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YANG-MILLS CONNECTIONS ON A COMPACT CONNECTED SEMISIMPLE LIE GROUP

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Abstract. Let G be a compact connected semisimple Lie group, $\mathfrak{g}$ the Lie algebra of G, g the canonical metric (the biinvariant Riemannian metric which is induced from the Killing form of $\mathfrak{g}$), and ∇ be the Levi-Civita connection for the metric g. Then, we get the fact that the Levi-Civita connection ∇ in the tangent bundle TG over (G, g) is a Yang-Mills connection.

1. Introduction

The problem of finding metrics and connections which are critical points of some functional plays an important role in global analysis and Riemannian geometry. A Yang-Mills connection is a critical point of the Yang-Mills functional

$$\mathcal{YM}(D) = \frac{1}{2}\int_M \|R^D\|^2 \, v_g \tag{1.1}$$

on the space $\mathfrak{C}_E$ of all connections in a smooth vector bundle E over a closed (compact and connected) Riemannian manifold (M, g), where R^D is the curvature of $D \in \mathfrak{C}_E$. Equivalently, D is a Yang-Mills connection if it satisfies the Yang-Mills equation (cf. [1, 5, 6])

$$\delta_D R^D = 0, \tag{1.2}$$

(the Euler-Lagrange equations of the variational principle associated with (1.1)). If D is a connection in a vector bundle E with bundle metric h over a Riemannian manifold (M, g), then the connection D^* given by

$$h({D^*}_X s, t) = X(h(s,t)) - h(s, D_X t), \quad (X \in \mathfrak{X}(M) \text{ and } s, t \in \Gamma(E)) \tag{1.3}$$

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is referred to *conjugate* (cf. [1, 5]) to D. A connection D in E is a Yang-Mills connection if and only if (cf. [1, 5, 6])

$$(\delta_D R^D)(X)s = -\sum_{i=1}^{n}(D^*_{e_i} R^D)(e_i, X)s = 0 \tag{1.4}$$

for arbitrary given $X \in \mathfrak{X}(M)$, where $s \in \Gamma(E)$ and $\{e_i\}_{i=1}^n$ is a local orthonormal frame on(M, g).

Recently using the concept of conjugate connection, the present author obtained the following:

Theorem [5]. *A connection D in a vector bundle E over a closed Riemannian manifold (M, g) is a Yang-Mills connection if and only if the conjugate connection D^* is a Yang-Mills connection.*

In this paper, using this conception we get the following main result:

Theorem 3.2. *Let G be a compact connected semisimple Lie group, g the canonical metric on G, ∇ the Levi-Civita connection for the metric g. Then, ∇ is a Yang-Mills connection in the tangent bundle over the base manifold (G, g).*

2. Yang-Mills connections in vector bundles over a Riemannian manifold

Let E be a vector bundle, with bundle metric h, over an n-dimensional closed (compact and connected) Riemannian manifold (M, g). Let $D \in \mathfrak{C}_E$ and ∇ the Levi-Civita connection of (M, g). The pair (D, ∇) induces a connection in product bundles $\bigwedge^p TM^* \otimes E$, also denoted by D. Set $A^p(E) := \Gamma(\bigwedge^p TM^* \otimes E)$. We consider the differential operator

$$d_D : A^p(E) \longrightarrow A^{p+1}(E),$$

$$(d_D\varphi)(X_1, X_2, \cdots, X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}(D_{X_i}\varphi)(X_1, \cdots, \widehat{X}_i, \cdots, X_{p+1}),$$

$$\varphi \in A^p(E),\ X_i \in \mathfrak{X}(M)\ (i = 1, 2, \cdots, p+1),$$

which are defined by

$$d_D(\omega \otimes \xi) := d\omega \otimes \xi + (-1)^p \omega \wedge D\xi,$$
$$D_X(\omega \otimes \xi) := (\nabla_X \omega) \otimes \xi + \omega \otimes D_X\xi,$$

for $\omega \in \Gamma(\bigwedge^p TM^*)$, $\xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$.

Let δ_D be the formal adjoint of d_D with respect to the L^2-inner product

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle v_g$$

for $\varphi, \psi \in A^p(E)$. Here $\langle\ ,\ \rangle$ is the bundle metric in $\bigwedge^p TM^* \otimes E$ induced by the pair (g, h) and v_g is the canonical volume form on (M, g). The following identity is elementary, yet crucial (cf. [1, 2])

$$\delta_D\varphi = (-1)^{p+1}(*^{-1} \cdot d_{D^*} \cdot *)(\varphi) = (-1)^{np+1}(* \cdot d_{D^*} \cdot *)(\varphi) \tag{2.1}$$

for any $\varphi \in A^{p+1}(E)$. Here, $* : A^q(E) \longrightarrow A^{n-q}(E), (0 \leq q \leq n)$, is the Hodge operator with respect to g. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame on (M, g). Note that (2.1) may also be written as (cf. [1])

$$(\delta_D\varphi)(X_1, \cdots, X_p) = -\sum_{i=1}^{n}(D^*_{e_i}\varphi)(e_i, X_1, \cdots, X_p). \tag{2.2}$$

The connections D, $D^* \in \mathfrak{C}_E$ naturally induce connections, denoted by the same symbols, in $\mathrm{End}(E)$ $(:= E \otimes E^*)$. Then, a straightforward argument shows that D, $D^* \in \mathfrak{C}_{\mathrm{End}(E)}$ are conjugate connections. Thus, we find from (1.3) and (2.2) that a connection D in E is a Yang-Mills connection if and only if (cf. [1, 5, 6])

$$(\delta_D R^D)(X)s = -\sum_{i=1}^{n}(D^*_{e_i}R^D)(e_i, X)s = 0 \tag{2.3}$$

for arbitrary given $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$.

3. Yang-Mills Levi-Civata connection on a compact connected semisimple Lie group

3.1. Let G be a compact connected Lie group, g a biinvariant Riemannian metric on G, and $\mathfrak{g}$ the Lie algebra of G. Here, $\mathfrak{g}$ is identified with the algebra of all left invariant vector fields on G. Then, the Levi-Civita connection ∇ for the metric g is given as follows (cf. [4, Theorem 13.1]):

$$\nabla_X Y = \frac{1}{2}[X, Y], \quad (X, Y \in \mathfrak{g}). \tag{3.1}$$

3.2. Let G be an n-dimensional compact connected semisimple Lie group. Then, minus the Killing form of its Lie algebra $\mathfrak{g}$ (the set of all left invariant vector fields on G) is said to be the *canonical metric* on the Lie group G. Let g be the canonical metric on the Lie group G. Then, g is bi-invariant on G. Let $\{X_i\}_{i=1}^n$ be an orthonormal basis of the semisimple Lie algebra $\mathfrak{g}$ with respect to the canonical metric g. Let $\{\theta^j\}_{j=1}^n$ be the dual basis of the basis $\{X_i\}_{i=1}^n$. Then each θ^j is left invariant, that is, $L_x^*(\theta^j) = \theta^j$ $(x \in G)$. From (3.1), the Levi-Civita connection ∇ for the metric g is given by

$$\theta^l(\nabla_{X_i}X_j) = \frac{1}{2}\, C_{ij}{}^l. \tag{3.2}$$

where $C_{ij}{}^l := \theta^l([X_i, X_j])$ for the orthonormal frame $\{X_i\}_{i=1}^n$. By virtue of (3.1) and properties of the Killing form on the semisimple Lie algebra $\mathfrak{g}$, we have for $X, Y, Z \in \mathfrak{g}$ (cf. [2, 3, 7])

$$g([X,Y],Z)+g(Y,[X,Z])=0, \quad R^{\nabla}(X,Y)=-\frac{1}{4}\ ad([X,Y]), \tag{3.3}$$

where ad is the adjoint representation of the semisimple Lie algebra $\mathfrak{g}$. From (3.3) and the definition of the Killing form B of the semisimple Lie algebra $\mathfrak{g}$ such that $(B(X,Y) := \text{Trace}\ (ad(X)\ ad(Y)) \quad (X,Y\in\mathfrak{g}))$, we get for $Y,Z\in\mathfrak{g}$ (cf. [2, 3, 7])

$$\sum_{i=1}^{n} g(R^{\nabla}(X_i,Y)Z,X_i)=\frac{1}{4}\ g(Y,Z), \tag{3.4}$$

that is, the Riemannian manifold (G,g) is an Einstein manifold of Ricci curvature $\frac{1}{4}$. From the fact $g(\nabla_{X_i}X_j,X_l)+g(X_j,\nabla_{X_i}X_l)=0$, (3.1) and (3.2), we have

$$C_{ij}{}^{k}=-C_{ik}{}^{j}=-C_{kj}{}^{i}. \tag{3.5}$$

By virtue of (3.3), (3.4) and (3.5), we get

$$\sum_{i,l=1}^{n} C_{il}{}^{k}C_{il}{}^{j}=\delta_{kj}. \tag{3.6}$$

Putting $R^{\nabla}(X_i,X_j)X_k =: \sum_t R_{ijk}{}^{t}X_t$, we have from (3.2)

$$R_{ijk}{}^{t}=\frac{1}{4}\sum_{s}(C_{jk}{}^{s}C_{is}{}^{t}-C_{ik}{}^{s}C_{js}{}^{t}-2C_{ij}{}^{s}C_{sk}{}^{t}). \tag{3.7}$$

From (2.3), we get

$$\begin{aligned}(\delta_{\nabla}R^{\nabla})(X_j)X_k = -\sum\{&\nabla_{X_i}(R^{\nabla}(X_i,X_j)X_k)-R^{\nabla}(\nabla_{X_i}X_i,X_j)X_k\\ &-R^{\nabla}(X_i,\nabla_{X_i}X_j)X_k-R^{\nabla}(X_i,X_j)\nabla_{X_i}X_k\}.\end{aligned} \tag{3.8}$$

By the helf of (3.2),(3.5)–(3.8), we have

$$(\delta_{\nabla}R^{\nabla})(X_j)X_k=-\frac{1}{2}\sum_{l}(C_{jk}{}^{l}-2\sum_{i,t}C_{ij}{}^{t}C_{tk}{}^{s}C_{si}{}^{l})X_l. \tag{3.9}$$

On the other hand, we get

Lemma 3.1.

$$2\sum_{i,s,t=1}^{n} C_{ij}{}^{t}C_{tk}{}^{s}C_{si}{}^{l}=C_{jk}{}^{l}.$$

Proof. By virtue of (3.3), (3.5) and (3.6),

$$\begin{aligned}
&\sum_{i,s,t} C_{ij}{}^t C_{tk}{}^s C_{si}{}^l \\
&\qquad = \sum_{i,s,t} g([[[X_i, X_j], X_k], X_i], X_l) \\
&\qquad = \sum_{i,s,t} g([[X_i, X_j], X_k], [X_i, X_l]) \\
&\qquad = -\sum_{i,s,t} g([[X_j, X_k], X_i] + [[X_k, X_i], X_j], [X_i, X_l]) \\
&\qquad = -\sum_{i,s,t} (C_{jk}{}^t C_{ti}{}^s C_{il}{}^s + C_{ki}{}^t C_{tj}{}^s C_{il}{}^s) \\
&\qquad = C_{jk}{}^l - \sum_{i,s,t} C_{tj} i C_{ks}{}^t C_{sl}{}^i = C_{jk}{}^l - \sum_{i,s,t} C_{ij}{}^t C_{tk}{}^s C_{si}{}^l.
\end{aligned}$$

Thus, the proof of this Lemma is completed.

By virtue of (3.9) and Lemma 3.1, we obtain

Theorem 3.2. *Let G be a compact connected semisimple Lie group, g the canonical metric on G, ∇ the Levi-Civita connection for the metric g. Then, ∇ is a Yang-Mills connection in the tangent bundle over the base manifold (G, g).* □

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