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VARIATIONAL-LIKE INCLUSION SYSTEMS VIA GENERAL MONOTONE OPERATORS WITH CONVERGENCE ANALYSIS

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Abstract. In this paper using Lipschitz continuity of the resolvent operator associated with general H-maximal m-relaxed η-monotone operators, existence and uniqueness of the solution of a variational inclusion system is proved. Also, an iterative algorithm and its convergence analysis is given.

1. Introduction

The concept of general H-maximal m-relaxed η-monotone operator (so-called the general G-η-monotone mapping in [3]) as a generalization of the general A-monotone mapping [3, 8, 13, 14], the general (H, η)-monotone operator [5, 6], general H-monotone operator [20] in Banach spaces, and also as a generalization of the (A, η)-maximal m-relaxed monotone operator [2], A-maximal m-relaxed monotone operator [1, 17, 19], G-η-monotone operator [22], (A, η)-monotone operator [18], A-monotone operator [16], (H, η)-monotone operator [12], H-monotone operator [7, 11], maximal η-monotone operator [10] and classical maximal monotone operator [21] in Hilbert spaces, is introduced and considered in [4]. At the mentioned paper the authors provided some examples and also they studied many properties of general H-maximal m-relaxed η-monotone operators. Further, the generalized resolvent operator associated with this type of monotone operators has been defined and some results about Lipschitz continuity of this type of monotone operators has been established. At the present paper, first we recall some notions, definitions, and results about monotone operators and their generalized versions. Using Lipschitz continuity of the resolvent operator associated with general H-maximal m-relaxed η-monotone operators, existence and uniqueness of the solution of a variational inclusion system is proved. Further, we construct an iterative algorithm and the convergence analysis of this algorithm is given.

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2. Preliminaries

Throughout in this paper, suppose X is a real Banach space with dual space X^* and with dual pair $\langle \cdot, \cdot \rangle$ between X and X^*. The single-valued mapping $\eta : X \times X \to X$ is called *γ-Lipschitz continuous*, if there exists a constant $\gamma > 0$ such that $\|\eta(x,y)\| \leq \gamma \|x-y\|$ for all $x, y \in X$. For a set-valued mapping $M : X \multimap Y$, the *domain* of M is

$$\mathrm{Dom}(M) = \{x \in X : \exists y \in Y, (x,y) \in M\},$$

the *inverse* M^{-1} of M is $\{(y,x) : (x,y) \in M\}$ and the *graph* of M is $Gph(M) = \{(x,y) : (x,y) \in M\}$. For a real number c, let $cM = \{(x, cy) : (x,y) \in M\}$. If M and N are any set-valued mappings, we define

$$M + N = \{(x, y+z) : (x,y) \in M, (x,z) \in N\}.$$

Let us recall definitions of some generalized types of monotone operators. For more details one can see ([1]–[22]) and the references cited therein.

Definition 1. A single-valued mapping $H : X \to X^*$ is said to be

(a) *monotone* if $\langle H(x) - H(y), x - y \rangle \geq 0$ for all $x, y \in X$.

(b) *η-monotone* if $\langle H(x) - H(y), \eta(x,y) \rangle \geq 0$ for all $x, y \in X$.

(c) *strictly monotone* if H is monotone and $\langle H(x) - H(y), x - y \rangle = 0$ if and only if $x = y$.

(d) *strictly η-monotone* if H is η-monotone and $\langle H(x) - H(y), \eta(x,y) \rangle = 0$ if and only if $x = y$.

(e) *r-strongly monotone* if there exists some constant $r > 0$ such that $\langle H(x) - H(y), x - y \rangle \geq r\|x-y\|^2$ for all $x, y \in X$.

(f) *r-strongly η-monotone* if there exists some constant $r > 0$ such that $\langle H(x) - H(y), \eta(x,y) \rangle \geq r\|x-y\|^2$ for all $x, y \in X$.

(g) *δ-Lipschitz continuous* if $\|H(x) - H(y)\| \leq \delta \|x-y\|$ for all $x, y \in X$.

Definition 2. Let $\mathcal{H}$ be a Hilbert space. A set-valued mapping $M : \mathcal{H} \multimap \mathcal{H}$ is said to be

(a) *maximal monotone* if M is monotone and $(I + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$.

(b) *maximal η-monotone* if M is η-monotone and $(I + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$, if and only if M is η-monotone and there is no other η-monotone set-valued mapping whose graph strictly contains the graph of M [9].

Definition 3. A set-valued mapping $M : X \multimap X^*$ is said to be

(a) *monotone* if $\langle x^* - y^*, x - y \rangle \geq 0$ for all $x, y \in \mathrm{Dom}(M)$ and all $x^* \in M(x)$, $y^* \in M(y)$.

(b) *η-monotone* if $\langle x^* - y^*, \eta(x,y) \rangle \geq 0$ for all $x, y \in \mathrm{Dom}(M)$ and all $x^* \in M(x)$, $y^* \in M(y)$.

(c) *r-strongly monotone* if there exists some constant $r > 0$ such that $\langle x^* - y^*, x - y \rangle \geq r\|x-y\|^2$ for all $x, y \in \mathrm{Dom}(M)$ and all $x^* \in M(x), y^* \in M(y)$.

(d) *r-strongly η-monotone* if there exists some constant $r > 0$ such that $\langle x^* - y^*, \eta(x,y)\rangle \geq r\|x-y\|^2$ for all $x, y \in \mathrm{Dom}(M)$ and all $x^* \in M(x)$, $y^* \in M(y)$.

(e) *m-relaxed monotone* if, there exists some constant $m > 0$ such that $\langle x^* - y^*, x - y\rangle \geq -m\|x-y\|^2$ for all $x, y \in \mathrm{Dom}(M)$ and all $x^* \in M(x)$, $y^* \in M(y)$.

(f) *m-relaxed η-monotone* if, there exists some constant $m > 0$ such that $\langle x^* - y^*, \eta(x,y)\rangle \geq -m\|x-y\|^2$ for all $x, y \in \mathrm{Dom}(M)$ and all $x^* \in M(x)$, $y^* \in M(y)$.

Definition 4. [5, 6] The set-valued mapping $M : X \multimap X^*$ is said to be *general (H,η)-monotone operator* if M is η-monotone and $(H + \lambda M)(X) = X^*$ holds for every $\lambda > 0$.

Definition 5. [3, 4] A set-valued mapping $M : X \multimap X^*$ satisfying $(H + \lambda M)(X) = X^*$ is said to be *general H-maximal m-relaxed η-monotone operator*, provided that it is m-relaxed η-monotone.

Theorem 2.1. [3, 4] *Suppose $H : X \to X^*$ is an r-strongly η-monotone mapping and $M : X \multimap X^*$ is a general H-maximal m-relaxed η-monotone operator. Then for $0 < \lambda < \frac{r}{m}$, the operator $(H + \lambda M)^{-1}$ from X^* to X is single-valued.*

Definition 6. [3, 4] For an r-strongly η-monotone mapping $H : X \to X^*$ and a general H-maximal m-relaxed η-monotone operator $M : X \multimap X^*$ and for $0 < \lambda < \frac{r}{m}$, the *generalized resolvent operator* $R^{H,m}_{M,\lambda,\eta} : X^* \to X$ is defined by $R^{H,m}_{M,\lambda,\eta}(x^*) = (H + \lambda M)^{-1}(x^*)$.

Theorem 2.2. [3, 4] *Suppose that $\eta : X \times X \to X$ is a γ-Lipschitz continuous mapping, $H : X \to X^*$ is an r-strongly η-monotone operator and $M : X \multimap X^*$ is a general H-maximal m-relaxed η-monotone operator. Then for $0 < \lambda < \frac{r}{m}$ the generalized resolvent operator $R^{H,m}_{M,\lambda,\eta} : X^* \to X$ is $\frac{\gamma}{r-\lambda m}$-Lipschitz continuous.*

Definition 7. [4] A set-valued mapping $M : X \multimap X^*$ satisfying $(H + \lambda M)(X) = X^*$ is said to be *general H-maximal β-strongly η-monotone operator*, provided that it is β-strongly η-monotone.

Fact 2.3. [5, 6] *Suppose that $H : X \to X^*$ is an r-strongly η-monotone operator and $M : X \multimap X^*$ is a general H-maximal β-strongly η-monotone operator. Then*

(a) *the operator $(H + \lambda M)^{-1}$ from X^* to X is single-valued;*

(b) *the generalized resolvent operator $R^{H,\beta}_{M,\lambda,\eta} : X^* \to X$ is defined by $R^{H,\beta}_{M,\lambda,\eta}(x^*) = (H + \lambda M)^{-1}(x^*)$;*

(c) *if $\eta : X \times X \to X$ is a γ-Lipschitz continuous mapping, then the generalized resolvent operator $R^{H,\beta}_{M,\lambda,\eta} : X^* \to X$ is $\frac{\gamma}{r+\lambda\beta}$-Lipschitz continuous.*

Fact 2.4. [5, 6] *Suppose $H : X \to X^*$ is a strictly η-monotone mapping and $M : X \multimap X^*$ is a general H-maximal β-strongly η-monotone operator. Then*

(a) *the operator $(H + \lambda M)^{-1}$ from X^* to X is single-valued;*

(b) *the generalized resolvent operator $R^{H,\beta}_{M,\lambda,\eta} : X^* \to X$ is defined by $R^{H,\beta}_{M,\lambda,\eta}(x^*) = (H + \lambda M)^{-1}(x^*)$;*

(c) *if $\eta : X \times X \to X$ is a γ-Lipschitz continuous mapping, then the generalized resolvent operator $R^{H,\beta}_{M,\lambda,\eta} : X^* \to X$ is $\frac{\gamma}{\lambda\beta}$-Lipschitz continuous.*

Fact 2.5. [5, 6] *Suppose $H : X \to X^*$ is an r-strongly η-monotone operator and $M : X \multimap X^*$ is an H-maximal η-monotone operator. Then*

(a) *the operator $(H + \lambda M)^{-1}$ from X^* to X is single-valued;*

(b) *the generalized resolvent operator $R^{H,\eta}_{M,\lambda} : X^* \to X$ is defined by $R^{H,\eta}_{M,\lambda}(x^*) = (H + \lambda M)^{-1}(x^*)$;*

(c) *if $\eta : X \times X \to X$ is a γ-Lipschitz continuous mapping, then the generalized resolvent operator $R^{H,\eta}_{M,\lambda} : X^* \to X$ is $\frac{\gamma}{r}$-Lipschitz continuous.*

3. Main Results

The *module of smoothness* of a Banach space X is the function $\rho_X : [0, +\infty) \longrightarrow [0, +\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| \le 1, \|y\| \le t\right\}.$$

A Banach space X is called *uniformly smooth* if there exists a constant $c > 0$ for which $\rho_X(t) \le ct^2$.

For $i = 1, 2$, suppose X_i is a uniformly smooth Banach space with dual space X_i^* and with $\rho_{X_i}(t) \le c_i t^2$ for some $c_i > 0$. Let $H_i : X_i \to X_i^*$, $A_1 : X_2 \to X_1^*$, $A_2 : X_1 \to X_2^*$ and $f_i : X_i \to X_i$ be six single-valued mappings and let $M_i : X_i \multimap X_i^*$ be two set-valued mappings. Our problem is finding $(x, y) \in X_1 \times X_2$ such that

$$\begin{cases} 0 \in A_1(y) + M_1(f_1(x)) \\ 0 \in A_2(x) + M_2(f_2(y)). \end{cases} \tag{1}$$

Note that for appropriate and suitable choices of X_i, H_i, A_i, f_i and M_i, one can obtain many known and new classes of variational inequality and variational inclusion systems and problems as special cases of the system (1). Some special cases can be found in ([1]–[22]) and the references cited therein.

Theorem 3.1. *Suppose that i, X_i, c_i, H_i, A_i, f_i, and M_i are the same as above and $\eta_i : X_i \times X_i \to X_i$ is single-valued mapping. If $H_i : X_i \to X_i^*$ is r_i-strongly η_i-monotone and M_i is a general H_i-maximal m_i-relaxed η_i-monotone operator, then the following statements are equivalent.*

(a) *(x, y) is a solution of system (1).*

(b) *$f_1(x) = R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(x)) - \lambda_1 A_1(y)]$ and $f_2(y) = R^{H_2,m_2}_{M_2,\lambda_2,\eta_2}[H_2(f_2(y)) - \lambda_2 A_2(x)]$.*

(c) *For all* $s, t \neq 0$

$$\begin{cases} x = (1-s)x + s(x - f_1(x) + R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(x)) - \lambda_1 A_1(y)]) \\ y = (1-t)y + t(y - f_2(y) + R^{H_2,m_2}_{M_2,\lambda_2,\eta_2}[H_2(f_2(y)) - \lambda_2 A_2(x)]). \end{cases} \tag{2}$$

(d) *There exist* $s, t \neq 0$ *such that* (2) *holds.*

Proof. The equivalence relations (b)$\Longleftrightarrow$(c)$\Longleftrightarrow$(d) are straightforward. We only prove (a)$\Longleftrightarrow$(b). To see this, (x, y) is a solution of system (1) if and only if $-\lambda_1 A_1(y) \in \lambda_1 M_1(f_1(x))$ and $-\lambda_2 A_2(x) \in \lambda_2 M_2(f_2(y))$ or equivalently

$$\begin{cases} H_1(f_1(x)) - \lambda_1 A_1(y) \in H_1(f_1(x)) + \lambda_1 M_1(f_1(x)) \\ H_2(f_2(y)) - \lambda_2 A_2(x) \in H_2(f_2(y)) + \lambda_2 M_2(f_2(y)) \end{cases} \tag{3}$$

which it is equivalent to

$$\begin{cases} f_1(x) = (H_1 + \lambda_1 M_1)^{-1}(H_1(f_1(x)) - \lambda_1 A_1(y)) \\ f_2(y) = (H_2 + \lambda_2 M_2)^{-1}(H_2(f_2(y)) - \lambda_2 A_2(x)), \end{cases} \tag{4}$$

we are done. □

Lemma 3.2. [4] *Suppose that* X *is a uniformly smooth Banach space with* $\rho_X(t) \leq ct^2$ *for some* $c > 0$. *If* $f : X \to X$ *is* κ*-strongly accretive and* α*-Lipschitz continuous mapping, then*

$$\|x - y - f(x) + f(y)\| \leq \sqrt{1 - 2\kappa + 64c\alpha^2}\|x - y\|.$$

Theorem 3.3. *Suppose that* i, X_i, c_i, H_i, A_i, f_i, *and* M_i *are the same as in* Theorem 3.1. *Further, suppose that*

(a) f_i *is* κ_i*-strongly accretive and* α_i*-Lipschitz continuous mapping.*

(b) $\eta_i : X_i \times X_i \to X_i$ *is* γ_i*-Lipschitz continuous.*

(c) H_i *is* θ_i*-Lipschitz continuous.*

(d) A_i *is* π_i*-Lipschitz continuous.*

(e) $\xi_1 = \sqrt{1 - 2\kappa_1 + 64c_1\alpha_1^2} + \frac{\alpha_1\theta_1\gamma_1}{r_1 - \lambda_1 m_1} + \frac{\gamma_2\pi_2\lambda_2}{r_2 - \lambda_2 m_2} < 1.$

(f) $\xi_2 = \sqrt{1 - 2\kappa_2 + 64c_2\alpha_2^2} + \frac{\alpha_2\theta_2\gamma_2}{r_2 - \lambda_2 m_2} + \frac{\gamma_1\pi_1\lambda_1}{r_1 - \lambda_1 m_1} < 1.$

Then the system of variational inclusions (1) *admits a unique solution.*

Proof. First define $\|(\cdot,\cdot)\|_\times : X_1 \times X_2 \to \mathbb{R}$ by $\|(x, y)\|_\times = \|x\| + \|y\|$ for all $(x, y) \in X_1 \times X_2$. It is well known that $(X_1 \times X_2, \|(\cdot,\cdot)\|_\times)$ is a Banach space. Now, consider the single-valued mapping $\Theta : X_1 \times X_2 \to X_1 \times X_2$ defined by

$$\Theta(x, y) = (F(x, y), G(x, y)) \tag{5}$$

for all $(x, y) \in X_1 \times X_2$, where

$$F(x, y) = x - f_1(x) + R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(x)) - \lambda_1 A_1(y)] \tag{6}$$

and

$$G(x, y) = y - f_2(y) + R^{H_2,m_2}_{M_2,\lambda_2,\eta_2}[H_2(f_2(y)) - \lambda_2 A_2(x)]. \tag{7}$$

It follows from our assumptions, Lemma 3.2, and Theorem 2.2 that

$$\begin{aligned}
&\| \ F(x,y) - F(u,v)\| \\
&= \|(x - f_1(x) + R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(x)) - \lambda_1 A_1(y)]) \\
&\qquad -(u - f_1(u) + R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(u)) - \lambda_1 A_1(v)])\| \\
&\leq \|x - u - f_1(x) + f_1(u)\| \\
&\quad +\|R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(x)) - \lambda_1 A_1(y)] - R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(u)) - \lambda_1 A_1(v)]\| \\
&\leq \sqrt{1 - 2\kappa_1 + 64 c_1 \alpha_1^2}\|x - u\| \qquad (8) \\
&\quad +\frac{\gamma_1}{r_1 - \lambda_1 m_1}(\|H_1(f_1(x)) - H_1(f_1(u))\| + \lambda_1\|A_1(y) - A_1(v)\|) \\
&\leq \left(\sqrt{1 - 2\kappa_1 + 64 c_1 \alpha_1^2} + \frac{\alpha_1\theta_1\gamma_1}{r_1 - \lambda_1 m_1}\right)\|x - u\| + \frac{\lambda_1\pi_1\gamma_1}{r_1 - \lambda_1 m_1}\|y - v\|
\end{aligned}$$

and similarly

$$\begin{aligned}
\|G(x,y) - G(u,v)\| \leq & \left(\sqrt{1 - 2\kappa_2 + 64 c_2 \alpha_2^2} + \frac{\alpha_2\theta_2\gamma_2}{r_2 - \lambda_2 m_2}\right)\|y - v\| \\
& +\frac{\lambda_2\pi_2\gamma_2}{r_2 - \lambda_2 m_2}\|x - u\|. \qquad (9)
\end{aligned}$$

Therefore, by (5)–(9) we get

$$\begin{aligned}
\|\Theta(x,y) - \Theta(u,v)\|_\times &= \|F(x,y) - F(u,v)\| + \|G(x,y) - G(u,v)\| \\
&\leq \xi_1\|x - u\| + \xi_2\|y - v\| \\
&\leq \max\{\xi_1, \xi_2\}\|(x,y) - (u,v)\|_\times.
\end{aligned}$$

By (e) and (f), we find that Θ is a contraction map and hence the Banach Contraction Theorem implies that Θ has a unique fixed point. That system (1) has a unique solution, follows from Theorem 3.1. □

Motivated and inspired by part (c) of Theorem 3.1 we get the following useful algorithm.

Algorithm 1. Suppose that i, X_i, c_i, H_i, A_i, f_i, M_i and η_i are the same as in Theorem 3.3. Let $\{d_n\} \subseteq X_1$, $\{e_n\} \subseteq X_1$, $\{k_n\} \subseteq X_2$, $\{l_n\} \subseteq X_2$ and $\{t_n\} \subseteq [0,1]$ be five sequences. For any given $(x_0, y_0) \in X_1 \times X_2$, we define an iterative sequence $\{(x_n, y_n)\}$ as follows

$$\left\{\begin{aligned}
x_{n+1} &= (1 - t_n)x_n + t_n[x_n - f_1(x_n) \\
&\qquad +R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(x_n)) - \lambda_1 A_1(y_n)] + d_n] + e_n \\
y_{n+1} &= (1 - t_n)y_n + t_n[y_n - f_2(y_n) \\
&\qquad +R^{H_2,m_2}_{M_2,\lambda_2,\eta_2}[H_2(f_2(y_n)) - \lambda_2 A_2(x_n)] + k_n] + l_n.
\end{aligned}\right. \qquad (10)$$

Here, $\{d_n\}$, $\{e_n\}$, $\{k_n\}$ and $\{l_n\}$ are four error sequences to take into account a possible inexact computation.

Lemma 3.4. [15] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences such that there exists a natural number n_0 such that*

$$a_{n+1} \leq (1-s_n)a_n + s_n b_n + c_n, \quad \forall n \geq n_0,$$

where $s_n \in [0,1]$, $\sum_{n=0}^{\infty} s_n = +\infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to 0} a_n = 0$.

Theorem 3.5. *Suppose that i, X_i, c_i, H_i, A_i, f_i, M_i, η_i, ξ_1, ξ_2, $\{d_n\}$, $\{e_n\}$, $\{k_n\}$, $\{l_n\}$ and $\{t_n\} \subseteq [0,1]$ are the same as in* Theorem 3.3 *and* Algorithm 1. *If $\|(d_n,k_n)\|_\times \to 0$, $\sum_{n=0}^{+\infty} \|(e_n,l_n)\|_\times < +\infty$ and $\sum_{n=0}^{+\infty} t_n = +\infty$, then the iterative sequence $\{(x_n,y_n)\}$ generated by* Algorithm 1 *converges strongly to the unique solution of system* (1).

Proof. All conditions of Theorem 3.3 hold and hence by Theorem 3.1 we find $(x,y) \in X_1 \times X_2$ such that

$$\begin{cases} x = (1-t_n)x + t_n(x - f_1(x) + R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(x)) - \lambda_1 A_1(y)]) \\ y = (1-t_n)y + t_n(y - f_2(y) + R^{H_2,m_2}_{M_2,\lambda_2,\eta_2}[H_2(f_2(y)) - \lambda_2 A_2(x)]). \end{cases} \tag{11}$$

It follows from our assumption, (10), (11), Lemma 3.2 and Theorem 2.2 that

$$\begin{aligned}
&\| \ x_{n+1} - x\| \\
&= \|(1-t_n)x_n + t_n(x_n - f_1(x_n) + R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(x_n)) - \lambda_1 A_1(y_n)] + d_n) \\
&\quad + e_n - [(1-t_n)x + t_n(x - f_1(x) + R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(x)) - \lambda_1 A_1(y)])]\| \\
&\leq (1-t_n)\|x_n - x\| + t_n\|x_n - x - f_1(x_n) + f_1(x)\| + t_n\|d_n\| + \|e_n\| + \\
&\quad t_n\|R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(x_n)) - \lambda_1 A_1(y_n)] - R^{H_1,m_1}_{M_1,\lambda_1,\eta_1}[H_1(f_1(x)) - \lambda_1 A_1(y)])]\| \\
&\leq (1-t_n)\|x_n - x\| + t_n\sqrt{1 - 2\kappa_1 + 64c_1\alpha_1^2}\|x_n - x\| + t_n\|d_n\| \qquad (12) \\
&\quad + \|e_n\| + t_n\frac{\gamma_1}{r_1 - \lambda_1 m_1}(\|H_1(f_1(x_n)) - H_1(f_1(x))\| + \lambda_1\|A_1(y_n) - A_1(y)\|) \\
&\leq \left[(1-t_n) + t_n\left(\sqrt{1 - 2\kappa_1 + 64c_1\alpha_1^2} + \frac{\alpha_1\theta_1\gamma_1}{r_1 - \lambda_1 m_1}\right)\right]\|x_n - x\| \\
&\qquad + t_n\frac{\lambda_1\pi_1\gamma_1}{r_1 - \lambda_1 m_1}\|y_n - y\| + t_n\|d_n\| + \|e_n\|
\end{aligned}$$

and similarly we get

$$\begin{aligned}
\|y_{n+1} - y\| &\leq \left[(1-t_n) + t_n\left(\sqrt{1 - 2\kappa_2 + 64c_2\alpha_2^2} + \frac{\alpha_2\theta_2\gamma_2}{r_2 - \lambda_2 m_1}\right)\right]\|y_n - y\| \\
&\quad + t_n\frac{\lambda_2\pi_2\gamma_2}{r_2 - \lambda_2 m_2}\|x_n - x\| + t_n\|k_n\| + \|l_n\|. \qquad (13)
\end{aligned}$$

Hence, by (12), (13), and also (e) and (f) of Theorem 3.3 we have

$$\begin{aligned} \|(x_{n+1}, y_{n+1}) - (x, y)\|_\times &= \|x_{n+1} - x\| + \|y_{n+1} - y\| \\ &\le [1 - (1 - \max\{\xi_1, \xi_2\})t_n]\|(x_n, y_n) - (x, y)\|_\times \\ &\quad + t_n\|(d_n, k_n)\|_\times + \|(e_n, l_n)\|_\times. \end{aligned} \tag{14}$$

Set $a_n = \|(x_n, y_n) - (x, y)\|_\times$, $s_n = (1 - \max\{\xi_1, \xi_2\})t_n$, $b_n = \frac{\|(d_n,k_n)\|_\times}{1-\max\{\xi_1,\xi_2\}}$ and $c_n = \|(e_n, l_n)\|_\times$. Therefore all conditions of Lemma 3.4 hold and hence $\lim_n \|(x_n, y_n) - (x, y)\|_\times = 0$; i.e., $\lim_n (x_n, y_n) = (x, y)$. □

Remark 1. It should be noticed that by using Fact 2.3 and Fact 2.4 (resp. Fact 2.5) instead of Theorem 2.2 one can achieve some advantages about existence of the unique solution and convergence analysis, but for general H-maximal β-strongly η-monotone (resp. H-maximal η-monotone) operators.

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