Journal of the Korean Data & Information Science Society
2010, **21**(3), 555–560

한국데이터정보과학회지

Various types of modelling for scale parameter in Weibull intensity function for two-dimensional warranty data

Jai Wook Baik[1] · Jin Nam Jo[2]

[1]Department of Information Statistics, Korea National Open University
[2]Department of Statistics and Information Science, Dongduk Women's University

Received 11 March 2010, revised 3 May 2010, accepted 10 May 2010

Abstract

One-dimensional approach to two-dimensional warranty data involves modeling usage as a function of time. Iskandar (1993) suggests a simple linear model for usage. However, simple linear form of intensity function is of limited value to model the situation where the intensity varies over time. In this study Weibull intensity is considered where the scale parameter is expressed in terms of different models. We will find out how each parameter in the model affects the warranty cost and which model gives a bigger number of failures within the two-dimensional warranty region.

Keywords: Scale parameter, two-dimensional warranty data, warranty cost, Weibull intensity function.

1. Introduction

A warranty is a manufacturer's assurance to a buyer that a product or service is or shall be as represented. It may be considered to be a contractual agreement between the buyer and manufacturer for the sale of the product or service. In broad terms, the purpose of a warranty is to establish liability between the two parties in the event that an item fails.

Warranty data provides useful information to assess product reliability in the field. This can then be used to assess future warranty costs and decisions to improve the reliability for the next generation to reduce warranty costs and customer dissatisfaction. For products sold with two-dimensional (age and usage) warranties, warranty data poses new challenges to assess product reliability.

Models play an important role in these evaluations and decision-making. Many different types of models (failure model, warranty cost model, servicing model, logistics model etc) are used and these can be found in Blischke and Murthy (1994, 1996). A more recent review

[1] Corresponding author: Professor, Department of Information Statistics, Korea National Open University, JongRoGoo DongSungDong 169, Seoul 110-791, Korea. E-mail: jbaik@knou.ac.kr
[2] Professor, Department of Statistics and Information Science, Dongduk Women's University, 23-1 Wolgok-Dong, Sungbuk-Ku, Seoul 136-714, Korea.

of models that have appeared since 1990 can be found in Murthy and Djamaludin (2002, 2008).

The case of one-dimensional warranties has been a lot of attention. See Kalbfleisch *et al.* (1991), Lawless and Kalbfleisch (1992), Lawless (1998) and Kalbfleisch and Lawless (1998). In contrast, the analysis and modeling of two-dimensional warranty data has received very little attention. See Iskandar (1993), Murthy *et al.* (1995), Singpurwalla and Wilson (1998), Kim and Rao (2000) and Yang and Nachlas (2001) for two-dimensional warranty data analysis. Optimization of cost following the expiration of warranty has been established by Jung (2008, 2006).

Item failures are points on a plane with one axis representing age (t) and the other representing usage (x). Two different approaches have been employed to modeling item failures. The first is to model item failures by a two-dimensional point process formulation. See Iskandar (1993), Murthy *et al.* (1995) and Blischke and Murthy (1996, Chapter7). The second approach involves modeling usage as a function of time. Iskandar (1993) suggests a linear model for usage given by $x(t) = rt$ where r is the usage rate and is modeled as a random variable to model the varying usage rate across the consumer population. For numerical computation, linear form of intensity function given usage rate is assumed to see the effect of the parameters.

However, simple linear form of intensity function is of limited value to model the situation where the intensity varies over time. In this paper, given a distribution for the usage rate in 2-dimensional warranty data and a Weibull intensity function warranty cost expressed in terms of the number of failures under two-dimensional warranty is sought. Weibull intensity function is assumed where the scale parameter of the intensity function is a function of the usage rate. In Section 2 model formulation and analysis are undertaken. In Section 3 numerical results and their interpretation are given. In the final section conclusion is drawn.

2. Model formulation and analysis

One-dimensional approach involves modeling usage as a function of time so that failures are effectively modeled by a one-dimensional point process formulation. Iskandar (1993) suggests a linear model for usage given by $x(t) = rt$ where r is a random variable for usage rate with the distribution function G(r) and is modeled to represent the varying usage across the consumer population. According to chapter 8 of Blischke and Murthy (1994) if we let $N(t|r)$ be the number of failures over the interval $[0, t)$ conditional on $R = r$ then

$$P\left\{N(t) = n\right\} = \int_0^\infty P\left\{N(t|r) = n\right\} dG(r)$$

where $P\left\{N(t|r) = n\right\} = \left\{\int_0^t \lambda_W(s|r)ds\right\}^n \exp\left\{-\int_0^t \lambda_W(s|r)ds\right\}/n!$ and $\lambda_W(t|r)$ is the intensity function of age t given usage rate r.

If we let $N(W, U|r)$ be the number of failures under warranty conditional on $R = r$ then

$$E(N(W,U)) = \int_0^{r_0} \Lambda_W(W|r)dG(r) + \int_{r_0}^\infty \Lambda_W(\frac{U}{r}|r)dG(r) \quad where \quad \Lambda_W(t|r) = \int_0^t \lambda_W(s|r)ds \quad and \quad r_0 = U/W.$$

A special case of linear form $\lambda(t|r) = \theta_0 + \theta_1 r + \theta_2 X_c(t) + \theta_3 Y_c(t)$ is assumed in Blischke and Murthy (1994) where $X_c(t) = t$ and $Y_c(t) = rt$. So the intensity function is given by $\lambda(t|r) = \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)t$ and $\Lambda_W(W|r) = (\theta_0 + \theta_1 r)W + (\theta_2 + \theta_3 r)W^2/2$. Numerical computations are given for some values of θ_i 's, U and W in Blischke and Murthy (1994).

However, simple linear form of intensity function is of limited value to model the situation where the intensity varies over time. Hence, we assume Weibull intensity function $\lambda_W(t|r) = (\beta/\alpha(r))\,(t/\alpha(r))^{\beta-1}$ where the scale parameter $\alpha(r)$ can be interpreted as a stress on warranty cost. Therefore the requirement for $\alpha(r)$ is that it decrease as the usage rate r increases. Note that $\Lambda_W(t|r) = (t/\alpha(r))^{\beta}$. Different types of model for $\alpha(r)$ can be considered. These three different types of models are considered since they are frequently used in reliability analysis: first model being still linear except it is inverse linear, second and third being popular non-linear relationship between r and $\alpha(r)$. The third model can model the feasible usage rate. In practice, given dataset with r we can determine which model fit best among the following three models.

Model 1: $\alpha(r) = \alpha_0 + \alpha_1/r$

Model 1 assumes a simple inverse linear relationship between r and $\alpha(r)$.

Model 2: $\alpha(r) = \alpha_0 e^{\alpha_1/r}$

Model 2 assumes an inverse exponential relationship (Arrhenius relationship) between r and $\alpha(r)$.

Model 3: $\alpha(r) = \alpha_0(r_0/r)^{\alpha_1}$ with $\alpha_1 > 1$, where the usage rate $r = a + by$ and y has a Beta distribution with range $[0, 1]$. That is, $f(y) = y^{\nu-1}(1-y)^{w-1}/B(\nu, w)$, $0 \le y \le 1$ where $B(\nu, w) = \int_0^1 u^{\nu-1}(1-u)^{w-1}du$.

Model 3 assumes an inverse power law relationship between r and $\alpha(r)$. Note that $\alpha(r)$ can be an accelerating (when $r > r_0$) or decelerating (when $r < r_0$) factor on the warranty cost.

3. Numerical results

For each of the models and a distribution for the usage rate r, $E(N(W, U))$ is of interest since it determines the warranty cost. For the numerical examples in this section the unit for usage x is 10^4 miles and for age t is years. It is assumed that $W = 5$ and $U = 10$. Therefore $r_0 = U/W = 2$. Hence, $W = 5$ and $U = 10$ corresponds to a time limit of 5 years and a usage limit of 100,000 miles whichever comes first. Note that if $r = r_0$ with probability 1 then the warranty expires at W. And if we set $\alpha(r_0) = 2.5$ then $E(N(W, U)) = \Lambda_W(W|r = r_0) = (W/\alpha(r_0))^{\beta} = (5/2.5)^2 = 4$.

We assume $\alpha(r)$ to be 2.5 on the average in models 1, 2 and 3 and see the effect of α_0 and α_1 on the warranty cost.

Iskandar and Murthy (1994) suggests Uniform and Gamma distributions as suitable models for $G(r)$. In this paper, Uniform distribution is assumed for the distribution of usage rate r for models 1 and 2; $r : U(0, u)$ where u is taken as $2 \times$ mean and the mean refers to the mean usage rate of 2 (20,000 miles per year). In this case $r : U(0, 4)$.

For Model 3 the usage rate $r = a + by$ and y has a Beta distribution with range $[0, 1]$. We can take $\alpha = 1$ and $\alpha = 4$ so that if $\nu = 1$, $w = 3$ then $E(r) = 2(= a + b\nu/(\nu + w) = 1 + 4(1/1 + 3))$ and r can take values from 1 $(= a)$ to 5 $(= a + b)$. So we can assume that the usage rate is 2 for Model 3 too. Note that with model 3 we can model very heavy usages group (as big as $r = 5$) as well as low usage group (as low as $r = 2$).

The density function for r is

$$f(r) = \frac{1}{4}\frac{\left\{\frac{1}{4}(r-1)\right\}^{\nu-1}\left\{1-\frac{1}{4}(r-1)\right\}^{w-1}}{B(\nu,w)}, 1 \le r \le 5.$$

In general if $r = a + by$ with $r \sim$ B(ν, w) then

$$f(r) = \frac{1}{b}\frac{\left\{\frac{1}{b}(r-1)\right\}^{\nu-1}\left\{1-\frac{1}{b}(r-1)\right\}^{w-1}}{B(\nu,w)}, a \le r \le a+b.$$

Then $E(N(W,U))$ for each of the models can be expressed as follows:

$$E(N(W,U)) = \frac{1}{u}\left\{\int_0^{r_0}(\frac{W}{\alpha_0+\alpha_1/r})^\beta dr + \int_{r_0}^{u}(\frac{U/r}{\alpha_0+\alpha_1/r})^\beta dr\right\} \text{ for Model 1}$$

$$E(N(W,U)) = \frac{1}{u}\left\{\int_0^{r_0}(\frac{W}{\alpha_0+e^{\alpha_1/r}})^\beta dr + \int_{r_0}^{u}(\frac{U/r}{\alpha_0+e^{\alpha_1/r}})^\beta dr\right\} \text{ for Model 2}$$

$$E(N(W,U)) = (\frac{W}{\alpha_0+r_0^{\alpha_1}})^\beta \int_\alpha^{r_0}(r^{\alpha_0})^\beta f(r)dr + (\frac{U}{\alpha_0+r_0^{\alpha_1}})^\beta \int_\alpha^{a+b}(r^{\alpha_1-1})^\beta f(r)dr$$

for Model 3.

In Model 1 $\alpha(r) = \alpha_0 + \alpha_1/r$ determines the scale parameter at some nominal usage rate r. This is held at a value of 2.5 for the numerical example. Different combinations of α_0 and α_1 correspond to different design options.

Computational Results for Model 1 with $\alpha(r) = 2.5$ and $\beta = 2$ are given in Table 3.1. Note that $\alpha(r) = 2.5$ gives specific combinations of α_0 and α_1 for an average usage rate of $r = 2$. We can tell from the equation above and Table 3.1 that with small values of α_0 or α_1 we observe a big number of failures. It is certain from the equation that warranty cost increases as β increases for a given value of $\alpha(r)$.

Table 3.1 $E(N(W,U))$ for models 1, 2 and 3 with $\alpha(r) = 2.5$, an average usage rate of $r = 2$, and $\beta = 2$

Models 1 and 2	Model 1		Model 2		Model 3 $a=1$, $b=4$, $\nu=1$, $w=3$		
α_0	α_1	$E(N(W,U))$	α_1	$E(N(W,U))$	α_0	α_1	$E(N(W,U))$
0.5	4	2.41	3.22	2.89	2.5	0.5	11.53
1	3	2.27	1.83	2.22	2.5	1.0	11.67
1.5	2	2.24	1.02	2.10	2.5	1.5	13.17
2	1	2.35	0.45	2.24	2.5	2.0	16.49

In Model 2 $\alpha(r) = \alpha_0 e^{\alpha_1/r}$ determines the scale parameter at some nominal usage rate r. This is held at a value of 2.5 for the numerical example. Different combinations of α_0 and α_1 correspond to different design options.

Computational Results for Model 2 with $\alpha(r) = 2.5$ and some values of β are given in Table 1 too. Note that $\alpha(r) = 2.5$ gives specific combinations of α_0 and α_1 for an average usage rate of $r = 2$. As seen in Model 1, small values of α_0 and α_1 yield a big number of failures. Warranty cost increases as β increases for a given value of $\alpha(r)$.

In Model 3 $\alpha(r) = \alpha_0(r_0/r)^{\alpha_1}$ determines the scale parameter at some nominal usage rate r. α_0 is taken as 2.5 for the numerical example. Different combinations of a, b, ν, w and α_1 correspond to different design options. Note that for a big value of α_1 (and β) the warranty cost gets big if $r > r_0$. Computational results for Model 3 with $a = 1$, $b = 4$, $\nu = 1$, $w = 3$ and with some values of α_1 are given in Table 3.1. It is shown in Table 3.1 that the number of failures increase as α_1 increases.

In this section the number of failures under two-dimensional warranty region is determined by the parameter values in the model. It is shown from the design point of view that if the number of failures is not large then we can use the inverse linear model (Model 1) or inverse exponential relationship (Model 2) with current levels of parameter values to model the effect of usage rate on the warranty cost. However, if the number of failures is large then inverse power law relationship is to be preferred.

4. Conclusions

Two-dimensional warranty data is not uncommon in today's industry. Previously simple linear form of intensity function has only been considered. However, in this study Weibull intensity function has been considered where the scale parameter is expressed in terms of 3 different models: inverse linear, inverse exponential and inverse power law relationships. In Section 3 the number of failures under two-dimensional warranty region is shown to be determined by the parameters in the model. Given the parameter values as in Section 3 inverse linear and inverse exponential models have a smaller number of failures than the inverse power law model.

Following warrants further study in relation to the current study.

- In this study different types of models for the scale parameter have been considered in order to incorporate the effect of usage rate. However, we may be given two-dimensional warranty data and want to know which model best fits the data.
- In this study we have looked at 3 different types of models such as inverse linear, inverse exponential and inverse power law relationships. But we may want to assume some other type of models such as Eyring relationship.
- In this study Weibull intensity function has been considered. However, some other forms of the intensity function may be of interest.

References

Blischke, W. R. and Murthy, D. N. P. (1994). *Warranty cost analysis*, Marcel Dekker Inc., NewYork.

Blischke, W. R. and Murthy, D. N. P. [Eds.] (1996). *Product warranty handbook*, Marcel Dekker, Inc., NewYork.

Iskandar, B. P. (1993). Modelling and analysis of two dimensional warranties policies, doctoral Dissertation, The University of Queensland, Australia.

Jung, K. M. (2006). Optimal preventive maintenance policy for a repairable system. *Journal of the Korean Data & Information Science Society*, **17**, 367-377.

Jung, K. M. (2008). Optimization of cost and downtime for periodic PM model following the expiration of warranty. *Journal of the Korean Data & Information Science Society*, **19**, 587-596.

Kalbfleisch, J. D. and Lawless, J. F. (1998). Estimation of reliability in field-performance Studies (with Discussion). *Technometrics*, **30**, 365-388.

Kalbfleisch, J. D., Lawless, J. F. and Robinson, J. A. (1991). Methods for the analysis and prediction of warranty claims. *Technometrics*, **33**, 273-28

Kim, H. G. and Rao, B. M. (2000). Expected warranty cost of a two-attribute free-replacement warranties based on a bivariate exponential distribution. *Computers and Industrial Engineering*, **38**, 425-434.

Lawless, J. F. (1998). Statistical analysis of product warranty data. *International Statistical Review*, **66**, 41-60.

Lawless, J. F. and Kalbfleisch, J.D. (1992). Some issues in the collection and analysis of field reliability data. *Survival Analysis: State of the Art (J. P. Klein and P. K. Goel eds.)*, Kluwer, Amsterdam, 141-152.

Murthy, D. N. P. and Djamaludin, I. (2002). New product warranty A literature review. *International Journal of Production Economics*, **79**, 231-260.

Murthy, D. N. P., Iskandar, B. P. and Wilson, R. J. (1995). Two dimensional failure free warranties: Two-dimensional point process models. *Operations Research*, **43**, 356-366.

Murthy, D. N. P., Rausand, M. and Osteras, T. (2008). *Product reliability (Specification and Performance)*, Springer Verlag.

Singpurwalla, N. D. and Wilson, S. P. (1998). Failure models indexed by two scales. *Advances in Applied Probability*, **30**, 1058-1072.

Yang, S. C. and Nachlas, J. A. (2001). Bivariate reliability and availability modelling. *IEEE Transactions on Reliability*, **50**, 26-35.