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On asymptotic stability in nonlinear differential system[†]

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Abstract

We obtain, in using generalized norms, some stability results for a very general system of differential equations using the method of cone-valued Lyapunov funtions and we obtain necessary and/or sufficient conditions for the uniformly asymptotic stability of the nonlinear differential system.

Keywords: Asymptotic stability, cone-valued Lyapunov funtion, differential equations, generalized norms, nonlinear differential system.

1. Preliminaries and definitions

Lyapunov second methods are now well established subjects as the most powerful techniques of analysis for the stability and qualitative properties of nonlinear differential equations $x' = f(t, x)$, $x(t_0) = x_0 \in R^N$. One of the original Lyapunov theorems is as follows:

Lyapunov Theorems. For $x' = f(t, x)$, assume that there exists a function $V : R_+ \times S_\rho \to R_+$ such that

(i) V is C^1-function and positive definite,

(ii) V is decresent,

(iii) $\frac{d}{dt}V(t, x) = V_t(t, x) + V_x \cdot f(t, x) \le -a(||x||)$ for $t \ge 0$, $x \in S_\rho$, where $S_\rho = \left\{x \in R^N ||x|| < \rho\right\}$ for $\rho > 0$, $a(r)$ is strictly increasing function with $a(0) = 0$.

Then the trivial solution $x(t) \equiv 0$ is uniformly asymptotically stable. The advantage of the method is that is that it does not require the knowledge of solutions to analyse the stability of the equations. However in practical sense, how to find suitable Laypunov functions V for given equations are the most difficult questions. Hence weakening the conditions (i), (ii), and (iii), and enlarging the class of Lyapunov functions are basic trends in Lyapunov

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stability theory (Brauer, 1965; Brauer and Nohel, 1969; Ki, 2000; Danan and Elaydi, 1969; Lee, *et al.*, 1999; Lee, 2007).

In the unified comparison frameworks for Ladde (1976), Moon and Shin (1997) of analysed the stability of comparison differential equations by using vector Lyapunov function methods.

Lakeshmikantham and Leela (1977) initiated the cone-valued Lyapunov function methods to avoid the quasimonotonicity assumption of comparison equations. They obtained various useful differential inequalities with cone-valued Lyapunov functions, Akpan and Akinyele (1992) extended and generalized the results of Lakshmikantham and Leela (1969) to the ϕ_0 -stability of the comparison differential equations by using the cone-valued Lyapunov functions.

Here we generalize, in some sense, the results of Akpan and Akinyele (1992) to the $\phi(t)$-stabilities of comparison equations below.

Let R^n denote the n-dimensional Euclidean space with any equivalent norm $||\cdot||$, and scalar product $(\ ,\ )$.

$R_+ = [0, \infty)$. $C\,[R_+ \times R^n, R^n]$ denotes the space of continuous functions from $R_+ \times R^n$ into R^n.

Definition 1.1. A proper subset K of R^n is called a cone if (i) $\lambda K \subset K$, $\lambda \geq 0$; (ii) $K + K \subset K$; (iii) $K = \bar{K}$; (iv) $K^\circ \neq \emptyset$; (v) $K \cap (K) = \{0\}$, where $\bar{K}$ and K° denote the closure and interior of K, respectively and ∂K denotes the boundary of K. The order relation on R^n induced by the cone K is defined as follow: for $x, y \in R^n$, $x \leq_k y$ iff $x - y \in K$, and $x \leq_{k^\circ} y$ iff $y - x \in K^\circ$.

Definition 1.2. The set $K^* = \{\phi \in R^n : (\phi, x) \geq 0)$, for all $x \in K\}$ is called the adjoint cone of K if K^* itself satisfies Definition 1.1.

Note that $x \in \partial K$ if and only if $(\phi, x) = 0$ for some $\phi \in K^*$, where $K_0 = K - \{0\}$.

Definition 1.3. A function $g : D \to R^n$, $D \subset R^n$ is said to be quasimonotone nondecreasing relative to the cone K when it satisfies that if $x, y \in D$ with $x \leq_K y$ and $(\phi_0, y - x) = 0$ for some $\phi_0 \in K_{0^*}$, then $(\phi, g(y) - g(x)) \geq 0$.

Definition 1.4. A generalized norm from R^n to cone $K (\subseteq R^s)$ is a mapping $||\ ||_G : R^n \to K$ defined by $||x||_G = (\alpha_1(x), \cdots, \alpha_s(x))(s \leq n)$ such that

(a) $||x||_G \geq_K 0$.

(b) $||x||_G = 0$ iff $x = 0$. (i.e., $\alpha_i(x) = 0$ iff $x = 0, i = 1, \cdots, s$)

(c) $||\lambda x||_G = |\lambda| ||x||_G$. (i.e., $\alpha_i(\lambda x) = |\lambda| \alpha_i(x), i = 1, \cdots, s$)

(d) $||x + y||_G \leq_K ||x||_G + ||y||_G$

If $R^s_+ = R^n_+$, then we have a special generalized norm $||u||_{G_n}$, for u $\in R^n$, defined by$||u||_{G_n} = (|u_1|, |u_2|, \cdots, |u_n|)$ where $u = (u_1, u_2, \cdots, u_n)$.

Assume that $||\cdot||_{K_N} : R^N \to K, ||\cdot||_{K_n} : R^n \to K$, and $||x||_{K_n} = x$ for all $x \in K$.

Consider the differential equation

$$x' = f(t, x),\ x(t_0) = x_0, \qquad t_0 \geq 0 \tag{1.1}$$

where $f \in C\left[R_+ \times R^N, R^N\right]$ and $f(t, 0) = 0$ for all $t \geq 0$. Let $S^*_\rho = \left\{x \in R^N : ||x||_{K_N} <_K \rho\right\}$, $\rho > 0$. Let $K \subset R^n$ be a cone in R^n, $n \leq N$. For $V \in C\left[R_+ \times S^*_\rho, K\right]$ at $(t, x) \in R_+ \times S^*_\rho$, let $D^{+}V(t,x) = \lim (1/h)\,[V(t+h, x+hf(t,x)) - V(t,x)]$ be a Dini derivative of V along the solution curves of the equations (1.1).

Consider a comparison differential equation

$$u' = g(t,u),\ u(t_0) = u_0, \quad t_0 \geq 0 \tag{1.2}$$

where $g \in C\,[R_+ \times K, R^n]$, $g(t,0) = 0$ for all $t \geq 0$ and K is a cone in R^n.

Let $S^*(\rho) = \{u \in K : ||u||_{K_n} <_K \rho\}$, $\rho > 0$. for $v \in C\,[R_+ \times S^*(\rho), K]$, at $(t,u) \in R_+ \times S^*(\rho)$, let $D^{+}v(t,u) = \lim (1/h)\,[v(t+h, u+hg(t,u)) - v(t,u)]$ be a Dini derivative of v along solution curves of the equation (1.2).

Definition 1.5. The trivial solution $x = 0$ of (1.1) is (S_1) equistable if for each $\varepsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ such that the inequality $||x_0|| < \delta$ implies $||x(t, t_0, x_0)|| < \varepsilon$, for all $t \geq t_0$.

Other stability notions $(S_2 \sim S_8)$ can be similarly defined V. Lakshmikantham and Leela (1977), and Lakshmikantham, Matrosov and Sivasundaram (1991).

Now we give cone-valued $\phi(t)$-stability definitions of the trival solution of (1.2). Let $\phi : [0,\infty] \to K*$ be a cone-valued function.

Definition 1.6. The trivial solution $u = 0$ of (1.2) is

(S_1^*) $\phi(t)$-equistable if for each $\varepsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ such that the inequality $(\phi(t_0), ||u_0||_{K_n}) < \delta$ implies $(\phi(t), ||r(t))||_{K_n}) < \varepsilon$, for all $t \geq t_0$ where $r(t)$ is a maximal solution of (1.2);

(S_2^*) uniformly $\phi(t)$-stable if the δ in (S1*) is independent of t_0; Other $\phi(t)$-stability notions $(S_3^* \sim S_8^*)$ can be similarly defined.

2. Stability theorems

Theorem 2.1. Assume that

(i) $v \in C\,[R_+ \times S^*(\rho), K]$, $v(t,0) = 0$, $v(t,u)$ is locally Lipschitzian in u relative to K, and for each $(t,u) \in R_+ \times S^*(\rho)$, $D^{+}v(t,x) \leq_K 0$,

(ii) $g \in C\,[R_+ \times K, R^n]$ and $g(t,u)$ is quasimonotone in u relative to K,

(iii) $\phi(t) \in K_0^*$ is a bounded continuous function on $[0,\infty)$ and $a[(\phi(t), ||r(t)||_{K_n})\ ] \leq (\phi(t), v(t, u(t)))$, $t \geq t_0 \geq 0$ for some funtion $a \in$ K.

Then the trivial solution $u = 0$ of (1.2) is $\phi(t)$-eqistable.

Proof: Let $\epsilon > 0$ be arbitrarily given and let M $=\sup\{||\phi(t)|| : t \geq t_0\}$. Since $a^{-1}(Ma(\eta))$ is continuous and $a^{-1}(Ma(0)) = 0$, there exists $\epsilon_1 > 0$ such that $a^{-1}(Ma(\eta)) \leq \epsilon$ for $0 \leq \eta \leq \epsilon_1$. Since $v(t,0) = 0$ and $v(t,u)$ is continuous in u, given $a(\epsilon_1) > 0$, $t_0 \in R^+$, there exists $\delta_1 = \delta_1(t, a(\epsilon_1))$ such that $||u_0|| < \delta_1$ implies $||v(t_0, u_0)|| < a(\epsilon_1)$.

Now for the bounded continuous function $\phi(t) \in K_0^*$, $(\phi(t_0), ||u_0||_{K_n}) \leq ||\phi(t_0)|| \cdot ||u_0|| \leq ||\phi(t_0)||\delta_1$ implies $(\phi(t), v(t_0, u_0)) < ||\phi(t)||a(\epsilon_1)$. Put $\delta = ||\phi(t_0)||\delta_1$. Then $(\phi(t_0), \|u_0\|_{K_n}) < \delta$ implies $(\phi(t), v(t_0, u_0)) \leq ||\phi(t)|| \cdot ||v(t_0, u_0)|| < Ma(\epsilon_1)$.

Let $u(t)$ be any solution of (2) such that $(\phi(t_0), \|u(t_0)\|_{K_n}) < \delta$. Then by(i), $v(t, u(t)) \leq_K v(t_0, u_0), t \geq t_0$. Thus $(\phi(t_0), ||u(t_0)||_{K_n}) < \delta$ implies $a[(\phi(t), ||r(t)||_{K_n}) \leq (\phi(t), v(t, u(t))) \leq (\phi(t_0), v(t, u(t_0))) < Ma(\epsilon_1)$. Hence $(\phi(t), ||r(t)||_{K_n}) \leq a^{-1}(M(a(\epsilon_1))) \leq \epsilon$ which completes the proof. □

Theorem 2.2. Let conditions (i) and (ii) of Theorem 2.1 hold. Assume further that for some bounded continuous function $\phi(t) \in K_0^*$, for each $(t, u) \in R_+ \times S^*(\rho)$, $D^+(\phi(t), v(t, u(t))) \leq 0$ and $a[(\phi(t), ||r(t)||_{K_n})] \leq (\phi(t), V(t, u(t))) \leq b[(\phi(t), ||r(t)||_{K_n})]$ $a, b \in K$. Then the trivial solution $u = 0$ of (1.2) is uniformly $\phi(t)$-stable.

Proof: For $\epsilon > 0$, let $\delta = b^{-1}[a(\epsilon)]$. Let $u(t)$ be any solution of (2) such that $(\phi(t_0), \|u_0\|_{K_n}) < \delta$. Then by the hypothesis, $(\phi(t), v(t, u(t)))$ is decreasing and so $(\phi(t), v(t, u(t))) \leq (\phi(t_0), v(t, u(t_0)))$ for all $t \geq t_0$. Thus

$$\begin{aligned} a[(\phi(t), ||r(t)||_{K_n}) \leq (\phi(t), v(t, u(t))) &\leq (\phi(t_0), v(t, u(t_0))) < b[(\phi(t_0), ||r(t)||_{K_n})] \\ &= b[(\phi(t_0), \|u_0\|_{K_n})] < b(\delta) = b(b^{-1}(a(\epsilon))) = a(\epsilon). \end{aligned}$$

Hence $(\phi(t_0), ||u(t_0)||_{K_n}) < \delta$ implies $(\phi(t), ||r(t)||_{K_n}) \leq \epsilon$ for each $t \geq t_0$. □

Theorem 2.3. Assume that

(i) $V \in C[R_+ \times S^*{}_\rho, K]$, $V(t, x)$ is locally Lipschitzian in x relative to K and for $(t, x) \in R_+ \times S^*{}_\rho$, $D^+V(t, x) \leq_K g(t, V(t, x))$,

(ii) $g \in C[R_+ \times K, R^n]$ and $g(t, u)$ is quasimonotone in u relative to K for each $t \in R_+$,

(iii) there exist $a, b \in K$ such that for some $\phi(t) \in K_{0*}$, for each $x \in S_\rho^*$, $b(||x||) \leq (\phi(t), V(t, x)) \leq a(||x||)$, $t \geq t_0 \geq 0$

Then the trivial solution $x = 0$ of (1.1) has the corresponding one of the stability $(S_1 \sim S_8)$ properties if the trivial solution $u = 0$ of (1.2) has each one of the $\phi(t)$ -stability $(S_1^* \sim S_8^*)$ properties in Definition 1.6.

Theorem 2.4. Assume that

(i) $g \in C[R_+ \times K, R^n]$, $g(t, 0) = 0$ and $g(t, u)$ is quasimonotone in u relative to K for each $t \in R_+$ and for $(t, u), (t, v) \in R_+ \times K$ and
$||g(t, u) - g(t, v)||_{K_n} \leq_K L(t)||u - v||_{K_n}$, $L \in C[R_+, R_+]$

(ii) $(\phi(t), ||r(t)||_{K_n}) \leq \beta(\phi(t), ||u(t)||_{K_n})$, $\beta \in K$, $u(t)$ is a solution of (1.2).

If the trivial solution $u = 0$ of (1.2) is uniformly asymptotically $\phi(t)$ -stable, if and only if there exists a cone-valued Lyapunov function v with the following properties:

(a) $v \in C[R_+ \times S^*(\rho), K]$, $v(t, 0) = 0$, and $v(t, u)$ is locally Lipschitzian in u relative to K for each $t \in R_+$.

(b) $a[(\phi(t), ||r(t)||_{K_n})] \leq (\phi(t), v(t, u(t))) \leq b[(\phi(t), ||r(t)||_{K_n})]$ for some $\phi(t) \in K_0^*$, $a, b \in K$, $(t, u) \in R_+ \times S^*(\rho)$.

(c) For $(t, u) \in R_+ \times S^*(\rho)$,and for $p(t)$ is increasing and bounded, $D^+v(\phi(t), v(t, u)) \leq_K -p'(t)(\phi(t), v(t, u))$, where $p'(t)$ exists.

Proof: The sufficiency is straightforward. Necessity. Let $u = u(t, 0, u_0)$ so that $u_0 = u(0, t, u)$. Define a cone-valued Lyapunov function $v(t, u(t))$ by

$$v(t, u(t)) = \exp(-p(t))C[(\phi(t), ||r(t)||_{K_N})]u(t, 0, \sigma_\omega(u)0, t, u))), \tag{2.1}$$

where $C[(\phi(t), ||r(t)||_{K_n})] = (1/D)[1 - \exp[-D((\phi(t), ||r(t)||_{K_n})]]$, $D > 0$, $p'(t)$ exists and $\sigma_\omega(x)$ is the function defined in Lakshmikantham and Leela (1969), and $u(t, u_0)$ is any solution of (1.2). When $u = 0$, then the right hand side of (2.1) vanishes so that $v(t, 0) = 0$.

Using (i) and Corollary 2.7.1 in Lakshmikantham and Leela (1969), and Lakshmikantham, *et al.* (1989) for $u_1, u_2 \in S^*(\rho)$,

$$\begin{aligned} ||v(t, u_1) - v(t, u_2)||_{K_n} &= || \exp(-p(t))C[(\phi(t), ||r(t)||_{Kn})]u_1(t, 0, \sigma_\omega(u_1(0, t, u_1))) \\ &\quad - \exp(-p(t))C[(\phi(t), \|r(t)\|_{K_n})]u_2(t, 0, \sigma_\omega(u_2(0, t, u_2)))||_{K_n} \\ &\leq_K ||N(t)||||\sigma_\omega(u_1(0, t, u_1)) - \sigma_\omega(u_2(0, t, u_2))||_{K_n} \exp \int_0^t L(s)ds \\ &\leq_K l(t)||\omega||||N(t)||||u_1 - u_2||_{K_n} \exp \int_0^t L(s)ds \\ &= \beta(t)||u_1 - u_2||_{K_n}, \end{aligned}$$

where $\beta(t) = l(t)||\omega|||| \exp(-p(t))C[(\phi(t), r(t))]$, $\exp \int_0^t L(s)ds \geq 0$. Now,

$$\begin{aligned} ||v(t + \delta, u^*) - v(t, u)||_{K_n} &\leq ||v(t + \delta, u^*) - v(t + \delta, u)||_{K_n} \\ &\quad + ||v(t + \delta, u) - v(t + \delta, u(t + \delta, t, u)))||_{K_n} \\ &\quad + ||v(t + \delta, u(t + \delta, u)) - v(t, u)||_{K_n}. \end{aligned}$$

Since $v(t, u)$ is locally Lipschitizan in u and u is continuous in δ, then the first two terms in the right hand side of the inequality are small whenever $||u - u^*||$ and δ are small.

Using (2.1) the third term tends to zero. Therefore $v(t, u)$ is continuous in all its arguments. Since $u = 0$ is uniformly asymptotically $\phi(t)$-stable, then given $\epsilon > 0$, there exist two number $\delta = \delta(\epsilon)$ and $T = T(\epsilon)$, independent of t_0 such that $(\phi(t_0), ||u_0||_{K_n}) < \delta \Rightarrow (\phi(t), ||r(t))||_{K_n}) < \varepsilon$ for $t > T(\epsilon)$. And so

$$\begin{aligned} (\phi(t), v(t, u(y))) &= \exp(-p(t))C\left[(\phi(t), ||r(t)||_{K_n})\right] u(t, 0, \sigma_\omega(u)0, t, u))) \\ &\leq \epsilon C\left[(\phi(t), ||r(t)||_{K_n})\right] \\ &= b\left[(\phi(t), ||r(t)||_{K_n})\right], \quad b \in \mathrm{K} \end{aligned}$$

and

$$\begin{aligned} (\phi(t), v(t, u(y))) &= \exp(-p(t))C[(\phi(t), ||r(t)||_{K_N})]u(t, 0, \sigma_\omega(u)0, t, u))) \\ &\leq qC[(\phi(t), ||r(t)||_{K_n})]\beta^{-1}[(\phi(t), ||r(t)||_{K_n})] \text{ by condition (b)} \\ &= a[(\phi(t), ||r(t)||_{K_n})], \quad a \in \mathrm{K}, \end{aligned}$$

since C, $\beta^{-1} \in \mathrm{K}$, where $q = \inf\left\{\exp(-p(t))\right\}$.

Hence $a[(\phi(t), ||r(t)||_{K_n})] \leq (\phi(t), v(t, u(t))) \leq b[(\phi(t), ||r(t)||_{K_n})]$.

$$\begin{aligned} v(t + h, u + hg(t, u)) - v(t, u) &\leq_K \beta(t)||u + hg(t, u) - u(t + h, t, u)||e(t, z, h) \\ &\quad + v(t + h, u(t + h, t, u)) - v(t, u), \end{aligned}$$

where $\limsup(1/h)e(t,z,h) = 0$.

Dividing both sides by $h > 0$ and taking $\limsup$ as $h \to 0^+$, using (2.1) and uniqueness of solution of (1.2) we obtain

$$\begin{aligned} D^+(\phi(t), v(t,u)) \quad &\leq \quad \limsup_{h\to\infty} \tfrac{1}{h}[\exp(-p(t+h))C[(\phi(t+h), r(t+h))] \\ &\qquad \times (\phi(t), u(t+h, 0, \sigma_\omega(u(0, t+h, u)))) \\ &\qquad - \exp(-p(t)C[(\phi(t), r(t)](\phi(t), h(t, 0, \sigma_\omega(u(0,t,u))))] \\ &= \quad \exp(-p(t)C[(\phi(t), r(t)](\phi(t), h(t, 0, \sigma_\omega(u(0,t,u))))] \\ &\qquad \times \limsup_{h\to 0^+}[\exp(p(t) - p(t+h)) - 1] \\ &= \quad -p'(t)(\phi(t), v(t,u)). \end{aligned}$$

□

Theorem 2.5. Assume that $g \in C[R_+ \times K, R^n]$, $g(t,0) = 0$, and $g(t,u)$ is quasimonotone in u relative to K for each $t \in R_+$ and that for each (t,u), $(t,v) \in R_+ \times K$ and $L \in C[R_+, R_+]$, $||g(t,u) - g(t,v)||_{K_n} \leq_K L(t)||u - v||_{K_n}$. Then we have the trivial solution $u = 0$ of (1.2) is generalized exponentially asymptotically $\phi(t)$-stable if and only if there exists a cone-valued Lyapunov function v with the following properties:

(a) $v \in C[R_+ \times S^*(\rho), K]$, $v(t,0) = 0$, and $v(t,u)$ is locally Lipschitzian in u relative to K for each $t \in R_+$ and for a continuous function $\beta(t) \geq 0$.

(b) $(\phi(t), r(t)) \leq (\phi(t), v(t,u)) \leq \sigma(t,t_0)(\phi(t), r(t)))$, for some $\phi(t) \in K_0^*$, $\sigma \in C\ [R_+ \times R_+, R_+]$, $(t,u) \in R_+ \times S^*(\rho)$.

(c) $D^+v(\phi(t), u(t,u)) \leq_K -p'(t)(\phi(t), v(t,u)$, for $(t,u) \in R_+ \times S^{AST}(\rho)$, $p'(t)$ exists, $p(t)$ is bounded and increasing.

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