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RATIONAL CURVES ARE NOT UNIT SPEED IN THE GENERAL EUCLIDEAN SPACE

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Abstract. We invoke the characterization of Pythagorean-hodograph polynomial curves and prove that it is impossible to parameterize any real curves, other than a straight line, by rational functions of its arc length.

1. Introduction

Let $\mathbf{r}(t) = (x_0(t), x_1(t), \ldots, x_n(t))$ be a rational curve in the Euclidean space $\mathbb{R}^{n+1}$ with $n \geq 0$. Unless $\mathbf{r}(t)$ is the trivial constant curve, there are polynomials $P_0(t), \ldots, P_n(t)$ and $Q(t)$ in t with real coefficients such that

$$\gcd(P_0, P_1, \ldots, P_n, Q) = 1 \quad \text{and} \quad \max\{\deg(P_0), \ldots, \deg(P_n), \deg(Q)\} \geq 1 \tag{1}$$

so that

$$x_0(t) = \frac{P_0(t)}{Q(t)},\ x_1(t) = \frac{P_1(t)}{Q(t)},\ \ldots,\ x_n(t) = \frac{P_n(t)}{Q(t)}. \tag{2}$$

Since the hodograph $\mathbf{r}'(t) = (x_0'(t), \ldots, x_n'(t))$ of $\mathbf{r}(t)$ is given by

$$x_0'(t) = \frac{P_0'(t)Q(t) - P_0(t)Q'(t)}{Q(t)^2},\ \ldots,\ x_n'(t) = \frac{P_n'(t)Q(t) - P_n(t)Q'(t)}{Q(t)^2}, \tag{3}$$

we can see that $\mathbf{r}(t)$ is unit speed if and only if the polynomial curve

$$\mathbf{S}(t) = (P_0'(t)Q(t) - P_0(t)Q'(t), \ldots, P_n'(t)Q(t) - P_n(t)Q'(t))$$

is Pythagorean with $Q(t)^2$, i. e.

$$[P_0'(t)Q(t) - P_0(t)Q'(t)]^2 + \cdots + [P_n'(t)Q(t) - P_n(t)Q'(t)]^2 = [Q(t)^2]^2.$$

Thus in order to find the characterization of unit-speed rational curves, it is of use to study the related Pythagorean polynomial curves. We present the theorem of the characterization of unit-speed ration curves.

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Theorem 1.1. *Let* $\mathbf{r}(t) = (x_0(t), x_1(t), \ldots, x_n(t))$ *be a rational curve in the Euclidean space* $\mathbb{R}^{n+1}$ *with* $n \geq 0$*. Then* $\mathbf{r}(t)$ *is unit speed if and only if it is parameterized by*

$$\mathbf{r}(t) = (a_0 t + c_0, a_1 t + c_1, \ldots, a_n t + c_n) \tag{4}$$

for some constants $a_0, \ldots, a_n$ *and* $c_0, \ldots, c_n$ *with* $a_0^2 + a_1^2 + \cdots + a_n^2 = 1$*.*

The above theorem was proved by Farouki and Sakkalis for the plane ($n = 1$) [3] in 1991 and for the space ($n = 2$) [4] in 2007. For the proof, Farouki and Sakkalis used the ideas from the integration theory and the complex residue theory, and the characterizations of Pythagorean triples [6] and quadruples [2] of polynomials.

Recently, Sakkalis, Farouki, and Vaserstein [8] prove this theorem for the general Euclidean space by a rather different approach. Because they do not have the charaterization for Pythagorean polynomials, they need to work some substitute for the characterization. In this paper, we follow the scheme of Farouki and Sakkalis [3] and prove Theorem 1.1 for the general Euclidean space by invoking the following characterization [5, 7] for Pythagorean $(n+2)$-tuples of polynomials for $n \geq 0$.

Proposition 1.2. [5, 7] *Let* $\mathbf{r}(t) = (x_0(t), x_1(t), \ldots, x_n(t))$ *be a polynomial curve in the Euclidean space* $\mathbb{R}^{n+1}$ *with* $n \geq 1$*. Then the polynomial curve* $\mathbf{r}(t)$ *is Pythagorean with a polynomial function* $\sigma(t)$*, i. e.*

$$x_0(t)^2 + x_1(t)^2 + \cdots + x_n(t)^2 = \sigma(t)^2,$$

if and only if there exist polynomial functions

$$U(t), V(t), A(t), B(t), W_1(t), \ldots, W_n(t), H(t)$$

with

$$\gcd(U, VB) = 1, \quad \gcd(V, UA) = 1, \quad \gcd(W_1, \ldots, W_n) = 1 \tag{5}$$

and

$$W_1(t)^2 + \cdots + W_n(t)^2 = A(t)B(t), \tag{6}$$

so that

$$\begin{aligned}
x_0(t) &= H(t)\left(U(t)^2 A(t) - V(t)^2 B(t)\right), \\
x_1(t) &= H(t)(2U(t)V(t)W_1(t)), \\
&\vdots \\
x_n(t) &= H(t)(2U(t)V(t)W_n(t)), \\
\sigma(t)^2 &= H(t)\left(U(t)^2 A(t) + V(t)^2 B(t)\right).
\end{aligned}$$

2. Proof of the main theorem

To prove Theorem 1.1, we need the following two propositions.

Proposition 2.1. [3] *Let $f(t)$ and $g(t)$ be real polynomials such that* $\gcd(f,g) = 1$ *and* $\deg(f) < \deg(g)$. *Suppose further that g has no real roots and* $\deg(g) > 1$. *Let $(\xi_j, \overline{\xi}_j)$ and m_j, for $j = 1, \ldots, N$, denote the distinct pairs of complex conjugate roots of $g(t)$ and their respective multiplicities. Then $\int_0^t \frac{f(s)}{g(s)}\,ds$ is a rational function if and only if for $j = 1, \ldots, N$,*

$$\mathrm{Res}\left(\frac{f(t)}{g(t)}, \xi_j\right) = 0 = \mathrm{Res}\left(\frac{f(t)}{g(t)}, \overline{\xi}_j\right),$$

where $\mathrm{Res}\left(\frac{f(t)}{g(t)}, \xi_j\right)$ $\mathrm{Res}\left(\frac{f(t)}{g(t)}, \overline{\xi}_j\right)$ *are the* residues *of $\frac{f(t)}{g(t)}$ at its conjugate poles ξ_j and $\overline{\xi}_j$, respectively.*

Proposition 2.2. [1] *For a real rational function $r(t)$ without real poles, if $r(t)$ has a zero at infinity of order 2 at least (i. e., if $r(t) = f(t)/g(t)$ for some polynomial $f(t)$ and $g(t)$ then* $\deg(f) + 2 \leq \deg(g)$*), then*

$$\int_{-\infty}^{+\infty} r(t)\,dt = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(r(t), \xi_k)$$

where $(\xi_k, \overline{\xi}_k)$ are the distinct pairs of complex conjugate poles of $r(t)$ with $\mathrm{Im}(\xi_k) > 0$ *for $k = 1, \ldots, N$.*

Now we prove Theorem 1.1. If $\mathbf{r}(t)$ is of the form (4), then $\mathbf{r}(t)$ is clearly unit speed.

Conversely we suppose that $\mathbf{r}(t)$ is unit speed. In the case of $n = 0$, the result is clearly true. So we will prove that for the case of $n \geq 1$. In this case, there are polynomials $P_0(t), \ldots, P_n(t)$ and $Q(t)$ in t with real coefficients such that (1) so that (2). The hodograph $\mathbf{r}'(t) = (x_0'(t), \ldots, x_n'(t))$ of $\mathbf{r}(t)$ is given by (3).

Since $\mathbf{r}(t)$ is unit speed, by Proposition 1.2 there are polynomials

$$U(t), V(t), A(t), B(t), W_1(t), \ldots, W_n(t), H(t)$$

with (5) and (6), so that

$$\begin{aligned}
P_0'(t)Q(t) - P_0(t)Q'(t) &= H(t)\left(U(t)^2A(t) - V(t)^2B(t)\right),\\
P_1'(t)Q(t) - P_1(t)Q'(t) &= H(t)(2U(t)V(t)W_1(t)),\\
&\ \ \vdots\\
P_n'(t)Q(t) - P_n(t)Q'(t) &= H(t)(2U(t)V(t)W_n(t)),\\
Q(t)^2 &= H(t)\left(U(t)^2A(t) + V(t)^2B(t)\right).
\end{aligned}$$

From

$$\frac{P_0'Q - P_0Q'}{Q^2} = \frac{U^2A - V^2B}{U^2A + V^2B} = \frac{(UA)^2 - (VW_1)^2 - \cdots - (VW_n)^2}{(UA)^2 + (VW_1)^2 + \cdots + (VW_n)^2}$$

and

$$\frac{P_k'Q - P_kQ'}{Q^2} = \frac{2UVW_k}{U^2A + V^2B} = \frac{2(UA)(VW_k)}{(UA)^2 + (VW_1)^2 + \cdots + (VW_n)^2}$$

for $k = 1, \ldots, n$, we have

$$x_0(t) - x_0(0) = \int_0^t \frac{(U(s)A(s))^2 - (V(s)W_1(s))^2 - \cdots - (V(s)W_n(s))^2}{(U(s)A(s))^2 + (V(s)W_1(s))^2 + \cdots + (V(s)W_n(s))^2}\, ds$$

and

$$x_k(t) - x_k(0) = \int_0^t \frac{2(U(s)A(s))(V(s)W_k(s))}{(U(s)A(s))^2 + (V(s)W_1(s))^2 + \cdots + (V(s)W_n(s))^2}\, ds$$

for $k = 1, \ldots, n$. Now for any real numbers $\lambda_0, \ldots, \lambda_n$, let

$$y(t) = \frac{[\lambda_1 U(t)A(t) + \lambda_0 V(t)W_1(t)]^2 + \cdots + [\lambda_n U(t)A(t) + \lambda_0 V(t)W_n(t)]^2}{(U(t)A(t))^2 + (V(t)W_1(t))^2 + \cdots + (V(t)W_n(t))^2} \tag{7}$$

Then since

$$\begin{aligned}\int_0^t y(s)\, ds = {} & \frac{\lambda_1^2 + \cdots + \lambda_n^2 - \lambda_0^2}{2}(x_0(t) - x_0(0)) \\ & + \lambda_0 \{\lambda_1(x_1(t) - x_1(0)) + \cdots + \lambda_n(x_n(t) - x_n(0))\} \\ & + \frac{\lambda_1^2 + \cdots + \lambda_n^2 + \lambda_0^2}{2} t,\end{aligned}$$

$\int_0^t y(s)\, ds$ must be rational.

We choose $\lambda_0, \ldots, \lambda_n$ so that the degree of the numerator in (7) is two or more less than that of denominator, in the following manner:

(a) In the case of $\deg(VW_k) > \deg(UA)$ for some $1 \le k \le n$ we set $\lambda_0 = 0$ and $\lambda_1 = \cdots = \lambda_n = 1$. Then since $\int_0^t y(s)\, ds$ is rational, by Proposition 2.1 and Proposition 2.2, the integral

$$\int_{-\infty}^{+\infty} \frac{n[U(s)A(s)]^2}{(U(s)A(s))^2 + (V(s)W_1(s))^2 + \cdots (V(s)W_n(s))^2}\, ds$$

must be zero. This implies that $UA = 0$, so that $U = 0$ since $A \neq 0$. Therefore we conclude that $x_0'(t) = -1$, $x_1'(t) = 0$, …, $x_n'(t) = 0$, which implies that $\mathbf{r}(t)$ is of the form (4).

(b) In case of $\deg(VW_k) \le \deg(UA)$ for all $1 \le k \le n$, we take $\lambda_0 = 1$ and

$$\lambda_k = \begin{cases} \text{any constant} & \text{if } UA = 0, \\ -b/a & \text{if } 0 \le \deg(VW_k) = \deg(UA), \\ 0 & \text{if } \deg(VW_k) < \deg(UA), \end{cases}$$

where a and b are the leading coefficients of UA and VW_k, respectively, so that $\deg(\lambda_k UA + VW_k) < \deg(UA)$ in the case of $0 \le \deg(VW_k) = \deg(UA)$. With these choices for $\lambda_0, \dots, \lambda_n$, since $\int_0^t y(s)\,ds$ is rational, by Proposition 2.1 and Proposition 2.2, the integral

$$\int_{-\infty}^{+\infty} \frac{[\lambda_1 U(s)A(s) + V(s)W_1(s)]^2 + \cdots + [\lambda_n U(s)A(s) + V(s)W_n(s)]^2}{(U(s)A(s))^2 + (V(s)W_1(s))^2 + \cdots + (V(s)W_n(s))^2}\, ds.$$

exists and must be zero. This induces that for $k = 1, \dots, n$, $\lambda_k U(t)A(t) + V(t)W_k(t) = 0$. For $k = 1, \dots, n$, if $\lambda_k \neq 0$, then UA, V, and W_k are constant, since $\lambda_k UA = -VW_k$ and $\gcd(UA, V) = 1$. If $\lambda_k = 0$, then $VW_k = 0$, which implies that $V = 0$ or $W_k = 0$. In this case, $\mathbf{r}(t)$ is of the form (4). The proof is done.

References

[1] L. V. Ahlfors, Complex Analysis, Third Edition, McGraw-Hill Book Co., New York, 1979.

[2] R. Dietz, J. Hoschek, and B. Jüttler, An algebraic approach to curves and surfaces on the sphere and on other quadrics, Comput. Aided Geom. Design **10** (1993), 211–229.

[3] R. T. Farouki and T. Sakkalis, Real rational curves are not 'unit speed', Comput. Aided Geom. Design **8** (1991), 151–157.

[4] ______, Rational space curves are not "unit speed", Comput. Aided Geom. Design **24** (2007), 238–240.

[5] G.-I. Kim, Higher dimensional PH curves, Proc. Japan Acad. **78** (2002), Ser. A., 185–187.

[6] K. K. Kubota, Pythagorean triples in unique factorization domains, Amer. Math. Monthly **79** (1972), 503–505.

[7] S. Lee and G.-I. Kim, Characterization of Minkowski Pythagorean-hodograph curves, J. Appl. Math. & Computing **24** (2007), 521–528.

[8] T. Sakkalis, R. T. Farouki, and L. Vaserstein, Non-existence of rational arc length parameterizations for curves in $\mathbb{R}^n$, J. Comput. Appl. Math. **228** (2009), 494–497.

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