

# Cusum of squares test for discretely observed sample from multidimensional diffusion processes

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## Abstract

In this paper, we extend the work by Lee *et al.* (2010) to multidimensional diffusion processes. A test statistic analogous to the one-dimensional case is proposed to investigate the joint stability of covariance matrix parameters and, under certain regularity conditions, is shown to have a limiting distribution of the sup of a multidimensional Brownian bridge. A simulation result is provided for illustration.

*Keywords:* Discretely observed sample, multidimensional diffusion process, residual based cusum test.

## 1. Introduction

Lee *et al.* (2010) proposed the cusum of squares test based upon discretely observed sample to detect the change points of the volatility parameter in one-dimensional diffusion processes. While being free from any drift changes, their test considered the case which has not been dealt with by Gregorio and Iacus (2008), i.e., the case with unknown drift parameters. In this paper, we extend the work by Lee *et al.* (2010) to multidimensional diffusion processes in order to investigate the joint stability of the covariance matrix parameters.

For the extension, we consider the following multidimensional stochastic differential equation:

$$d\mathbf{X}_t = \mathbf{a}(\mathbf{X}_t; \boldsymbol{\theta})dt + \boldsymbol{\sigma}d\mathbf{W}_t, \quad \mathbf{X}_0 = \mathbf{x}_0, \quad (1.1)$$

where  $\boldsymbol{\theta} \in \bar{\Theta}$ ,  $\Theta$  is a bounded convex domain in  $R^m$ , and  $\boldsymbol{\sigma} \in R^d \otimes R^r$  are unknown parameters,  $\mathbf{a}(\cdot; \boldsymbol{\theta})$  is an  $R^d$ -valued function defined on  $R^d \otimes \bar{\Theta}$  and  $\mathbf{W}$  is an  $r$ -dimensional standard Brownian motion. As in the one-dimensional case, we suppose that  $\mathbf{X}_{t_i}$ ,  $t_i = ih_n$ ,  $i = 1, \dots, n$ , are observed, where  $\{h_n\}$  is a sequence of positive real numbers such that  $h_n \rightarrow 0$  and  $(nh_n)^{-1} = O(1)$ . Now our main purpose is to develop a statistical procedure to test if there exists a change point of the covariance matrix, i.e.,

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$H_0$ :  $\mathbf{S}$  is constant over  $i = 1, \dots, n$  vs.  $H_1$ : not  $H_0$ ,

where  $\mathbf{S} = \boldsymbol{\sigma}\boldsymbol{\sigma}'$ .

This idea of testing the stability of the correlation/covariance structure is not completely new. In financial time series, this topic has long been an issue because the correlation/covariance matrix serves as one of the inputs for the computation of a trading portfolio. In particular, the correlation from international equity returns has been studied under the heading of “contagion”, meaning abnormally high correlation during financial crisis. For example, Longin and Solnik (1995) found that the correlation rises in periods of high volatility by analyzing monthly excess returns over the period 1960-90. And, recently, Rodriguez (2007) used switching-parameter copulas to model dependence structure of Asian countries’ daily returns. We refer the interested readers to Rodriguez (2007) and the papers cited therein. In this paper, however, we are more interested in analyzing the interdependence structure of high frequency data, say, intraday returns from KOSPI and KOSDAQ, in order to gain some insight about the cross-asset dynamics in the short-term horizon. In fact, this is not a simple problem and not much research has been performed. Thus we decided to adopt the SDE (1.1), a simplified version of Yoshida (1992), as the starting point of our investigation and plan to extend the current result to a more realistic model in the near future. Here the detecting tool will be the multidimensional extension of the cusum test used in Lee *et al.* (2010). (See Park and Lee (2006, 2007) and Song *et al.* (2007) for more applications of the cusum test.) The paper is organized as follows. In Section 2, we propose a cusum of squares test based on residuals and show that, under certain regularity conditions, the test has a limiting distribution of the sup of a multidimensional Brownian bridge. In Section 3, we illustrate our results with simulation study. And, in the Appendix, we provide the proof of Lemma 2.1.

## 2. Main result

First, let us assume that

(A1) There exists a constant  $L$  such that

$$\|\mathbf{a}(\mathbf{x}, \boldsymbol{\theta})\| \leq L(1 + \|\mathbf{x}\|).$$

(A2) There exists a constant  $L$  such that

$$\|\mathbf{a}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{a}(\mathbf{y}, \boldsymbol{\theta})\| \leq L\|\mathbf{x} - \mathbf{y}\|.$$

(A3) Under  $H_0$ ,  $\sup_t E\|\mathbf{X}_t\|^p < \infty$  for all  $p > 0$ .

(A4)  $nh_n^2 \rightarrow 0$  as  $n \rightarrow \infty$ .

Instead of the residuals used in Lee *et al.* (2010), we are going to consider the i.i.d random vectors  $\boldsymbol{\eta}_i = \boldsymbol{\sigma}(\mathbf{W}_{t_i} - \mathbf{W}_{t_{i-1}})h_n^{-1/2}$  and the residuals of the form  $\hat{\boldsymbol{\eta}}_i = (\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}})h_n^{-1/2}$ . Note that the former have  $d$ -dimensional multivariate normal distributions with mean 0 and covariance matrix  $\mathbf{S}$ . Based upon the residuals  $\hat{\boldsymbol{\eta}}_i$ , we define our test statistic as

$$T_n = \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left\| \text{vech} \left( \sum_{i=1}^k \hat{\boldsymbol{\eta}}_i \hat{\boldsymbol{\eta}}_i' - \frac{k}{n} \sum_{i=1}^n \hat{\boldsymbol{\eta}}_i \hat{\boldsymbol{\eta}}_i' \right) \right\|^2, \quad (2.1)$$

where the *vech* operator on a matrix stacks the lower triangular portion of a  $d \times d$  matrix as a  $\frac{d(d+1)}{2} \times 1$  vector. As an illustration, for a symmetric matrix

$$\mathbf{X} = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

$vech(\mathbf{X}) = (a, b, c)'$  (Henderson and Searle, 1979). Finally, we denote by  $\mathbf{l}_i$  and  $\hat{\mathbf{l}}_i$  the  $\frac{d(d+1)}{2}$ -dimensional random vectors  $vech(\boldsymbol{\eta}_i \boldsymbol{\eta}'_i)$  and  $vech(\hat{\boldsymbol{\eta}}_i \hat{\boldsymbol{\eta}}'_i)$ , respectively.

The intuition behind (2.1) is simple: because  $\hat{\mathbf{S}}_n = \frac{1}{n} \sum_{i=1}^n \hat{\boldsymbol{\eta}}_i \hat{\boldsymbol{\eta}}'_i$  is a consistent estimator of  $\mathbf{S}$ , one can expect that both  $\hat{\mathbf{S}}_n$  and  $\hat{\mathbf{S}}_k$  would be close to  $\mathbf{S}$  under  $H_0$ , and hence, the bracketed term inside of (2.1),

$$\frac{k}{\sqrt{n}} (\hat{\mathbf{S}}_k - \hat{\mathbf{S}}_n) = \frac{k}{\sqrt{n}} \left( \frac{1}{k} \sum_{i=1}^k \hat{\boldsymbol{\eta}}_i \hat{\boldsymbol{\eta}}'_i - \frac{1}{n} \sum_{i=1}^n \hat{\boldsymbol{\eta}}_i \hat{\boldsymbol{\eta}}'_i \right),$$

would converge to a multivariate Brownian bridge. See, for example, Lee *et al.* (2003). Note that our residual  $\hat{\boldsymbol{\eta}}'_i$  and the test statistic  $T_n$  do not depend on the functional form of  $\mathbf{a}(\mathbf{x}; \cdot)$  and the parameter value of  $\boldsymbol{\theta}$ . This would allow us for a great deal of flexibility in simulation/empirical study. We also remark that our main results would hold for the same type of residuals in Lee *et al.* (2010) with a slight modification.

Our first result is concerned with the negligibility of the difference between  $\mathbf{l}_i$  and  $\hat{\mathbf{l}}_i$ .

**Lemma 2.1** Assume that (A1) - (A4) hold. Under  $H_0$ ,

$$\frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^k \hat{\mathbf{l}}_i - \sum_{i=1}^k \mathbf{l}_i \right\| = o_P(1). \tag{2.2}$$

**Proof.** See Appendix for the proof. □

Before we state our main result, we define the  $\frac{d(d+1)}{2}$ -dimensional Brownian bridge with covariance matrix  $\boldsymbol{\Gamma}$  by

$$\mathbf{B}_{\boldsymbol{\Gamma}}^0(t) = \boldsymbol{\Gamma}^{1/2} \mathbf{B}^0(t),$$

where  $\mathbf{B}^0(t)$  is a  $\frac{d(d+1)}{2}$ -dimensional standard Brownian bridge.

**Theorem 2.1** Assume that (A1) - (A4) hold. Under  $H_0$ ,

$$T_n \xrightarrow{w} \sup_{0 \leq u \leq 1} \left\| \mathbf{B}_{\boldsymbol{\Gamma}}^0(u) \right\|^2 \quad \text{as } n \rightarrow \infty,$$

where the covariance matrix  $\boldsymbol{\Gamma} = Var(\mathbf{l}_i)$ .

**Proof.** Since  $\mathbf{l}_i$ 's are  $\frac{d(d+1)}{2}$ -dimensional i.i.d. random vectors, the functional central limit theorem can be applied to yield

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nu]} \mathbf{l}_i \xrightarrow{w} \boldsymbol{\Gamma}^{1/2} \mathbf{B}(u) \quad \text{as } n \rightarrow \infty, \tag{2.3}$$

where  $\mathbf{B}(t)$  is a  $\frac{d(d+1)}{2}$ -dimensional standard Brownian motion. Combining Lemma 2.1 and (2.3), we establish the theorem.  $\square$

By defining

$$T_n(\mathbf{\Gamma}) = \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left\| \mathbf{\Gamma}^{-1/2} \text{vech} \left( \sum_{i=1}^k \hat{\boldsymbol{\eta}}_i \hat{\boldsymbol{\eta}}_i' - \frac{k}{n} \sum_{i=1}^n \hat{\boldsymbol{\eta}}_i \hat{\boldsymbol{\eta}}_i' \right) \right\|^2,$$

we obtain the following as an immediate consequence of Theorem 2.1.

**Corollary 2.1** Assume that (A1) - (A4) hold. Under  $H_0$ ,

i) if the covariance matrix  $\mathbf{\Gamma}$  is invertible,

$$T_n(\mathbf{\Gamma}) \xrightarrow{w} \sup_{0 \leq u \leq 1} \|\mathbf{B}^0(u)\|^2 \quad \text{as } n \rightarrow \infty,$$

ii) and, in addition, if  $\hat{\mathbf{\Gamma}}$  is a consistent estimator of  $\mathbf{\Gamma}$ ,

$$T_n(\hat{\mathbf{\Gamma}}) \xrightarrow{w} \sup_{0 \leq u \leq 1} \|\mathbf{B}^0(u)\|^2 \quad \text{as } n \rightarrow \infty.$$

### 3. Simulation study

In this section, we evaluate the performance of the cusum of squares test through a simulation study. We consider the two-dimensional diffusion processes of the form

$$\begin{pmatrix} dX_{1t} \\ dX_{2t} \end{pmatrix} = \begin{pmatrix} \mu_1 - \beta_1 X_{1t} \\ \mu_2 - \beta_2 X_{2t} \end{pmatrix} dt + \mathbf{S}^{1/2} \begin{pmatrix} dW_{1t} \\ dW_{2t} \end{pmatrix}$$

with the following 6 scenarios:

1.  $(\mu_1, \mu_2, \beta_1, \beta_2)' = (0, 0, 0, 0)'$  and  $\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,
2.  $(\mu_1, \mu_2, \beta_1, \beta_2)' = (1, -1, 0, 0)'$  and  $\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,
3.  $(\mu_1, \mu_2, \beta_1, \beta_2)' = (1, 1, 0.5, 0)'$  and  $\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,
4.  $(\mu_1, \mu_2, \beta_1, \beta_2)' = (0, 0, 0, 0)'$  and  $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ ,
5.  $(\mu_1, \mu_2, \beta_1, \beta_2)' = (1, -1, 0, 0)'$  and  $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ ,
6.  $(\mu_1, \mu_2, \beta_1, \beta_2)' = (1, 1, 0.5, 0)'$  and  $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ .

**Table 3.1**  $(\mu_1, \mu_2, \beta_1, \beta_2)'$  and  $\mathbf{S}$  do not change.

$n$	$h = 1/n$			$h = n^{-0.6}$		
	100	500	1000	100	500	1000
Scenario 1	0.022	0.031	0.040	0.009	0.048	0.052
Scenario 2	0.021	0.049	0.046	0.023	0.053	0.041
Scenario 3	0.025	0.037	0.055	0.017	0.034	0.059
Scenario 4	0.022	0.044	0.049	0.015	0.046	0.057
Scenario 5	0.021	0.044	0.044	0.027	0.038	0.037
Scenario 6	0.022	0.031	0.046	0.024	0.043	0.064

**Table 3.2**  $(\mu_1, \mu_2, \beta_1, \beta_2)'$  changes at  $n/2$  and  $\mathbf{S}$  does not change.

$n$	$h = 1/n$			$h = n^{-0.6}$		
	100	500	1000	100	500	1000
Scenario 1 $\rightarrow$ 2	0.029	0.040	0.063	0.028	0.057	0.044
Scenario 1 $\rightarrow$ 3	0.021	0.052	0.049	0.027	0.044	0.054
Scenario 4 $\rightarrow$ 5	0.022	0.044	0.042	0.047	0.079	0.071
Scenario 4 $\rightarrow$ 6	0.026	0.052	0.050	0.033	0.054	0.068

**Table 3.3**  $(\mu_1, \mu_2, \beta_1, \beta_2)'$  does not change and  $\mathbf{S}$  changes at  $n\tau_0$ .

$\tau_0$	$n$	$h = 1/n$					$h = n^{-0.6}$				
		0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
Scenario 1 $\rightarrow$ 4											
	100	0.092	0.702	0.864	0.675	0.091	0.092	0.710	0.882	0.676	0.068
	500	0.860	1.000	1.000	1.000	0.900	0.875	1.000	1.000	1.000	0.858
	1000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Scenario 2 $\rightarrow$ 5											
	100	0.065	0.711	0.878	0.670	0.082	0.061	0.578	0.812	0.653	0.088
	500	0.861	1.000	1.000	1.000	0.864	0.797	1.000	1.000	1.000	0.870
	1000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Scenario 3 $\rightarrow$ 6											
	100	0.076	0.676	0.868	0.646	0.081	0.051	0.599	0.815	0.631	0.062
	500	0.874	1.000	1.000	1.000	0.873	0.810	1.000	1.000	1.000	0.839
	1000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999

**Table 3.4**  $(\mu_1, \mu_2, \beta_1, \beta_2)'$  and  $\mathbf{S}$  change at  $n/2$ .

$n$	$h = 1/n$			$h = n^{-0.6}$		
	100	500	1000	100	500	1000
Scenario 1 $\rightarrow$ 5	0.875	1.000	1.000	0.762	1.000	1.000
Scenario 1 $\rightarrow$ 6	0.868	1.000	1.000	0.863	1.000	1.000
Scenario 4 $\rightarrow$ 2	0.901	1.000	1.000	0.926	1.000	1.000
Scenario 4 $\rightarrow$ 3	0.878	1.000	1.000	0.876	1.000	1.000

For easier exposition, we assume that the two processes are independent in the first half scenarios but dependent with increased volatility in the latter. In order to calculate the empirical sizes and powers, we examine the following 4 cases:

1. Neither  $(\mu_1, \mu_2, \beta_1, \beta_2)'$  nor  $\mathbf{S}$  changes.
2.  $(\mu_1, \mu_2, \beta_1, \beta_2)'$  changes at  $n/2$  and  $\mathbf{S}$  does not change.
3.  $(\mu_1, \mu_2, \beta_1, \beta_2)'$  does not change and  $\mathbf{S}$  changes at  $n\tau_0$ .
4. Both  $(\mu_1, \mu_2, \beta_1, \beta_2)'$  and  $\mathbf{S}$  change at  $n/2$ .

Each simulation result is summarized in Tables 3.1 through 3.4. Here we have used the test statistic in Corollary 2.1 at the nominal level of  $\alpha = 0.05$ , i.e., the critical value of 3.004 (Table 1 of Lee *et al.*, 2003), with the sample sizes of  $n = 100, 500, 1000$  and the sampling time lengths of  $h = n^{-1}$  and  $h = n^{-0.6}$ . As illustrated in Tables 3.1 and 3.2, there seems to be no severe size distortion except for  $n = 100$ . And Tables 3.3 and 3.4 show that the empirical powers are close to 1 except for the case when  $n = 100$  and the changes occur at  $\tau_0 = 0.1$  or  $0.9$ . Although not included in Table 3.3, almost the same result could be obtained for the opposite directions, say, scenario  $4 \rightarrow 1$ , etc. To conclude, the cusum of squares test is a proper tool to investigate the joint stability of the covariance matrix parameters of the multidimensional diffusion processes.

## Appendix.

**Proof of Lemma 2.1** First, note that

$$\hat{\eta}_{ij}\hat{\eta}_{il} - \eta_{ij}\eta_{il} = (\hat{\eta}_{ij} - \eta_{ij})(\hat{\eta}_{il} - \eta_{il}) + (\hat{\eta}_{ij} - \eta_{ij})\eta_{il} + (\hat{\eta}_{il} - \eta_{il})\eta_{ij}.$$

Hence the left-hand side of (2.2) can be bounded as follows:

$$\begin{aligned} & \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^k \hat{l}_i - \sum_{i=1}^k l_i \right\| \\ & \leq \sqrt{d} \max_{1 \leq j \leq l \leq d} \left( \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (\hat{\eta}_{ij}\hat{\eta}_{il} - \eta_{ij}\eta_{il}) \right| \right) \\ & \leq \sqrt{d} \max_{1 \leq j \leq d} \left[ \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \sum_{i=1}^k (\hat{\eta}_{ij} - \eta_{ij})^2 \right] \\ & \quad + \sqrt{d} \max_{1 \leq j, l \leq d} \left[ \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \sum_{i=1}^k (\hat{\eta}_{ij} - \eta_{ij}) \eta_{il} \right]. \end{aligned} \tag{A.1}$$

For the term inside the first square bracket of (A.1), it holds that

$$\begin{aligned} \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \sum_{i=1}^k (\hat{\eta}_{ij} - \eta_{ij})^2 &= \frac{1}{\sqrt{n}} \sum_{i=1}^n (\hat{\eta}_{ij} - \eta_{ij})^2 \\ &\leq \frac{1}{\sqrt{n}} \sum_{i=1}^n \|\hat{\boldsymbol{\eta}}_i - \boldsymbol{\eta}_i\|^2 \\ &\leq \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} L^2 (1 + \|\mathbf{X}_s\|)^2 ds. \end{aligned}$$

Therefore, under assumptions (A1), (A3), and (A4), we obtain that

$$E \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} L^2 (1 + \|\mathbf{X}_s\|)^2 ds \right] = O(\sqrt{nh_n^2}) = o(1). \tag{A.2}$$

And, for the term inside the second square bracket of (A.1), we put

$$\Delta_i = \int_{t_{i-1}}^{t_i} \{\mathbf{a}(\mathbf{X}_s; \boldsymbol{\theta}) - \mathbf{a}(\mathbf{X}_{t_{i-1}}; \boldsymbol{\theta})\} ds, \quad \mathbf{A}_i = \mathbf{a}(\mathbf{X}_{t_{i-1}}; \boldsymbol{\theta}).$$

Since

$$\begin{aligned} \hat{\boldsymbol{\eta}}_i - \boldsymbol{\eta}_i &= \frac{1}{\sqrt{h_n}} \int_{t_{i-1}}^{t_i} \mathbf{a}(\mathbf{X}_s; \boldsymbol{\theta}) ds \\ &= \frac{1}{\sqrt{h_n}} \int_{t_{i-1}}^{t_i} \{\mathbf{a}(\mathbf{X}_s; \boldsymbol{\theta}) - \mathbf{a}(\mathbf{X}_{t_{i-1}}; \boldsymbol{\theta})\} ds + \sqrt{h_n} \mathbf{a}(\mathbf{X}_{t_{i-1}}; \boldsymbol{\theta}), \end{aligned}$$

we can write

$$(\hat{\eta}_{ij} - \eta_{ij}) \eta_{il} = \frac{1}{h_n} \Delta_{ij} \eta_{il} + \sqrt{h_n} A_{ij} \eta_{il}.$$

By using the functional central limit theorem and the fact that  $\sum_{i=1}^k A_{ij} \eta_{il}$  is a martingale, we get

$$\frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \sqrt{h_n} A_{ij} \eta_{il} \right| = o_P(1). \tag{A.3}$$

Moreover,

$$\begin{aligned} E \left[ \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \frac{1}{\sqrt{h_n}} \Delta_{ij} \eta_{il} \right| \right] &\leq \frac{1}{\sqrt{n}} E \left[ \max_{1 \leq k \leq n} \left( \sum_{i=1}^k h_n^{-1} \Delta_{ij}^2 \right)^{1/2} \left( \sum_{i=1}^k \eta_{il}^2 \right)^{1/2} \right] \\ &\leq \frac{1}{\sqrt{n}} \left( E \left[ \sum_{i=1}^n h_n^{-1} \Delta_{ij}^2 \right] \right)^{1/2} \left( E \left[ \sum_{i=1}^n \eta_{il}^2 \right] \right)^{1/2} \\ &= O(\sqrt{nh_n^2}) = o(1). \end{aligned} \tag{A.4}$$

Here, in the last step, we have used the fact that

$$\begin{aligned} E[\Delta_{ij}^2] &\leq E[\|\Delta_i\|^2] \\ &\leq h_n \int_{t_{i-1}}^{t_i} E[\|\mathbf{a}(\mathbf{X}_s; \boldsymbol{\theta}) - \mathbf{a}(\mathbf{X}_{t_{i-1}}; \boldsymbol{\theta})\|^2] ds \\ &\leq h_n \int_{t_{i-1}}^{t_i} E[L^2 \|\mathbf{X}_s - \mathbf{X}_{t_{i-1}}\|^2] ds \leq Ch_n^3 \end{aligned}$$

(Yoshida, 1992). Now it follows from (A.3) and (A.4) that

$$\frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^k (\hat{\eta}_{ij} - \eta_{ij}) \eta_{il} \right| = o_P(1). \quad (\text{A.5})$$

Combining (A.1), (A.2), and (A.5) completes the proof.  $\square$

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