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Estimates for parameter changes in a uniform model with a generalized uniform outlier

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Abstract

We shall propose several estimators for the scale parameter in a uniform distribution with a generalized uniform outlier when the scale parameter is a function of a known exposure level, and obtain expectations and variances for their proposed estimators. And we shall compare numerically efficiencies for proposed estimators of changed parameters of the scale in the small sample sizes.

Keywords: Closer estimator, efficiency, generalized uniform distribution, outlier, parameter change.

1. Introduction

The uniform distribution is used to represent the distribution of roundoff errors and arise as a result of the probability integral transformation. And the uniform distribution often provide useful tools for constructing goodness-of fit statistics, simulation of complex statistical procedures, and testing the quality of pseudorandom number generators. Here, we shall consider parametric estimations in a uniform distribution with a generalized uniform outlier when its scale parameter is a function of a known exposure level t, which often occurs in the engineering and physical phenomena.

Woo and Ali (1994) studied the jackknife parametric estimations in the exponential distribution when its scale and location parameters change functions of environment dosage. Woo and Lee (2000) studied an application of the Weibull distribution to the strength of materials when its shape and scale parameters are functions of a known exposure level. Kim and Lee (2002) considered unified estimations for parameter changes in a generalized uniform distribution. Lee *et al.* (2003) studied unified estimations for parameter changes in the uniform distribution. Lee *et al.* (2006) considered jackknife estimation in a truncated exponential distribution with a uniform outlier. Lee *et al.* (2007) studied unified estimates for parameter changes in a Pareto model with an exponential outlier. Lee *et al.* (2008)

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considered estimations for the uniform scale parameter in the presence of a half-triangle outlier.

In this paper, we shall propose several estimators for the scale parameter in a uniform distribution with a generalized uniform outlier when the scale parameter changes a function of an environment dosage, say t. and obtain expectations and variances for their proposed estimators. Throughout the numerical values of relative efficiencies of proposed estimators for the scale parameter in the small sample sizes when the scale parameter changes a function of an environment dosage, we shall compare its efficiencies.

2. Estimations for parameter changes

We shall consider the uniform distribution with the p.d.f.

$$f(x;\theta(t)) = \begin{cases} \dfrac{1}{\theta(t)}, & 0 < x < \theta(t), \\ \\ 0, & \text{elsewhere}, \end{cases}$$

where $\theta(t)$ are positive for $t > 0$, denoted by $X \sim UNIF(0, \theta(t))$. Gibbons (1974) investigated parametric estimators of the scale parameter in a population uniformly distributed over (0, θ). Fan (1991) considered a property of the interior and exterior trimmed means in the uniform distribution over (0, θ).

Here, we shall consider unified estimation for the parameter change of exposure levels in the uniform distribution with a generalized uniform outlier when the scale parameter $\theta(t)$ is a polynomial of t;

$$\theta(t) = b_0 + b_1 t + \cdots + b_r t^r, t > 0 \text{ and } b_i > 0, \text{ for all } i = 0, 1, \cdots, r.$$

Assume $X_{1j}, \cdots, X_{n_j j}$ be independent random variable such that all but one of them are from $X \sim UNIF(0, \theta(t_j))$ and one remaining random variable is from $GUNIF(\alpha, \theta(t_j))$, $j = 1, 2, \cdots r+1$, where $GUNIF$ denotes a generalized uniform distribution with the density function (Proctor, 1987);

$$f(x : \alpha, \theta(t)) = \frac{\alpha + 1}{\theta(t)^{\alpha+1}} x^{\alpha}, \quad 0 < x < \theta(t),$$

where α ($\neq 0$) is a known real number greater than -1. And $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_{r+1}$ be independent, and $t_i \neq t_k$ for $i \neq k$.

Let $Y_{ij} = X_{ij}/\theta(t_j)$, $i = 1, \cdots, n_j$. Then $Y_{1j}, \cdots, Y_{n_j j}$ are independent random variables such that all but one of them are from $UNIF(0, 1)$ and one remaining random variable is from $GUNIF(\alpha, 1)$. So let $Y_{(1)j}, \cdots, Y_{(n_j)j}$ be the corresponding order statistics for $Y_{1j}, \cdots, Y_{n_j j}$.

Then the density function of the largest order statistics $Y_{(n_j)j}$ is

$$f_{n_j}(y) = (n_j + \alpha) y^{n_j + \alpha - 1}, \quad 0 < y < 1, \tag{2.1}$$

which is $GUNIF(n_j+\alpha-1,1)$. Define the following notation:

$$\det[t_i^0,\cdots,t_i^r]=\begin{vmatrix}1 & t_1 & t_1^2 & \cdots & t_1^r\\ 1 & t_2 & t_2^2 & \cdots & t_2^r\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 1 & t_{r+1} & t_{r+1}^2 & \cdots & t_{r+1}^r\end{vmatrix}.$$

By the maximum likelihood (ML) method, we can obtain the ML estimators for b_j, $j=0,1,\cdots,r$, as follows;

$$\widehat{b}_j^{(1)}=\frac{\det[t_i^0,\cdots,t_i^{j-1},X_{(n_i)i},t_i^j+1,\cdots,t_i^r]}{\det[t_i^0,\cdots,t_i^r]}.$$

Note that

$$\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdot & \cdot & \cdots & \cdot\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}=a_{k1}A_{k1}+a_{k2}A_{k2}+\cdots+a_{kn}A_{kn}, \tag{2.2}$$

where $A_{kj}=(-1)^{k+j}D_{kj}$ and D_{kj} is a minor determinant for a_{kj} eliminated k -row and j -column.

From (2.1) and (2.2), expectations and variances of these MLE's $\widehat{b}_j^{(1)}$ for b_j can be obtained by

$$E(\widehat{b}_j^{(1)})=b_j-\sum_{k=1}^{r+1}\frac{\det[t_i^0,\cdots,t_i^{j-1},t_i^{j+1},\cdots,t_i^r]_{i\neq k}}{\det[t_i^0,\cdots,t_i^r]}\cdot\frac{\theta(t_k)}{n_k+\alpha+1} \tag{2.3}$$

and

$$VAR(\widehat{b}_j^{(1)})=\sum_{k=1}^{r+1}\frac{\det^2[t_i^0,\cdots,t_i^{j-1},t_i^{j+1},\cdots,t_i^r]_{i\neq k}}{\det^2[t_i^0,\cdots,t_i^r]}\cdot\frac{(n_k+\alpha)\theta^2(t_k)}{(n_k+\alpha+1)^2(n_k+\alpha+2)},$$

where $\det[t_i^0,\cdots,t_i^{j-1},t_i^{j+1},\cdots,t_i^r]_{i\neq k}$ is a minor determinant eliminated k -row and j -column in the determinant, $\det[t_i^0,\cdots,t_i^r]$.

Since MLE's $\widehat{b}_j^{(1)}$ for b_j are biased estimators, we propose unbiased estimators $\widehat{b}_j^{(2)}$ for b_j, $j=0,1,\cdots,r$, in an assumed uniform distribution with a generalized uniform outlier as follows:

$$\widehat{b}_j^{(2)}=\frac{\det[t_i^0,\cdots,t_i^{j-1},\dfrac{n_i+\alpha+1}{n_i+\alpha}X_{(n_i)i},t_i^{j+1},\cdots,t_i^r]}{\det[t_i^0,\cdots,t_i^r]}.$$

From (2.1) and (2.2), variances of these MLE's $\widehat{b}_j^{(2)}$ for b_j, $j=0,1,\cdots,r$, can be obtained by

$$VAR(\widehat{b}_j^{(2)})=\sum_{k=1}^{r+1}\frac{\det^2[t_i^0,\cdots,t_i^{j-1},t_i^{j+1},\cdots,t_i^r]_{i\neq k}}{\det^2[t_i^0,\cdots,t_i^r]}\cdot\frac{\theta^2(t_k)}{(n_k+\alpha)(n_k+\alpha+2)}. \tag{2.4}$$

Since the UMVUE for the scale parameter $\theta(t_j)$ in the assumed uniform distribution when an outlier doesn't present is $((n_{j+1})/n_j)X_{(n_j)j}$, we can propose following estimator $\widehat{b}_j^{(3)}$ for b_j, $j = 0, 1, \cdots, r$:

$$\widehat{b}_j^{(3)} = \frac{\det[t_i^0, \cdots, t_i^{j-1}, \dfrac{n_{i+1}}{n}{}_i X_{(n_i)i}, t_i^{j+1}, \cdots, t_i^r]}{\det[t_i^0, \cdots, t_i^r]}.$$

From (2.1) and (2.2), expectations and variances of these estimators $\widehat{b}_j^{(3)}$ for b_j can be obtained by

$$E(\widehat{b}_j^{(3)}) = b_j + \sum_{k=1}^{r+1} \frac{\det[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i \neq k}}{\det[t_i^0, \cdots, t_i^r]} \cdot \frac{\alpha\theta(t_k)}{n_k(n_k + \alpha + 1)} \tag{2.5}$$

and

$$VAR(\widehat{b}_j^{(3)}) = \sum_{k=1}^{r+1} \frac{\det^2[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i \neq k}}{\det^2[t_i^0, \cdots, t_i^r]} \cdot \frac{(n_k + 1)^2(n_k + \alpha)\theta^2(t_k)}{n_k^2(n_k + \alpha + 1)^2(n_k + \alpha + 2)}.$$

Also, the minimum risk estimator for the scale parameter $\theta(t_j)$ in the assumed uniform distribution when an outlier doesn't present is $(n_j + 2)/(n_j + 1)X_{(n_j)j}$, we can propose following estimators $\widehat{b}_j^{(4)}$ for b_j, $j = 0, 1, \cdots, r$:

$$\widehat{b}_j^{(4)} = \frac{\det[t_i^0, \cdots, t_i^{j-1}, \dfrac{n_i + 2}{n_{i+1}} X_{(n_i)i}, t_i^{j+1}, \cdots, t_i^r]}{\det[t_i^0, \cdots, t_i^r]}.$$

From (2.1) and (2.2), expectations and variances of these estimators $\widehat{b}_j^{(4)}$ for b_j can be obtained by

$$E(\widehat{b}_j^{(4)}) = b_j + \sum_{k=1}^{r+1} \frac{\det[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i \neq k}}{\det[t_i^0, \cdots, t_i^r]} \cdot \frac{(\alpha - 1)\theta(t_k)}{(n_k + 1)(n_k + \alpha + 1)} \tag{2.6}$$

and

$$VAR(\widehat{b}_j^{(4)}) = \sum_{k=1}^{r+1} \frac{\det^2[t_i^0, \cdots, t_i^{j-1}, t_i^{j-1}, \cdots, t_i^r]_{i \neq k}}{\det^2[t_i^0, \cdots, t_i^r]} \cdot \frac{(n_k + 2)^2(n_k + \alpha)\theta^2(t_k)}{(n_k + 1)^2(n_k + \alpha + 1)^2(n_k + \alpha + 2)}.$$

As Johnson (1950) proposed closer estimator for the scale parameter $\theta(t_j)$ than MRE in the uniform distribution over (0, $\theta(t_j)$), estimator's $\widehat{b}_j^{(5)}$ for b_j are defined as follows;

$$\widehat{b}_j^{(5)} = \frac{\det[t_i^0, \cdots, t_i^{j-1}, 2^{1/n_i} \cdot X_{(n_i)i}, t_i^{j+1}, \cdots, t_i^r]}{\det[t_i^0, \cdots, t_i^r]}.$$

From (2.1) and (2.2), expectations and variances of these estimators $\widehat{b}_j^{(5)}$ for b_j can be obtained by

$$E(\widehat{b}_j^{(5)}) = \sum_{k=1}^{r+1} \frac{\det[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i\neq k}}{\det[t_i^0, \cdots, t_i^r]} \cdot \frac{(n_k+\alpha)2^{1/n_k}\theta(t_k)}{n_k+\alpha+1} \tag{2.7}$$

and

$$VAR(\widehat{b}_j^{(5)}) = \sum_{k=1}^{r+1} \frac{\det^2[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r]_{i\neq k}}{\det^2[t_i^0, \cdots, t_i^r]} \cdot \frac{(n_k+\alpha)2^{2/n_k}\theta^2(t_k)}{(n_k+\alpha+1)^2(n_k+\alpha+2)}.$$

From the results (2.3) through (2.7), the proposed estimators $\widehat{b}_j^{(i)}$, $i = 1, \cdots, 5$, are MSE-consistent for b_j, respectively.

3. Concluding remarks

In this paper, we proposed several estimators for the scale parameter in the uniform distribution with a generalized uniform outlier when its scale parameter is a function of a known exposure level t and derived expectations and variances for their estimators. From the results (2.3) through (2.7), Table 3.1 shows the numerical values of mean squared error (MSE) of $\widehat{b}_j^{(i)}$, $i = 1, \cdots, 5$, in the assumed uniform distribution with a generalized uniform outlier for sample sizes n_1 =10(5)30, n_2 =10(5)30, $b_0 = 0$, $b_1 = 1$, and $t_1 = 1$, $t_2 = 2$ when $r = 1$ and $\alpha = 1/2$.

From Table 3.1, the MLE $\widehat{b}_0^{(1)}$ is more efficient than other proposed estimators for b_0 when values for sample sizes n_1 and n_2 are similar but the closer estimator $\widehat{b}_0^{(5)}$ is more efficient than other proposed estimators for b_0 as differences between n_1 and n_2 are increasing. And the closer estimator $\widehat{b}_1^{(5)}$ is more efficient than other proposed estimators for b_1 when values for sample sizes n_1 and n_2 are similar. But, for fixed n_1, the MLE $\widehat{b}_1^{(1)}$ is more efficient than other proposed estimators for b_1 as n_2 are increasing.

Table 3.1 MSE's for proposed estimators for changing parameter in the uniform distribution with a generalized uniform outlier

size		parameter	MSE				
n_1	n_2		$\widehat{b}_j^{(1)}$	$\widehat{b}_j^{(2)}$	$\widehat{\mathbf{b}}_{\mathbf{j}}^{(\mathbf{3})}$	$\widehat{b}_j^{(4)}$	$\widehat{b}_j^{(5)}$
10	10	b_0	0.05081	0.06095	0.06148	0.06047	0.05837
		b_1	0.03932	0.03809	0.03844	0.03781	0.03693
	15	b_0	0.04119	0.04522	0.04556	0.04494	0.04356
		b_1	0.02053	0.02236	0.02249	0.02224	0.02168
	20	b_0	0.03983	0.03914	0.03947	0.03892	0.03792
		b_1	0.01427	0.01629	0.01638	0.01621	0.01576
	25	b_0	0.04037	0.03618	0.03650	0.03597	0.03522
		b_1	0.01176	0.01332	0.01340	0.01326	0.01287
	30	b_0	0.04138	0.03451	0.03483	0.03431	0.03375
		b_1	0.01068	0.01165	0.01173	0.01159	0.01126

Table 3.2 MSE's for proposed estimators for changing parameter in the uniform distribution with a generalized uniform outlier

size		parameter	MSE				
n_1	n_2		$\widehat{b}_j^{(1)}$	$\widehat{b}_j^{(2)}$	$\widehat{\mathbf{b}}_{\mathbf{j}}^{(\mathbf{3})}$	$\widehat{b}_j^{(4)}$	$\widehat{b}_j^{(5)}$
15	10	b_0	0.04119	0.04522	0.04556	0.04494	0.04356
		b_1	0.04149	0.03416	0.03448	0.03394	0.03346
	15	b_0	0.02602	0.02949	0.02961	0.02938	0.02854
		b_1	0.01993	0.01843	0.01851	0.01836	0.01810
	20	b_0	0.02169	0.02341	0.02350	0.02334	0.02276
		b_1	0.01218	0.01235	0.01239	0.01232	0.01211
	25	b_0	0.02038	0.02045	0.02052	0.02039	0.01997
		b_1	0.00875	0.00939	0.00941	0.00936	0.00917
	30	b_0	0.02012	0.01878	0.01885	0.01872	0.01843
		b_1	0.00704	0.00772	0.00774	0.00770	0.00753
20	10	b_0	0.03983	0.03914	0.03947	0.03892	0.03792
		b_1	0.04360	0.03264	0.03297	0.03244	0.03219
	15	b_0	0.02169	0.02341	0.02350	0.02334	0.02276
		b_1	0.02056	0.01691	0.01698	0.01686	0.01676
	20	b_0	0.01576	0.01734	0.01738	0.01730	0.01690
		b_1	0.01201	0.01084	0.01086	0.01081	0.01072
	25	b_0	0.01347	0.01437	0.01440	0.01434	0.01405
		b_1	0.00809	0.00787	0.00788	0.00785	0.00776
	30	b_0	0.01253	0.01270	0.01273	0.01268	0.01246
		b_1	0.00604	0.00620	0.00621	0.00619	0.00610
25	10	b_0	0.04037	0.03618	0.03650	0.03597	0.03522
		b_1	0.04527	0.03190	0.03223	0.03171	0.03161
	15	b_0	0.02038	0.02045	0.02052	0.02039	0.01997
		b_1	0.02130	0.01617	0.01624	0.01612	0.01613
	20	b_0	0.01347	0.01437	0.01440	0.01434	0.01405
		b_1	0.01226	0.01009	0.01012	0.01008	0.01007
	25	b_0	0.01056	0.01140	0.01142	0.01139	0.01116
		b_1	0.00802	0.00713	0.00714	0.00712	0.00709
	30	b_0	0.00920	0.00973	0.00975	0.00972	0.00955
		b_1	0.00576	0.00546	0.00546	0.00545	0.00541

Table 3.3 MSE's for proposed estimators for changing parameter in the uniform distribution with a generalized uniform outlier

size		parameter	MSE				
n_1	n_2		$\widehat{b}_j^{(1)}$	$\widehat{b}_j^{(2)}$	$\widehat{\mathbf{b}}_{\mathbf{j}}^{(\mathbf{3})}$	$\widehat{b}_j^{(4)}$	$\widehat{b}_j^{(5)}$
30	10	b_0	0.04138	0.03451	0.03483	0.03431	0.03375
		b_1	0.04656	0.03148	0.03181	0.03129	0.03131
	15	b_0	0.02012	0.01878	0.01885	0.01872	0.01843
		b_1	0.02196	0.01575	0.01582	0.01570	0.01580
	20	b_0	0.01253	0.01270	0.01273	0.01268	0.01246
		b_1	0.01258	0.00968	0.00970	0.00966	0.00966
	25	b_0	0.00920	0.00973	0.00975	0.00972	0.00955
		b_1	0.00813	0.00671	0.00672	0.00670	0.00670
	30	b_0	0.00756	0.00807	0.00807	0.00806	0.00792
		b_1	0.00573	0.00504	0.00504	0.00503	0.00503

References

Fan, D. Y. (1991). On a property of the order statistics of the uniform distribution. *Communications in*

Statistics, Theory and Method, **20**, 1903-1909.

Gibbons, J. D. (1974). Estimation of the unknown upper limit of a uniform distribution. *Sankya, Series B*, **36**, 29-40.

Johnson, N. L. (1950). On the comparison of estimators. *Biometrika*, **37**, 281-287.

Kim, J. D. and Lee, J. C. (2002). Unified estimations for parameter changes in a generalized uniform Distribution. *Journal of Korean Data & Information Science Scociety*, **13**, 295-305.

Lee, C. S., Chang, C. S. and Park, Y. W. (2003). Unified estimations for parameter changes in the uniform distribution. *The Korean Communications in Statistics*, **10**, 145-151.

Lee, C. S., Chang, C. S. and Park, Y. W. (2006). Jackknife estimation in a truncated exponential distribution with an uniform outlier. *Journal of the Korean Data & Information Sciences Society*, **17**, 1021-1028.

Lee, C. S., Kim, K. H. and Park, Y. W. (2008). Estimations for an uniform scale parameter in the presence of a half-triangle outlier. *Journal of the Korean Data & Information Sciences Society*, **19**, 959-965.

Lee, C. S., Ryu, S. G. and Chang, C. S. (2007). Unified estimates for parameter changes in a Pareto model with an exponential outlier. *Journal of the Korean Data & Information Science Society*, **18**, 507-514.

Proctor, J. W. (1987). Estimation of two generalized curves covering the pearson system. *Proceedings of ASA Computing Section*, 287-292.

Woo, J. S. and Ali, M. M. (1994). Unified jackknife estimation for parameter changes in the exponential distribution. *Journal of Statistical Studies*, **14**, 20-23

Woo, J. S. and Lee, C. S. (2000). Jackknife estimates for parameter changes in the weibull distribution. *The Korean Communications in Statistics*, **7**, 199-209.