remark* 3.5. The union of two $e$-$\theta$-closed sets is not necessarily $e$-$\theta$-closed as shown by the following example.

**Example 3.6.** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. The subsets $\{a\}$ and $\{b\}$ are $e$-$\theta$-closed in $(X, \tau)$ but $\{a, b\}$ is not $e$-$\theta$-closed.

**Corollary 3.7.** *Let $A$ and $A_\alpha(\alpha \in \Lambda)$ be any subsets of a space $X$. Then the following properties hold*:

(a) *$A$ is e-$\theta$-open in $X$ if and only if for each $x \in A$ there exists $U \in eR(X, x)$ such that $x \in U \subset A$,*

(b) *$e\text{-}cl_\theta(A)$ is e-$\theta$-closed,*

(c) *If $A_\alpha$ is e-$\theta$-open in $X$ for each $\alpha \in \Lambda$, then $\cup_{\alpha\in\Lambda} A_\alpha$ is e-$\theta$-open in $X$.*

**Theorem 3.8.** *For a subset $A$ of a space $X$, the following properties hold*:

(a) *If $A \in eO(X)$, then $e\text{-}cl(A) = e\text{-}cl_\theta(A)$,*

(b) *$A \in eR(X)$ if and only if $A$ is e-$\theta$-open and e-$\theta$-closed.*

*Proof.* (a) Generally we have $e\text{-}cl(B) \subset e\text{-}cl_\theta(B)$ for every subset $B$ of $X$. Let $A \in eO(X)$ and suppose that $x \notin e\text{-}cl(A)$. Then there exists $U \in eO(X, x)$ such that $U \cap A = \emptyset$. Since $A \in eO(X)$, we have $e\text{-}cl(U) \cap A = \emptyset$. This shows that $x \notin e\text{-}cl_\theta(A)$. Hence we obtain $e\text{-}cl(A) = e\text{-}cl_\theta(A)$.

(b) Let $A \in eR(X)$. Then $A \in eO(X)$ and by (a), $A = e\text{-}cl(A) = e\text{-}cl_\theta(A)$. Therefore, $A$ is $e$-$\theta$-closed. Since $X \backslash A \in eR(X)$, by the argument above, $X \backslash A$ is $e$-$\theta$-closed and hence $A$ is $e$-$\theta$-open. The converse is obvious. □

*Remark* 3.9. It can be easily shown that $e$-regular $\Rightarrow$ $e$-$\theta$-open $\Rightarrow$ $e$-open. But the converses are not necessarily true as shown by the following examples.

**Example 3.10.** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then the subset $\{a, b\}$ is $e$-$\theta$-open in $X$ but not $e$-regular.

**Example 3.11.** Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Then the subset $\{a\}$ is $e$-open in $X$ but not $e$-$\theta$-open.

## 4. Strongly $\theta$-$e$-continuous functions and some properties

In this section, we introduce a new type of continuous functions and look into some relations with other types.

**Definition 4.1.** A function $f : X \to Y$ is said to be strongly $\theta$-$e$-continuous (briefly, st.$\theta$.$e.c.$) if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists an $e$-open set $U$ of $X$ containing $x$ such that $f(e\text{-}cl(U)) \subset V$.

**Definition 4.2.** A function $f : X \to Y$ is said to be

(a) strongly $\theta$-continuous [14] if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists an open set $U$ of $X$ containing $x$ such that $f(cl(U)) \subset V$;

(b) strongly $\theta$-semicontinuous [9] if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists a semi-open set $U$ of $X$ containing $x$ such that $f(scl(U)) \subset V$;

(c) strongly $\theta$-precontinuous [15] if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists a preopen set $U$ of $X$ containing $x$ such that $f(pcl(U)) \subset V$;

(d) strongly $\theta$-$\beta$-continuous [16] if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists a semi-preopen set $U$ of $X$ containing $x$ such that $f(spcl(U)) \subset V$;

(e) strongly $\theta$-$b$-continuous [17] if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists a $b$-open set $U$ of $X$ containing $x$ such that $f(bcl(U)) \subset V$;

(f) $b$-continuous [7] if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in BO(X, x)$ such that $f(U) \subset V$.

*Remark* 4.1. From Definitions 4.1 and 4.2, we have the following diagram:

$$
\begin{array}{ccccccc}
 & & \text{st. } \theta\text{-precon.} & \rightarrow & \text{st. } \theta\text{-}e\text{-con.} & & \\
 & \nearrow & & \searrow & & \searrow & \\
\text{st. } \theta\text{-con.} & & & & \text{st. } \theta\text{-}b\text{-con.} & \rightarrow & \text{st. } \theta\text{-}\beta\text{-con.} \\
 & \searrow & & \nearrow & & & \\
 & & \text{st. } \theta\text{-semicon.} & & & &
\end{array}
$$

However, none of these implications is reversible as shown by the following examples.

**Example 4.2.** Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$ and $\sigma = \{\emptyset, X, \{b, c, d\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is both $b$-continuous and strongly $\theta$-$b$-continuous but not strongly $\theta$-$e$-continuous.

**Example 4.3.** Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$ and $\sigma = \{\emptyset, X, \{d\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is strongly $\theta$-$e$-continuous but not strongly $\theta$-$b$-continuous.

**Example 4.4.** Let $\tau$ be the usual topology for $\mathbb{R}$ and $\sigma = \{[0, 1] \cup ((1, 2) \cap \mathbb{Q}), \emptyset, \mathbb{R}\}$, where $\mathbb{Q}$ denotes the set of rational numbers. Then the identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ is strongly $\theta$-$e$-continuous but neither strongly $\theta$-precontinuous nor strongly $\theta$-semicontinuous.

**Example 4.5.** Let $\tau$ be the usual topology for $\mathbb{R}$ and $\sigma = \{\emptyset, \mathbb{R}, [0, 1) \cap \mathbb{Q}\}$. Then the identity function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ is strongly $\theta$-$\beta$-continuous but not strongly $\theta$-$e$-continuous.

**Theorem 4.6.** *For a function $f : X \to Y$, the following are equivalent*:

(a) *$f$ is strongly $\theta$-e-continuous,*

(b) *For each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in eR(X,x)$ such that $f(U) \subset V$,*

(c) *$f^{-1}(V)$ is e-$\theta$-open in $X$ for each open set $V$ of $Y$,*

(d) *$f^{-1}(F)$ is e-$\theta$-closed in $X$ for each closed set $F$ of $Y$,*

(e) *$f(e\text{-}cl_\theta(A)) \subset cl(f(A))$ for each subset $A$ of $X$,*

(f) *$e\text{-}cl_\theta(f^{-1}(B)) \subset f^{-1}(cl(B))$ for each subset $B$ of $Y$.*

*Proof.* (a) $\Rightarrow$ (b). It follows from Theorem 3.1.

(b) $\Rightarrow$ (c). Let $V$ be any open set of $Y$ and $x \in f^{-1}(V)$. There exists $U \in eR(X,x)$ such that $f(U) \subset V$. Therefore, we have $x \in U \subset f^{-1}(V)$. Hence by Corollary 3.7.(a), $f^{-1}(V)$ is $e$-$\theta$-open in $X$.

(c) $\Rightarrow$ (d). This is obvious.

(d) $\Rightarrow$ (e). Let $A$ be any subset of $X$. Since $cl(f(A))$ is closed in $Y$, by (d) $f^{-1}(cl(f(A)))$ is $e$-$\theta$-closed and we have

$$e\text{-}cl_\theta(A) \subset e\text{-}cl_\theta(f^{-1}(f(A))) \subset e\text{-}cl_\theta(f^{-1}(cl(f(A)))) = f^{-1}(cl(f(A))).$$

Therefore, we obtain $f(e\text{-}cl_\theta(A)) \subset cl(f(A))$.

(e) $\Rightarrow$ (f). Let $B$ be any subset of $Y$. By (e), we obtain $f(e\text{-}cl_\theta(f^{-1}(B))) \subset cl(f(f^{-1}(B))) \subset cl(B)$ and hence $e\text{-}cl_\theta(f^{-1}(B)) \subset f^{-1}(cl(B))$.

(f) $\Rightarrow$ (a). Let $x \in X$ and $V$ be any open set of $Y$ containing $f(x)$. Since $Y \backslash V$ is closed in $Y$, we have $e\text{-}cl_\theta(f^{-1}(Y\backslash V)) \subset f^{-1}(cl(Y\backslash V)) = f^{-1}(Y\backslash V)$. Therefore, $f^{-1}(Y\backslash V)$ is $e$-$\theta$-closed in $X$ and $f^{-1}(V)$ is an $e$-$\theta$-open set containing $x$. There exists $U \in eO(X,x)$ such that $e\text{-}cl(U) \subset f^{-1}(V)$ and hence $f(e\text{-}cl(U)) \subset V$. This shows that $f$ is st.$\theta$.$e$.c. □

**Theorem 4.7.** *Let $Y$ be a regular space. Then $f : X \to Y$ is st.$\theta$.e.c. if and only if $f$ is e-continuous.*

*Proof.* Let $x \in X$ and $V$ an open set of $Y$ containing $f(x)$. Since $Y$ is regular, there exists an open set $W$ such that $f(x) \in W \subset cl(W) \subset V$. If $f$ is $e$-continuous, there exists $U \in eO(X,x)$ such that $f(U) \subset W$. We shall show that $f(e\text{-}cl(U)) \subset cl(W)$. Suppose that $y \notin cl(W)$. There exists an open set $G$ containing $y$ such that $G \cap W = \emptyset$. Since $f$ is $e$-continuous, $f^{-1}(G) \in eO(X)$ and $f^{-1}(G) \cap U = \emptyset$ and hence $f^{-1}(G) \cap e\text{-}cl(U) = \emptyset$. Therefore, we obtain $G \cap f(e\text{-}cl(U)) = \emptyset$ and $y \notin f(e\text{-}cl(U))$. Consequently, we have $f(e\text{-}cl(U)) \subset cl(W) \subset V$. The converse is obvious. □

**Definition 4.3.** A space $X$ is said to be $e$-regular if for each closed set $F$ and each point $x \in X\backslash F$, there exist disjoint $e$-open sets $U$ and $V$ such that $x \in U$ and $F \subset V$.

**Lemma 4.8.** *For a space $X$ the following are equivalent*:

(a) *$X$ is e-regular*;

(b) *For each point $x \in X$ and for each open set $U$ of $X$ containing $x$, there exists $V \in eO(X)$ such that $x \in V \subset e\text{-}cl(V) \subset U$;*

(c) *For each subset $A$ of $X$ and each closed set $F$ such that $A \cap F = \emptyset$, there exist disjoint $U, V \in eO(X)$ such that $A \cap U \neq \emptyset$ and $F \subset V$;*

(d) *For each closed set $F$ of $X$, $F = \cap \{e\text{-}cl(V) | F \subset V,\ V \in eO(X)\}$.*

**Theorem 4.9.** *A continuous function $f : X \to Y$ is st.$\theta$.e.c. if and only if $X$ is e-regular.*

*Proof. Necessity.* Let $f : X \to X$ be the identity function. Then $f$ is continuous and st.$\theta$.$e$.c. by our hypothesis. For any open set $U$ of $X$ and any point $x \in U$, we have $f(x) = x \in U$ and there exists $V \in eO(X, x)$ such that $f(e\text{-}cl(V)) \subset U$. Therefore, we have $x \in V \subset e\text{-}cl(V) \subset U$. It follows from Lemma 4.8 that $X$ is $e$-regular.

*Sufficiency.* Suppose that $f : X \to Y$ is continuous and $X$ is $e$-regular. For any $x \in X$ and open set $V$ containing $f(x), f^{-1}(V)$ is an open set containing $x$. Since $X$ is $e$-regular, there exists $U \in eO(X)$ such that $x \in U \subset e\text{-}cl(U) \subset f^{-1}(V)$. Therefore, we have $f(e\text{-}cl(U)) \subset V$. This shows that $f$ is st.$\theta$.$e$.c. □

**Theorem 4.10.** *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ be the graph function of $f$. If $g$ is st.$\theta$.e.c., then $f$ is st.$\theta$.e.c. and $X$ is e-regular.*

*Proof.* First, we show that $f$ is st.$\theta$.$e$.c. Let $x \in X$ and $V$ an open set of $Y$ containing $f(x)$. Then $X \times V$ is an open set of $X \times Y$ containing $g(x)$. Since $g$ is st.$\theta$.$e$.c., there exists $U \in eO(X, x)$ such that $g(e\text{-}cl(U)) \subset X \times V$. Therefore, we obtain $f(e\text{-}cl(U)) \subset V$. Next, we show that $X$ is $e$-regular. Let $U$ be any open set of $X$ and $x \in U$. Since $g(x) \in U \times Y$ and $U \times Y$ is open in $X \times Y$, there exists $G \in eO(X, x)$ such that $g(e\text{-}cl(G)) \subset U \times Y$. Therefore, we obtain $x \in G \subset e\text{-}cl(G) \subset U$ and hence $X$ is $e$-regular. □

**Theorem 4.11.** *Let $f : X \to Y$ and $g : Y \to Z$ be functions. If $f$ is st.$\theta$.e.c. and $g$ is continuous, then the composition $g \circ f : X \to Z$ is st.$\theta$.e.c.*

*Proof.* It is clear from Theorem 4.6. □

We recall that a space $X$ is said to be *submaximal* [20] if each dense subset of $X$ is open in $X$. It is shown in [20] that a space $X$ is submaximal if and only if every preopen set of $X$ is open. A space $X$ is said to be *extremally disconnected* [4] if the closure of each open set of $X$ is open. Note that an extremally disconnected space is exactly the space where every semiopen set is $\alpha$-open.

**Theorem 4.12.** *Let $X$ be a submaximal extremally disconnected space. Then the following properties are equivalent for a function $f : X \to Y$.*

(a) *$f$ is strongly $\theta$-continuous*;

(b) *$f$ is strongly $\theta$-semicontinuous*;

(c) *$f$ is strongly $\theta$-precontinuous*;

(d) *$f$ is strongly $\theta$-b-continuous*;

(e) *$f$ is strongly $\theta$-e-continuous*;

(f) *$f$ is strongly $\theta$-$\beta$-continuous.*

*Proof.* It follows from the fact that if $X$ is submaximal extremally disconnected, then open set, preopen set, semiopen set, $b$-open set, $e$-open set and semi-preopen set are equivalent. $\square$

## 5. Separation axioms

We introduce a new type of space called $e$-$T_2$ and some properties between strongly $\theta$-$e$-continuous and $e$-$T_2$ are obtained.

**Definition 5.1.** A space $X$ is said to be $e$-$T_2$ if for each pair of distinct points $x$ and $y$ in $X$, there exist $U \in eO(X, x)$ and $V \in eO(X, y)$ such that $U \cap V = \emptyset$.

**Lemma 5.1.** *A space $X$ is $e$-$T_2$ if and only if for each pair of distinct points $x$ and $y$ in $X$, there exist $U \in eO(X, x)$ and $V \in eO(X, y)$ such that $e\text{-}cl(U) \cap e\text{-}cl(V) = \emptyset$.*

**Theorem 5.2.** *If $f : X \to Y$ is a st.$\theta$.e.c. injection and $Y$ is $T_0$, then $X$ is $e$-$T_2$.*

*Proof.* For any distinct points $x$ and $y$ of $X$, by hypothesis $f(x) \neq f(y)$ and there exists either an open set $V$ containing $f(x)$ not containing $f(y)$ or an open set $W$ containing $f(y)$ not containing $f(x)$. If the first case holds, then there exists $U \in eO(X, x)$ such that $f(e\text{-}cl(U)) \subset V$. Thus, we obtain $f(y) \notin f(e\text{-}cl(U))$ and hence $X \backslash e\text{-}cl(U) \in eO(X, y)$. If the second case holds, then we obtain a similar result. Thus, $X$ is $e$-$T_2$. $\square$

**Theorem 5.3.** *If $f : X \to Y$ is a st.$\theta$.e.c. function and $Y$ is Hausdorff, then the subset $A = \{(x, y) | f(x) = f(y)\}$ is $e$-$\theta$-closed in $X \times X$.*

*Proof.* It is clear that $f(x) \neq f(y)$ for each $(x, y) \notin A$. Since $Y$ is Hausdorff, there exist open sets $V$ and $W$ of $Y$ containing $f(x)$ and $f(y)$, respectively, such that $V \cap W = \emptyset$. Since $f$ is st.$\theta$.e.c., there exist $U \in eO(X, x)$ and $G \in eO(X, y)$ such that $f(e\text{-}cl(U)) \subset V$ and $f(e\text{-}cl(G)) \subset W$. Put $D = f(e\text{-}cl(U)) \times f(e\text{-}cl(G))$. It follows that $(x, y) \in D \in eR(X \times X)$ and $D \cap A = \emptyset$. This means that $e\text{-}cl_\theta(A) \subset A$ and thus, $A$ is $e$-$\theta$-closed in $X \times X$. $\square$

We recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) | x \in X\}$ of $X \times Y$ is called the graph of $f$ and is denoted by $G(f)$.

**Definition 5.2.** The graph $G(f)$ of a function $f : X \to Y$ is said to be strongly $e$-closed if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in eO(X, x)$ and an open set $V$ in $Y$ containing $y$ such that $(e\text{-}cl(U) \times V) \cap G(f) = \emptyset$.

**Lemma 5.4.** *The graph $G(f)$ of a function $f : X \to Y$ is strongly $e$-closed if and only if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in eO(X, x)$ and an open set $V$ in $Y$ containing $y$ such that $f(e\text{-}cl(U)) \cap V = \emptyset$.*

**Theorem 5.5.** *If $f : X \to Y$ is st.$\theta$.e.c. and $Y$ is Hausdorff, then $G(f)$ is strongly $e$-closed in $X \times Y$.*

*Proof.* It is clear that $f(x) \neq y$ for each $(x, y) \in (X \times Y)\backslash G(f)$. Since $Y$ is Hausdorff, there exist open sets $V$ and $W$ in $Y$ containing $f(x)$ and $y$, respectively, such that $V \cap W = \emptyset$. Since $f$ is st.$\theta$.e.c., there exists $U \in eO(X, x)$ such that $f(e\text{-}cl(U)) \subset V$. Thus, $f(e\text{-}cl(U)) \cap W = \emptyset$ and then by Lemma 5.4, $G(f)$ is strongly $e$-closed in $X \times Y$. □

## 6. Covering properties

**Definition 6.1.** A space $X$ is said to be

(a) $e$-closed if every cover of $X$ by $e$-open sets has a finite subcover whose preclosures cover $X$;

(b) countably $e$-closed if every countable cover of $X$ by $e$-open sets has a finite subcover whose preclosures cover $X$.

A subset $A$ of a space $X$ is said to be $e$-closed relative to $X$ if for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of $A$ by $e$-open sets of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $A \subset \cup\{e\text{-}cl(V_\alpha) : \alpha \in \Lambda_0\}$.

**Theorem 6.1.** *If $f : X \to Y$ is a st.$\theta$.e.c. function and $A$ is e-closed relative to $X$, then $f(A)$ is a compact set of $Y$.*

*Proof.* Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of $f(A)$ by open sets of $Y$. For each point $x \in A$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since $f$ is st.$\theta$.e.c., there exists $U_x \in eO(X, x)$ such that $f(e\text{-}cl(U_x)) \subset V_{\alpha(x)}$. The family $\{U_x : x \in A\}$ is a cover of $A$ by $e$-open sets of $X$ and hence there exists a finite subset $A_0$ of $A$ such that $A \subset \cup_{x \in A_0} e\text{-}cl(U_x)$. Therefore, we obtain $f(A) \subset \cup_{x \in A_0} V_{\alpha(x)}$. This shows that $f(A)$ is compact. □

**Corollary 6.2.** *Let $f : X \to Y$ be a st.$\theta$.e.c. surjection. Then the following properties hold*:

(a) *If $X$ is e-closed, then $Y$ is compact,*

(b)*If $X$ is countably e-closed, then $Y$ is countably compact.*

**Theorem 6.3.** *If a function $f : X \to Y$ has a strongly e-closed graph, then $f(A)$ is closed in $Y$ for each subset $A$ which is e-closed relative to $X$.*

*Proof.* Let $A$ be $e$-closed relative to $X$ and $y \in Y\backslash f(A)$. Then for each $x \in A$ we have $(x, y) \notin G(f)$ and by Lemma 5.4 there exist $U_x \in eO(X, x)$ and an open set $V_x$ of $Y$ containing $y$ such that $f(e\text{-}cl(U_x)) \cap V_x = \emptyset$. The family $\{U_x : x \in A\}$ is a cover of $A$ by $e$-open sets of $X$. Since $A$ is $e$-closed relative to $X$, there exists a finite subset $A_0$ of $A$ such that $A \subset \cup\{e\text{-}cl(U_x) : x \in A_0\}$. Put $V = \cap\{V_x : x \in A_0\}$. Then $V$ is an open set containing $y$ and $f(A) \cap V \subset [\cup_{x \in A_0} f(e\text{-}cl(U_x))] \cap V \subset \cup_{x \in A_0}[f(e\text{-}cl(U_x)) \cap V_x] = \emptyset$. Therefore, we have $y \notin cl(f(A))$ and hence $f(A)$ is closed in $Y$. □

**Theorem 6.4.** *Let $X$ be a submaximal extremally disconnected space. If a function $f : X \to Y$ has a strongly e-closed graph, then $f^{-1}(A)$ is $\theta$-closed in $X$ for each compact set $A$ of $Y$.*

*Proof.* Let $A$ be a compact set of $Y$ and $x \notin f^{-1}(A)$. Then for each $y \in A$ we have $(x, y) \notin G(f)$ and by Lemma 5.4 there exist $U_y \in eO(X, x)$ and an open set $V_y$ of $Y$ containing $y$ such that $f(e\text{-}cl(U_y)) \cap V_y = \emptyset$. The family $\{V_y : y \in A\}$ is an open cover of $A$ and there exists a finite subset $A_0$ of $A$ such that $A \subset \cup_{y \in A_0} V_y$. Since $X$ is submaximal extremally disconnected, each $U_y$ is open in $X$ and $e\text{-}cl(U_y) = cl(U_y)$. Set $U = \cap_{y \in A_0} U_y$, then $U$ is an open set containing $x$ and

$$f(cl(U)) \cap A \subset \cup_{x \in A_0}[f(cl(U)) \cap V_y] \subset \cup_{x \in A_0}[f(e\text{-}cl(U_y)) \cap V_y] = \emptyset.$$

Therefore, we have $cl(U) \cap f^{-1}(A) = \emptyset$ and hence $x \notin cl_\theta(f^{-1}(A))$. This shows that $f^{-1}(A)$ is $\theta$-closed in $X$. □

**Corollary 6.5.** *Let $X$ be a submaximal extremally disconnected space and $Y$ be a compact Hausdorff space. For a function $f : X \to Y$, the following properties are equivalent*:

(a) *$f$ is st.$\theta$.e.c.,*
(b) *$G(f)$ is strongly e-closed in $X \times Y$,*
(c) *$f$ is strongly $\theta$-continuous,*
(d) *$f$ is continuous,*
(e) *$f$ is e-continuous.*

*Proof.* (a) ⇒ (b). It follows from Theorem 5.5.
(b) ⇒ (c). It follows from Theorem 6.4.
(c) ⇒ (d) ⇒ (e). These are obvious.
(e) ⇒ (a). Since $Y$ is regular, it follows from Theorem 4.7. □

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Murad Özkoç
Ege University Faculty of Science Department of Mathematics
35100 Bornova-İZMİR, Turkey
*E-mail address*: murad.ozkoc@ege.edu.tr

Gülhan Aslim
Ege University Faculty of Science Department of Mathematics
35100 Bornova-İZMİR, Turkey
*E-mail address*: gulhan.aslim@ege.edu.tr