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FOURIER TRANSFORM AND L_p-MIXED PROJECTION BODIES

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Abstract. In this paper we define the L_p-mixed curvature function of a convex body. We develop a formula connection the support function of L_p-mixed projection body with Fourier transform of the L_p-mixed curvature function. Using this formula we solve an analog of the Shephard projection problem for L_p-mixed projection bodies.

1. Introduction

As usual, $\text{vol}_i(\cdot)$ denotes the i-dimensional Lebesgue measure and S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. Let B_n be the origin-symmetric standard unit ball in $\mathbb{R}^n$, and write ω_n for $\text{vol}_n(B_n)$. Note that

$$\omega_n = \pi^{\frac{n}{2}}/\Gamma(1+\frac{n}{2})$$

defines ω_n for all non-negative real n (not just the positive integers).

The Shephard problem states as follows: Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. Suppose that, for every $\theta \in S^{n-1}$,

$$\text{vol}_{n-1}(K|\theta^\perp) \leq \text{vol}_{n-1}(L|\theta^\perp).$$

Does it follow that

$$\text{vol}_n(K) \leq \text{vol}_n(L)?$$

This problem was solved independently by Petty [13] and Schneider [15], who showed that the answer is affirmative if $n \leq 2$ and negative if $n \geq 3$. It is also well-known ([16] pp. 422–423), that the Shephard problem has an affirmative answer if L is a projection body, i.e., if $\forall\theta \in S^{n-1}$

$$h_L(\theta) = \text{vol}_{n-1}(K|\theta^\perp) = \frac{1}{2}\int_{S^{n-1}} |\theta \cdot u| dS(K,u) \tag{1.1}$$

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for some convex body K. Here $S(K,\cdot)$ is the surface area measure (see [1]), and $h_L(x) = \max\{x\cdot y : y \in L\}$ is the support function of L. On the other hand, the existence of a convex body, which is not a projection body leads, to a counterexample. Thus the concept of the projection body represents one of the crucial steps in the solution of the Shephard problem.

Lutwak, Yang, and Zhang based on the classical projection body, first introduced the notion of L_p-projection body (see [11]). Let $\Pi_p K,\ p \geq 1$ denote the compact convex symmetric set whose support function is given by

$$h(\Pi_p K,\theta)^p = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |\theta\cdot u|^p dS_p(K,u), \forall\theta \in S^{n-1}, \tag{1.2}$$

where

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2\omega_n\omega_{p-1}}.$$

Here $S_p(K,\cdot)$ is the L_p-surface area measure, and its definition can be seen in [9] due to the L_p-Brunn-Minkowski Theory. A convex body M is called L_p-projection body if there is a convex body K such that $M = \Pi_p K$.

Projection body and intersection body are two basic concepts in geometric tomography. Since Koldobsky found the Fourier analytic characterization of intersection bodies, the Fourier analytic approach to Busemann-Petty problem has recently been developed and has led to many results (see [2, 5, 6, 7, 12, 17]). As the duality to intersection body, Koldobsky, Ryabogin, and Zvavitch [8] characterized the projection body with the Fourier transform. Moreover, Ryabogin and Zvavitch [14] proved that if the surface area measure of a convex body K is absolutely continuous, then

$$h(\Pi_p K,\xi)^p = \frac{n\omega_n c_{n-2,p}}{4\pi C_p} \widehat{f_p(K,\cdot)}(\xi). \tag{1.3}$$

Here $p \geq 1, p$ is not an even integer, $f_p(K,\cdot)$ is the L_p-curvature function of the body K, and C_p is a constant depending only on p. Using this formula, Ryabogin and Zvavitch considered the Shephard problem for L_p-projection bodies as follows.

Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ and $p \geq 1$. Suppose

$$\Pi_p K \subseteq \Pi_p L.$$

Does it follow that

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L) \text{ for } 1 \leq p < n$$

and

$$\mathrm{vol}_n(K) \geq \mathrm{vol}_n(L) \text{ for } p > n?$$

By using the Fourier transform analytic approach, they generalized the Shephard problem for L_p-projection body as follows:

Theorem A. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ and $p \geq 1$, $p \neq n$, p is not an even integer. Suppose that the support function h_L is infinitely smooth and the functions $C_p \hat{h_L^p}(\theta) \geq 0$ for all $\theta \in S^{n-1}$. If*

$$\Pi_p K \subseteq \Pi_p L,$$

then

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L) \text{ for } 1 \leq p < n,$$

and

$$\mathrm{vol}_n(K) \geq \mathrm{vol}_n(L) \text{ for } p > n.$$

Theorem B. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$ and $p \geq 1$, $p \neq n$, p is not an even integer. If the curvature function f_K is positive on S^{n-1} and $C_p \hat{h_K^p}(\theta)$ is negative on an open subset of S^{n-1}, then there exists a convex body D so that*

$$\Pi_p K \subseteq \Pi_p D,$$

but

$$\mathrm{vol}_n(K) > \mathrm{vol}_n(D) \text{ for } 1 \leq p < n$$

and

$$\mathrm{vol}_n(K) < \mathrm{vol}_n(D) \text{ for } p > n.$$

At the same time, if p is an even integer, Ryabogin and Zvavitch showed the answer to the Shephard problem for L_p-projection bodies is negative by perturbing a convex body.

The main object of this article is the ith L_p-mixed projection body $\Pi_{p,i}K$. Let $\Pi_{p,i}K, i = 0, 1, \ldots, n-1, p \geq 1$, denote the compact convex set whose support function is given by

$$h(\Pi_{p,i}K, \theta)^p = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |\theta \cdot u|^p dS_{p,i}(K, u), \forall \theta \in S^{n-1}. \tag{1.4}$$

Here $S_{p,i}(K, \cdot)$ is the ith L_p-mixed surface area measure with $n-i-1$ copies of K and i copies of B. More precisely, the Borel measure $S_{p,i}(K, \cdot)$ on S^{n-1}, is defined by ([9])

$$S_{p,i}(K, \omega) = \int_\omega h_K^{1-p}(u) dS_i(K, u)$$

for each Borel $\omega \subset S^{n-1}$. If $i = 0$, $S_{p,i}(K, \cdot)$ is just $S_p(K, \cdot)$. A convex body M is called the ith L_p-mixed projection body if there is a convex body K such that $M = \Pi_{p,i}K$. Obviously, $\Pi_{p,0}K = \Pi_p K$.

In this article, we consider the more general Shephard projection problem:

Shephard problem for L_p-mixed projection bodies. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ and $i = 0, 1, \ldots, n-1, p \geq 1$. Suppose

$$\Pi_{p,i}K \subseteq \Pi_{p,i}L.$$

Does it follow that

$$W_i(K) \leq W_i(L) \text{ for } 1 \leq p < n - i$$

and

$$W_i(K) \geq W_i(L) \text{ for } p > n - i?$$

By using the Fourier analytic formula for L_p-mixed projection body, we will generalize Theorem A and Theorem B, respectively.

Theorem 1. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, $i = 0, 1, \ldots, n-1$ and $p \geq 1$, $p \neq n-i$, p is not an even integer. Suppose that the support function h_L is infinitely smooth and the functions $C_p \hat{h_L^p}(\theta) \geq 0$ for all $\theta \in S^{n-1}$. If*

$$\Pi_{p,i} K \subseteq \Pi_{p,i} L,$$

then

$$W_i(K) \leq W_i(L) \text{ for } 1 \leq p < n - i$$

and

$$W_i(K) \geq W_i(L) \text{ for } p > n - i.$$

Theorem 2. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$, and $i = 0, 1, \ldots, n-1$, $p \geq 1$, $p \neq n-i$, p is not an even integer. If the mixed curvature function $f_i(K, \cdot)$ is positive on S^{n-1} and $C_p \hat{h_K^p}(\theta)$ is negative on an open subset of S^{n-1}, then there exists a convex body D so that*

$$\Pi_{p,i} K \subseteq \Pi_{p,i} D,$$

but

$$W_i(K) > W_i(D) \text{ for } 1 \leq p < n - i$$

and

$$W_i(K) < W_i(D) \text{ for } p > n - i.$$

2. Notation and preliminaries

2.1. Fourier transform and Parseval's formula

Koldobsky's book [7] is an excellent general reference for the Fourier transform. Some basic notions and the background material are required. As usual, we denote by $S(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable test functions on $\mathbb{R}^n$, and by $S'(\mathbb{R}^n)$ the space of distributions over $S(\mathbb{R}^n)$. The Fourier transform $\hat{f}$ of a distribution $f \in S'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function ϕ, where

$$\hat{\phi}(y) = \int \phi(x) \exp(-i\langle x, y \rangle) dx. \tag{2.1}$$

A distribution f is called even homogeneous of degree $p \in \mathbb{R}$ if $\langle f, \phi(\cdot/\alpha) \rangle = |\alpha|^{n+p} \langle f, \phi \rangle$ for every $\alpha \in \mathbb{R}, \alpha \neq 0$. The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$. A distribution f is called positive if $\langle f, \phi \rangle \geq 0$ for every $\phi \geq 0$, implying that f is necessarily a non-negative Borel measure on $\mathbb{R}^n$. We use Schwartz's generalization of Bochner's theorem (see [4]) as a definition, and call a homogeneous distribution positive definite if its Fourier transform is a positive distribution.

Let K and L be infinitely smooth origin-symmetric star bodies in $\mathbb{R}^n$, and let $0 < p < n$. Then a version of Parseval's formula on the sphere ([7] p. 66) can be expressed by

$$\int_{S^{n-1}} (\|\cdot\|_K^{-p})^\wedge(\theta)(\|\cdot\|_L^{-n+p})^\wedge(\theta)d\theta = (2\pi)^n \int_{S^{n-1}} \|\theta\|_K^{-p} \|\theta\|_L^{-n+p}\, d\theta. \tag{2.2}$$

Let μ be a finite Borel measure on the unit sphere S^{n-1}. We extend μ to a homogeneous distribution of degree $-n-p$. A distribution $\mu_{p,e}$ is called the L_p extended measure of μ if, for every even test function $\phi \in S(\mathbb{R}^n)$,

$$\langle \mu_{p,e}, \phi\rangle = \int_{S^{n-1}} \langle r_+^{-1-p}, \phi(r\xi)\rangle d\mu(\xi). \tag{2.3}$$

In most cases we are only interested in even test functions supported outside of the origin, for which

$$\langle r_+^{-1-p}, \phi(r\xi)\rangle = \int_{\mathbb{R}} r_+^{-1-p}\phi(r\xi)dr = \frac{1}{2}\int_{\mathbb{R}} |r|^{-1-p}\phi(r\xi)dr \tag{2.4}$$

(see [3]) for the general definition of $\langle r_+^{-1-p}, \phi(r\xi)\rangle$.

If μ is absolutely continuous with density $g \in L_1(S^{n-1})$, we define the extension $g(x), x \in \mathbb{R}^n\backslash\{0\}$ as a homogeneous function of degree $-n-p$: $g(x) = |x|^{-n-p}g(x/|x|)$, and identify $\widehat{\mu_{p,e}}$ with $\hat{g}$.

2.2. L_p-mixed curvature functions and L_p-mixed quermassintegrals

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors in $\mathbb{R}^n$, we write $\mathcal{K}_0^n$.

For $K, L \in \mathcal{K}^n$, and $\varepsilon > 0$, the Minkowski linear combination $K + \varepsilon L \in \mathcal{K}^n$ is defined by ([1])

$$h(K + \varepsilon L, \cdot) = h(K, \cdot) + \varepsilon h(L, \cdot). \tag{2.5}$$

For $K, L \in \mathcal{K}_0^n$, and $\varepsilon > 0$, the Firey L_p-combination $K +_p \varepsilon \cdot L \in \mathcal{K}_0^n$ is defined by ([9])

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p, \tag{2.6}$$

where " $\cdot$ " in $\varepsilon \cdot L$ denotes the Firey scalar multiplication, i.e., $\varepsilon \cdot L = \varepsilon^{\frac{1}{p}} L$.

If $K, L \in \mathcal{K}_0^n$ in $\mathbb{R}^n$, then for $p \geq 1$, the L_p-mixed volume $V_p(K, L)$ of K and L is defined by ([9])

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.7}$$

Corresponding to each $K \in \mathcal{K}_0^n$, there is a positive Borel measure $S_p(K,\cdot)$ on S^{n-1} such that ([9])

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p dS_p(K,u) \tag{2.8}$$

for each $L \in \mathcal{K}_0^n$. The measure $S_p(K,\cdot)$ is just the L_p-surface area measure of K, which is absolutely continuous with respect to classical surface area measure $S(K,\cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)} = h(K,\cdot)^{1-p}. \tag{2.9}$$

A convex body $K \in \mathcal{K}^n$ is said to have a L_p-curvature function (see [10]) $f_p(K,\cdot) : S^{n-1} \to \mathbb{R}$, if its L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S and the Radon-Nikodym derivative

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot). \tag{2.10}$$

The mixed quermassintegral $W_i(K,L)$ with $n-i-1$ copies of K, i copies of B_n $(i = 0,1,\ldots,n-1)$ is defined by ([9])

$$(n-i)W_i(K,L) = \lim_{\varepsilon\to 0^+} \frac{W_i(K+\varepsilon L) - W_i(K)}{\varepsilon}. \tag{2.11}$$

For $K \in \mathcal{K}^n$ and $i = 0,1,\ldots,n-1$ there exists a regular Borel measure $S_i(K,\cdot)$ on S^{n-1}(see [9] or [15]), such that the mixed quermassintegral $W_i(K,L)$ has the following integral representation:

$$W_i(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u) dS_i(K,u) \tag{2.12}$$

for all $L \in \mathcal{K}^n$. As a general reference for the mixed surface area measure we recommend the article by Lutwak [9]. From the fact that $S_i(K,\cdot)$ is generated only by i copies of B_n and $(n-1-i)$ copies of K, we know that the measure $S_{n-1}(K,\cdot)$ is independent of the body K, and is just ordinary Lebesgue measure S on S^{n-1}. In fact, the ith surface area measure of the unit ball, $S_i(B_n,\cdot) = S$ for all i. The surface area measure $S_0(K,\cdot)$ will frequently be written simply as $S(K,\cdot)$. If ∂K is a regular C^2-hypersurface with everywhere positive principal curvatures, then $S(K,\cdot)$ is absolutely continuous with respect to S, and the Radon-Nikodym derivative is

$$\frac{dS(K,\cdot)}{dS} = f(K,\cdot). \tag{2.13}$$

Suppose that $\mathbb{R}$ is the set of real numbers. A convex body $K \in \mathcal{K}^n$ is said to have a continuous ith curvature function $f_i(K,\cdot) : S^{n-1} \to \mathbb{R}$, if its mixed surface area measure $S_i(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S and the Radon-Nikodym derivative

$$\frac{dS_i(K,\cdot)}{dS} = f_i(K,\cdot). \tag{2.14}$$

For $K, L \in \mathcal{K}_0^n$ and $p \geq 1, i = 0, 1, \ldots, n-1$, the L_p-mixed quermassintegral $W_{p,i}(K, L)$ with $n-i-1$ copies of K, i copies of B is defined by ([9])

$$\frac{n-i}{p} W_{p,i}(K, L) = \lim_{\varepsilon \to 0^+} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}. \tag{2.15}$$

Moreover, Lutwak [9] proved there exists a regular Borel measure $S_{p,i}(K, \cdot)$, such that the L_p-mixed quermassintegral $W_{p,i}(K, L)$ has the following integral representation:

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_{p,i}(K, u) \tag{2.16}$$

for all $L \in \mathcal{K}_0^n$. And the measure $S_{p,i}(K, \cdot)$ is absolutely continuous with respect to $S_i(K, \cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h(K, \cdot)^{1-p}. \tag{2.17}$$

Of course

$$S_{p,0}(K, \cdot) = S_p(K, \cdot), \tag{2.18}$$

$$S_{1,i}(K, \cdot) = S_i(K, \cdot), \tag{2.19}$$

$$S_{1,0}(K, \cdot) = S(K, \cdot). \tag{2.20}$$

For $K, L \in \mathcal{K}_0^n$ and $p \geq 1, i = 0, 1, \ldots, n-1$, the L_p-mixed curvature function $f_{p,i}(K, \cdot)$ is defined by

$$f_{p,i}(K, \cdot) = \frac{dS_{p,i}(K, \cdot)}{dS}. \tag{2.21}$$

If the mixed surface area measure $S_i(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, we have

$$f_{p,i}(K, u) = f_i(K, u) h(K, u)^{1-p}. \tag{2.22}$$

Obviously, from (2.10), (2.13), (2.14), (2.18), (2.19) and (2.20), we have that

$$f_{p,0}(K, u) = f_p(K, u), \tag{2.23}$$

$$f_{1,i}(K, u) = f_i(K, u), \tag{2.24}$$

$$f_{1,0}(K, u) = f(K, u). \tag{2.25}$$

3. Shephard problem for L_p-mixed projection bodies

In order to prove our main results, the following results are required.

Lemma 3.1 ([14])**.** *Let $p > -1, p \neq 2k, k \in N \cup \{0\}$. For every $\theta \in S^{n-1}$,*

$$\widehat{\mu_{p,e}}(\theta) = \frac{1}{4\pi C_p} \int_{S^{n-1}} |\theta \cdot y|^p d\mu(y), \tag{3.1}$$

where the constant

$$C_p = \frac{2^{p+1}\sqrt{\pi}\Gamma((p+1)/2)}{\Gamma(-p/2)}$$

is positive for each $p \in (4k-2, 4k)$ and negative for each $p \in (4k, 4k+2)$.

Lemma 3.2 ([9])**.** *If $K, L \in \mathcal{K}_0^n, i = 0, 1, \ldots, n-1$ and $p > 1$, then*

$$W_{p,i}(K,L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p, \tag{3.2}$$

with equality if and only if K and L are dilates.

The following statement follows from (1.4) and Lemma 3.1.

Lemma 3.3. *Let $p \geq 1$, p is not an even integer and $i = 0, 1, \ldots, n-1$. Then for every $\theta \in S^{n-1}$,*

$$\widehat{S_{p,i}(K,\cdot)}(\theta) = \frac{n\omega_n c_{n-2,p}}{4\pi C_p} h(\Pi_{p,i}K, \theta)^p, \tag{3.3}$$

where C_p is as above. In particular, if $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure, then

$$\widehat{f_{p,i}(K,\cdot)}(\theta) = \frac{n\omega_n c_{n-2,p}}{4\pi C_p} h(\Pi_{p,i}K, \theta)^p. \tag{3.4}$$

Taking $i = 0$ to Lemma 3.3, we immediately obtain that:

Corollary 3.1 ([14])**.** *Let $p \geq 1$, p is not an even integral. If $S_p(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure, then for every $\theta \in S^{n-1}$,*

$$\widehat{f_p(K,\cdot)}(\theta) = \frac{n\omega_n c_{n-2,p}}{4\pi C_p} h(\Pi_p K, \theta)^p. \tag{3.5}$$

Theorem 3.1. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, $i = 0, 1, \ldots, n-1$ and $p \geq 1$, $p \neq n-i$, p is not an even integer. If $\Pi_{p,i}K = \Pi_{p,i}L$, then $K = L$.*

Proof. Applying (1.4) and the uniqueness theorem of the Fourier transform, we have $S_{p,i,e}(K,\cdot) = S_{p,i,e}(L,\cdot)$. By homogeneity, $S_{p,i}(K,\cdot) = S_{p,i}(L,\cdot)$ is the same as $S_{p,i,e}(K,\cdot) = S_{p,i,e}(L,\cdot)$. It remains to use the uniqueness property of L_p-mixed surface area measures for $p \neq n-i$ (see [9]). □

Remark 1. Taking $p = 1, i = 0$ to Theorem 3.1, it is just Aleksandrov's projection theorem.

Remark 2. In the case $p = n - i$ and p is not an even integer, it follows that $\Pi_{n-i,i}K = \Pi_{n-i,i}L$ implies K and L are dilates. Theorem 3.1 is not true for even values of p. Indeed, one can perturb $S_{p,i}(K, \cdot)$ (i.e., to perturb a body K) without changing $h(\Pi_{p,i}K, \xi)$ (see the following theorem).

Theorem 3.2. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$, $i = 0, 1, \ldots,$ $n - 1$ and $p \geq 1$, $p \neq n - i$. If p is an even integer, then there exists an origin-symmetric convex body L, such that $\Pi_{p,i}K = \Pi_{p,i}L$, but $W_i(K) \neq W_i(L)$.*

Proof. Then there exists a nonzero continuous even function g on S^{n-1} such that

$$\int_{S^{n-1}} |x \cdot \xi|^p g(x)dx = 0, \ \xi \in S^{n-1}. \tag{3.6}$$

Indeed, if $p = 2k$, then $|x \cdot \xi|^{2k}$ is a polynomial of degree $2k$ with coefficients depending on ξ. So, it is enough to construct a nontrivial even function g, satisfying

$$\int_{S^{n-1}} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} g(x)dx = 0 \tag{3.7}$$

for all integer power $0 \leq i_j \leq 2k$ such that $\sum_{j=1}^n i_j = 2k$.

Taking $g(x) = \sum_{l=1}^n c_l x_1^{2l}$ and solving the system of linear equations, one can find a nontrivial solution $c_1, \ldots, c_m$ provided m is big enough.

Consider an origin-symmetric convex body K in $\mathbb{R}^n$ with a strictly positive ith L_p-mixed curvature function (i.e., $f_{p,i}(K, \xi) > 0$ for all $\xi \in S^{n-1}$). We may assume that

$$\int_{S^{n-1}} h_K^p(\xi)g(\xi)d\xi \geq 0 \tag{3.8}$$

(otherwise consider $-g(\xi)$ instead of $g(\xi)$). Choose $\varepsilon > 0$ such that

$$f_{p,i}(K, \xi) - \varepsilon g(\xi) > 0. \tag{3.9}$$

Then we may use the existence theorem for L_p-mixed curvature functions to conclude that there exists an origin-symmetric convex body L in $\mathbb{R}^n$ such that

$$f_{p,i}(L, \xi) = f_{p,i}(K, \xi) - \varepsilon g(\xi). \tag{3.10}$$

Applying (1.4) and (2.21), we obtain that

$$\begin{aligned} h(\Pi_{p,i}L, \xi)^p &= \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |\theta \cdot \xi|^p dS_{p,i}(L, \xi) \\ &= \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |\theta \cdot \xi|^p f_{p,i}(L, \xi)dS(\xi) \\ &= \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |\theta \cdot \xi|^p f_{p,i}(K, \xi)dS(\xi) \\ &\quad - \frac{\varepsilon}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |\theta \cdot \xi|^p g(\xi)dS(\xi) \end{aligned}$$

$$= \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |\theta \cdot \xi|^p f_{p,i}(K,\xi) dS(\xi)$$
$$= \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |\theta \cdot \xi|^p dS_{p,i}(K,\xi)$$
$$= h(\Pi_{p,i}K,\xi)^p,$$

it is just to say

$$\Pi_{p,i}L = \Pi_{p,i}K. \tag{3.11}$$

But

$$\begin{aligned} W_i(K) &= W_{p,i}(K,K) \\ &= \frac{1}{n}\int_{S^{n-1}} h_K^p(\xi) f_{p,i}(K,\xi) d\xi \\ &= \frac{1}{n}\int_{S^{n-1}} h_K^p(\xi) f_{p,i}(L,\xi) d\xi + \frac{\varepsilon}{n}\int_{S^{n-1}} h_K^p(\xi) g(\xi) d\xi \\ &\geq \frac{1}{n}\int_{S^{n-1}} h_K^p(\xi) f_{p,i}(L,\xi) d\xi \\ &= W_{p,i}(L,K). \end{aligned} \tag{3.12}$$

From Lemma 3.2 and (3.12), we have

$$W_i(K) \geq W_{p,i}(L,K) \geq W_i(K)^{\frac{n-i-p}{n-i}} W_i(L)^{\frac{p}{n-i}}. \tag{3.13}$$

So if $W_i(K) = W_i(L)$, then there is an equality in (3.2) and then L and K are dilates. This contradicts the construction of the body L. □

Remark 3. The proof of Theorem 3.2 is exact copy of a similar result which was proved by Ryabogin and Zvavitch in [14]. If $i = 0$, Theorem 3.2 is just their result.

Proof of Theorem 1. From $C_p\widehat{f_{p,i}(K,\cdot)}(\theta) \leq C_p\widehat{f_{p,i}(L,\cdot)}(\theta)$ and $C_p\widehat{h_L^p}(\theta) \geq 0$, $\forall\ \theta \in S^{n-1}$, we get

$$\int_{S^{n-1}} \widehat{h_L^p}(\theta)\widehat{f_{p,i}(K,\cdot)}(\theta) d\theta \leq \int_{S^{n-1}} \widehat{h_L^p}(\theta)\widehat{f_{p,i}(L,\cdot)}(\theta) d\theta = (*). \tag{3.14}$$

Using Parseval's formula on the sphere, one can have

$$\begin{aligned} (*) &= (2\pi)^n \int_{S^{n-1}} h_L^p(\theta) f_{p,i}(L,\theta) d\theta \\ &= (2\pi)^n \int_{S^{n-1}} h_L^p(\theta) dS_{p,i}(L,\theta) \\ &= n(2\pi)^n W_{p,i}(L,L) \\ &= n(2\pi)^n W_i(L). \end{aligned} \tag{3.15}$$

But

$$\int_{S^{n-1}} \widehat{h_L^p}(\theta)\widehat{f_{p,i}(K,\cdot)}(\theta)d\theta = (2\pi)^n \int_{S^{n-1}} h_L^p(\theta) f_{p,i}(K,\theta)d\theta = n(2\pi)^n W_{p,i}(K,L). \tag{3.16}$$

Thus

$$W_{p,i}(K,L) \le W_i(L). \tag{3.17}$$

Applying the Lemma (3.2), we get

$$W_i(K)^{n-p-i} \le W_i(L)^{n-p-i}. \tag{3.18}$$

Finally

$$W_i(K) \le W_i(L) \text{ for } 1 \le p < n-i, \tag{3.19}$$

and

$$W_i(K) \ge W_i(L) \text{ for } p > n-i. \tag{3.20}$$

□

Proof of Theorem 2. Let $\Omega = \{\theta \in S^{n-1} : C_p\widehat{h_K^p}(\theta) < 0\}$. Consider a function $\nu \in C^\infty(S^{n-1})$ such that $C_p\nu$ is a positive even function supported on Ω, ν is not identically zero. We extend ν to a homogeneous function $r^p\nu(\theta)$ of degree p on $\mathbb{R}^n$. Then the Fourier transform of $r^p\nu(\theta)$ is a homogeneous function of degree $-n-p : \widehat{r^p\nu(\theta)} = r^{-n-p}g(\theta)$, where g is an infinitely smooth function on S^{n-1}.

Since g is bounded on S^{n-1} and $f_{p,i}(K,\theta) = h_K^{1-p}(\theta)f_i(K,\theta) > 0$, one can choose a small $\varepsilon > 0$ so that, for every $\theta \in S^{n-1}$ and $r > 0$,

$$f_{p,i}(L,r\theta) = f_{p,i}(K,r\theta) + \varepsilon r^{-n-p}g(\theta) > 0. \tag{3.21}$$

By Lutwak's extension of the Minkowski's existence theorem, $f_{p,i}(L,\theta)$ defines a convex body $L \in \mathbb{R}^n$. By the definition of the function ν,

$$C_p\widehat{f_{p,i}(L,\cdot)}(r\theta) = C_p\widehat{f_{p,i}(K,\cdot)}(r\theta) + \varepsilon C_p\nu(\theta) \ge C_p\widehat{f_{p,i}(K,\cdot)}(r\theta). \tag{3.22}$$

Next, since $C_p\nu$ is supported and is positive in the set where $C_p\widehat{h_K^p} < 0$,

$$\begin{aligned} &\int_{S^{n-1}} \widehat{h_K^p}(\theta)\widehat{f_{p,i}(L,\cdot)}(\theta)d\theta \\ &= \int_{S^{n-1}} \widehat{h_K^p}(\theta)\widehat{f_{p,i}(K,\cdot)}(\theta)d\theta + \int_{S^{n-1}} \widehat{h_K^p}(\theta)\varepsilon\nu(\theta)d\theta \\ &< \int_{S^{n-1}} \widehat{h_K^p}(\theta)\widehat{f_{p,i}(K,\cdot)}(\theta)d\theta = (*). \end{aligned} \tag{3.23}$$

Now the Parseval's formula gives

$$\begin{aligned}(*) &= (2\pi)^n \int_{S^{n-1}} h_K^p(\theta) f_{p,i}(K,\theta)d\theta \\ &= (2\pi)^n \int_{S^{n-1}} h_K^p(\theta) dS_{p,i}(K,\theta) \\ &= n(2\pi)^n W_{p,i}(K,K) = n(2\pi)^n W_i(K).\end{aligned} \tag{3.24}$$

And

$$\begin{aligned}\int_{S^{n-1}} \widehat{h_K^p}(\theta)\widehat{f_{p,i}(L,\cdot)}(\theta)d\theta &= (2\pi)^n \int_{S^{n-1}} h_K^p(\theta) f_{p,i}(L,\theta)d\theta \\ &= (2\pi)^n \int_{S^{n-1}} h_K^p(\theta) dS_{p,i}(L,\theta) \\ &= n(2\pi)^n W_{p,i}(L,K).\end{aligned} \tag{3.25}$$

Thus

$$W_{p,i}(L,K) < W_i(K). \tag{3.26}$$

As in the previous lemma, this implies

$$W_i(L) < W_i(K) \text{ for } 1 \le p < n-i, \tag{3.27}$$

and

$$W_i(L) > W_i(K) \text{ for } p > n-i. \tag{3.28}$$

□

Taking $p = 1$ to Theorem 1 and Theorem 2, respectively, we obtain:

Corollary 3.2. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, $i = 0, 1, \ldots, n-1$. Suppose that the support function h_L is infinitely smooth and the functions $\hat{h}_L(\theta) \le 0$ for all $\theta \in S^{n-1}$. If $\Pi_i K \subseteq \Pi_i L$, then $W_i(K) \le W_i(L)$.*

Corollary 3.3. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$, and $i = 0, 1, \ldots, n-1$. If the mixed curvature function $f_i(K,\cdot)$ is positive on S^{n-1} and $\hat{h}_K(\theta)$ is positive on an open subset of S^{n-1}, then there exists a convex body D so that $\Pi_i K \subseteq \Pi_i D$, but $W_i(K) > W_i(D)$.*

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